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Row stochastic inverse eigenvalue problem

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Abstract

In this paper, we give sufficient conditions or realizability criteria for the existence of a row stochastic matrix with a given spectrum $\Lambda = \{\lambda_1, ..., \lambda_n\} = \Lambda_1 \cup ... \cup \Lambda_m \cup \Lambda_{m+1}$, $m > 0$; where $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kp_k}\} = \{\lambda_{k1}, \omega_k e^{2\pi i/p_k}, \omega_k e^{4\pi i/p_k}, ..., \omega_k e^{2(p_k-1)\pi i/p_k}\}$ (p_k is an integer greater than 1), $\lambda_{k1} = \lambda_k > 0$, $1 = \lambda_1 \geq \omega_k > 0$, $k = 1, ..., m$; $\Lambda_{m+1} = \{\lambda_m+1\}$, $\omega_{m+1} \equiv \lambda_1 + ... + \lambda_n \leq \lambda_1$, $\omega_k \geq \lambda_k$, $\omega_1 \geq \lambda_k$, $k = 2, ..., m + 1$. In the case when $p_1, ..., p_m$ are all equal to 2, Λ becomes a list of $2m + 1$ real numbers for any positive integer m, and our result gives sufficient conditions for a list of $2m + 1$ real numbers to be realizable by a row stochastic matrix.
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Keywords: row stochastic matrices, inverse eigenvalue problem, row stochastic inverse eigenvalue problem

1 Introduction and preliminaries

A list of complex numbers Λ is realizable by a matrix A if Λ is the spectrum of A. An $n \times n$ nonnegative matrix $A = (a_{mk})$ = is a row stochastic matrix if $\sum_{k=1}^{n} a_{mk} = 1$, $m = 1, ..., n$. The row stochastic inverse eigenvalue problem is the problem of characterizing all possible spectrum of row stochastic matrices. Since row stochastic matrices are important in applications this kind of inverse eigenvalue problems should be interesting. There are also some papers that contribute to the study of the doubly stochastic inverse eigenvalue problem (e.g., [1,2] and references therein).

In this paper, we give sufficient conditions for some lists of complex numbers, including lists of $2m + 1$ real numbers, to be realizable by a row stochastic matrix.

We use $A \in \mathcal{CS}_r$ to denote the fact that the $n \times n$ real square matrix $A = (a_{mk})$ satisfies $\sum_{k=1}^{n} a_{mk} = r$, $m = 1, ..., n$; use $A = \text{diag}(A_1, ..., A_t)$ to denote the fact that A is a block diagonal matrix with diagonal blocks $A_1, ..., A_t$; use $\sigma(A)$ to denote the spectrum of A; use $P(n)$ to denote the permutation matrix of order n the kth row of which is the $(k + 1)$th row of I_n with the first row being $(1, 0, ..., 0)$. Later, we will make use of the fact that the spectrum of $\omega P(n)$ is $\{\omega, \omega e^{2\pi i/n}, \omega e^{4\pi i/n}, ..., \omega e^{(n-1)\pi i/n}\}$.

Since our results are based on the following theorem from [3], we restate it with the proof.

Theorem 1 (Brauer extended) [3] Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, ..., \lambda_n$ and $X = (x_1, ..., x_t)$ be such that $\text{rank}(X) = t$ and $Ax_k = \lambda_k x_k$, $k = 1, ..., t$, $t \leq n$. Let C be a $t \times n$ arbitrary matrix. Then the matrix $A + XC$ has eigenvalues $\mu_1, ..., \mu_t, \lambda_{t+1}, ..., \lambda_n$, where $\mu_1, ..., \mu_t$ are eigenvalues of the matrix $D + CX$ with $D = \text{diag}(\lambda_1, ..., \lambda_t)$.

Proof Let $S = (X, Y)$ be a nonsingular matrix with $S^{-1} = \begin{pmatrix} U \\ V \end{pmatrix}$. Then $UX = I_t$, $VY = I_{n-t}$, $VX = UY = 0$. Let $C = C_1, C_2)$, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ where both C_1, X_1 are both $t \times t$ and Y_1 is $t \times (n - t)$. since $AX = XD$, we have

$$\begin{aligned} S^{-1}AS &= \begin{pmatrix} U \\ V \end{pmatrix} (XD\ AY) = \begin{pmatrix} D & UAY \\ 0 & VAY \end{pmatrix} \\ S^{-1}XCS &= \begin{pmatrix} I_t \\ 0 \end{pmatrix} (C_1\ C_2) S = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix} = \begin{pmatrix} CX & CY \\ 0 & 0 \end{pmatrix} \\ S^{-1}(A + XC)S &= S^{-1}AS + S^{-1}XCS = \begin{pmatrix} D + CX & UAY + CY \\ 0 & VAY \end{pmatrix}. \end{aligned} \quad (1)$$

Now from ([**?**]) we have $\sigma(VAY) = \sigma(A)\backslash\sigma(D)$ and therefore

$$\sigma(A + XC) = \sigma(D + CX) \bigcup (\sigma(A)\backslash\sigma(D)).$$

◊

Lemma 1 If

$$0 \le \omega_k \le \lambda_1 = 1,\ k = 1, \ldots, t; \quad (2)$$

$$\omega_1 + \cdots + \omega_t = \lambda_1 + \cdots + \lambda_t; \quad (3)$$

$$\omega_k \ge \lambda_k,\ \omega_1 \ge \lambda_k,\ k = 2, \ldots, t, \quad (4)$$

then the following matrix

$$B = \begin{pmatrix} \omega_1 & \omega_2 - \lambda_2 & \omega_3 - \lambda_3 & \cdots & \omega_t - \lambda_t \\ \omega_1 - \lambda_2 & \omega_2 & \omega_3 - \lambda_3 & \cdots & \omega_t - \lambda_t \\ \omega_1 - \lambda_3 & \omega_2 - \lambda_2 & \omega_3 & \cdots & \omega_t - \lambda_t \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega_1 - \lambda_t & \omega_2 - \lambda_2 & \omega_3 - \lambda_3 & \cdots & \omega_t \end{pmatrix} \quad (5)$$

is a row stochastic matrix with eigenvalues $\lambda_1, ..., \lambda_t$ and diagonal entries $\omega_1, ..., \omega_t$.

Proof It is clear that $B \in \mathcal{CS}_{\lambda_1} = \mathcal{CS}_1$ has diagonal entries $\omega_1, ..., \omega_t$ and is a nonnegative matrix by (2) and (4). In addition, the eigenpolynomial of B is factorized as

$$det(\lambda I - B) = (\lambda - \lambda_1) det \begin{pmatrix} 1 & \lambda_2 - \omega_2 & \cdots & \lambda_t - \omega_t \\ 1 & \lambda - \omega_2 & \cdots & \lambda_t - \omega_t \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_2 - \omega_2 & \cdots & \lambda - \omega_t \end{pmatrix} = \prod_{k=1}^{t} (\lambda - \lambda_k).$$

◊

2 Main results

Since the set Λ in Theorem 2 is assumed to be a list of complex numbers, it could be considered as a generalization of Theorem 8 of [3] (Λ in Theorem 8 of [3] is assumed to be a real list). But the representation and the proof of these two theorems almost have no difference.

Theorem 2 [3] Let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ be a list of complex numbers. If there exists a partition $\Lambda = \Lambda_1 \cup ... \cup \Lambda_t$, $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ $k = 1, \ldots, t$ with $\lambda_{11} = \lambda_1 \geq \lambda_{21} \geq \lambda_{31} \geq ... \geq \lambda_{t1} > 0$ and for each Λ_k we associate a corresponding list $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $0 \leq \omega_k \leq \omega_1$ which is realizable by a nonnegative matrix $A_k \in \mathcal{CS}_{\omega_k}$ of order p_k, $k = 1, ..., t$, as well as there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ of order t, which has eigenvalues $\{\lambda_1, \lambda_{21}, ..., \lambda_{t1}\}$ and diagonal entries $\{\omega_1, ..., \omega_t\}$, then Λ is realizable by an $n \times n$ nonnegative matrix $M \in \mathcal{CS}_{\lambda_1}$.

Proof Note the p_k-dimensional vector $(1, ..., 1)^{\mathrm{T}}$ is an eigenvector of $A_k \in \mathcal{CS}_{\omega_k}$ corresponding to the eigenvalue ω_k. Let $X = (X_1, ..., X_t)$, where $X_k = (0, ... 0, 1, ..., 1, 0, ..., 0)^{\mathrm{T}}$ is an n-dimensional vector with p_k ones from the position $p_1 + ... + p_{k-1} + 1$ to $p_1 + ... + p_k$ and zeros elsewhere. Let $A = \mathrm{diag}(A_1, ..., A_t)$, $D = \mathrm{diag}(\omega_1, ..., \omega_t)$, then X is of rank t and $AX = XD$.

Let $C = (C_1, ..., C_t)$, where C_k is the $t \times p_k$ matrix whose first column is $(c_{1k}, c_{2k}, ..., c_{tk})^{\mathrm{T}}$ and whose other entries are all zero. Then

$$CX = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1t} \\ c_{21} & c_{22} & \cdots & c_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t1} & c_{t2} & \cdots & c_{tt} \end{pmatrix}, \quad XC = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1t} \\ c_{21} & c_{22} & \cdots & c_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t1} & c_{t2} & \cdots & c_{tt} \end{pmatrix},$$

where C_{mk} is the $p_m \times p_k$ matrix whose first column is $(c_{mk}, c_{mk}, ..., c_{mk})^{\mathrm{T}}$ and whose other entries are all zero. Now we chose C with $c_{11}, ..., c_{tt} = 0$ so that the matrix $D + CX = B \in \mathcal{CS}_{\lambda_1}$. Then for this choice of C, we conclude that $M = A + XC \in \mathcal{CS}_{\lambda_1}$ is nonnegative with spectrum Λ by Theorem 1. ◊

Theorem 3 Let a list of complex numbers $\Lambda = \{\lambda_1, ..., \lambda_n\} = \Lambda_1 \cup ... \cup \Lambda_m \cup \Lambda_{m+1}$, $m > 0$; $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kp_k}\} = \{\lambda_{k1}, \omega_k e^{2\pi i/p_k}, \omega_k e^{4\pi i/p_k}, ..., \omega_k e^{2(p_k-1)\pi i/p_k}\}$ (p_k is an integer greater than 1), $\lambda_{k1} = \lambda_k > 0$, $k = 1, ..., m$; $\Lambda_{m+1} = \{\lambda_{m+1}\}$ be such that $1 = \lambda_1 \geq \omega_k > 0$, $k = 1, 2, ..., m$. Let

$$\omega_{m+1} = s = \lambda_1 + \cdots + \lambda_n, \ p_{m+1} = 1. \tag{6}$$

If

$$\lambda_1 \geq s, \tag{7}$$

$$\omega_k \geq \lambda_k, \ \omega_1 \geq \lambda_k, \ k = 2, \ldots, m+1, \tag{8}$$

then Λ is realizable by the following $n \times n$ row stochastic matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1,m+1} \\ M_{21} & M_{22} & \cdots & M_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m+1,1} & M_{m+1,2} & \cdots & M_{m+1,m+1} \end{pmatrix}, \tag{9}$$

where $M_{kk} = \omega_k P(p_k)$, $k = 1, ..., m + 1$; M_{kj} is the $p_k \times p_j$ matrix whose first column is $(\omega_j - \lambda_j, ..., \omega_j - \lambda_j)^{\mathrm{T}}$, $k \neq j$, $j = 2, ..., m + 1$ and whose other entries are all zero; M_{k1} is the $p_k \times p_1$ matrix whose first column is $(\omega_1 - \lambda_k, ..., \omega_1 - \lambda_k)^{\mathrm{T}}$, $k = 2, ..., m + 1$ and whose other entries are all zero.

Proof It is clear that $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ is realizable by the nonnegative matrix $A_k = \omega_k P(p_k) \in \mathcal{CS}_{\omega_k}$, $k = 1, ..., m + 1$. Since

$\omega_k e^{2\pi i/p_k} + \omega_k e^{4\pi i/p_k} + \cdots + \omega_k e^{2(p_k-1)\pi i/p_k} = -\omega_k$, $k = 1, ..., m$, we have $\omega_{m+1} = s = \lambda_1 + ... + \lambda_n = \lambda_1 - \omega_1 + ... + \lambda_m - \omega_m + \lambda_{m+1}$ by Condition (6) and hence $\lambda_1 + ... + \lambda_{m+1} = \omega_1 + ... + \omega_{m+1}$. Meanwhile if (7) and (8) hold, then all conditions of Lemma 1 are satisfied and hence the row stochastic matrix B defined in (5) with $t = m + 1$ has eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_{m+1}\}$ and diagonal entries $\{\omega_1, ..., \omega_{m+1}\}$. Therefore, the list Λ must be realizable by an $n \times n$ row stochastic matrix M by Theorem 2. Applying Theorem 2, we compute the solution matrix M and get the result as defined in (9). ◊

When $p_k \le 2$ for all $k = 1, ..., m + 1$, the set Λ in Theorem 3 becomes a list of real numbers. In this case, applying Theorem 3, we have the following result for real row stochastic inverse eigenvalues problem.

Theorem 4 If a list of real numbers $\Lambda = \{\lambda_1, ..., \lambda_{2m+1}\} = \Lambda_1 \cup ... \cup \Lambda_m \cup \{\lambda_{m+1}\}$; $\Lambda_k = \{\lambda_k, \lambda_{2m+2-k}\}$, $k = 1, ..., m$ satisfies

$$1 = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{m+1} > 0 > \lambda_{m+2} \ge \cdots \ge \lambda_{2m+1} \ge -1, \tag{10}$$

$$s = \lambda_1 + \lambda_2 + \cdots + \lambda_{2m+1} > \lambda_{m+1} \tag{11}$$

$$\lambda_{2m+2-k} > \lambda_k, \; k = 2, \ldots, m, \tag{12}$$

then Λ is realizable by the following row stochastic matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1m} & M_{1,m+1} \\ M_{21} & M_{22} & \cdots & M_{2m} & M_{1,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mm} & M_{m,m+1} \\ M_{m+1,1} & M_{m+2,2} & \cdots & M_{m+1,m} & s \end{pmatrix} \tag{13}$$

where

$$M_{kk} = \begin{pmatrix} 0 & -\lambda_{2m+2-k} \\ -\lambda_{2m+2-k} & 0 \end{pmatrix}, \; k = 1, \ldots, m;$$

$$M_{k1} = \begin{pmatrix} -\lambda_{2m+1} - \lambda_k & 0 \\ -\lambda_{2m+1} - \lambda_k & 0 \end{pmatrix}, \; k = 2, \ldots, m; \quad M_{m+1,1} = \begin{pmatrix} -\lambda_{2m+1} - s & 0 \end{pmatrix},$$

$$M_{kj} = \begin{pmatrix} -\lambda_{2m+2-j} - \lambda_j & 0 \\ -\lambda_{2m+2-j} - \lambda_j & 0 \end{pmatrix}; \quad M_{m+1,j} = \begin{pmatrix} -\lambda_{2m+2-j} - \lambda_j & 0 \end{pmatrix},$$

$$j = 2, \ldots, m, j \ne k; \quad M_{k,m+1} = \begin{pmatrix} s - \lambda_{m+1} \\ s - \lambda_{m+1} \end{pmatrix}, \quad k = 1, \ldots, m.$$

Proof Let $\mu_k = \lambda_k$, $\omega_k = -\lambda_{2m+2-k}$, $p_k = 2$, $k = 1, ..., m$, $\omega_{m+1} = s$, $p_{m+1} = 1$, then all the conditions of Theorem 3 are satisfied and Λ is realizable by the row stochastic matrix M defined in (9) by Theorem 3. In the case of Theorem 4, the matrix in (9) becomes the matrix in (13). Therefore, Λ is realizable by the row stochastic matrix M in (13). ◊

Remark Theorem 10 of [3] gives sufficient conditions only for a list of 5 real numbers to be the spectrum of some 5×5 nonnegative matrix $M \in \mathcal{CS}_{\lambda_1}$; our Theorem 4 gives sufficient conditions for a list of $2m + 1$ real numbers for any integer $m > 0$ to be the spectrum of some row stochastic matrix. In addition, the conditions of Theorem 4 are more easily handled.

3 Examples

Example 1 $\Lambda = \{\lambda_1, ..., \lambda_7\} = \{1, 0.75, 0.7, 0.1, -0.75, -0.8, -0.8\}$ satisfies Conditions (10), (11) and (12) of Theorem 4 with $m = 3$, $s = 0.2$, $-\lambda_5 - \lambda_2 = -\lambda_6 - \lambda_3 = 0.05$, $s - \lambda_4 = 0.1$, $-\lambda_7 - \lambda_2 = 0.05$, $-\lambda_7 - \lambda_3 = 0.1$, $-\lambda_7 - \lambda_4 = 0.7$. Therefore Λ is realizable by the following row stochastic matrix:

$$M = \begin{pmatrix} 0 & 0.8 & 0.05 & 0 & 0.05 & 0 & 0.1 \\ 0.8 & 0 & 0.05 & 0 & 0.05 & 0 & 0.1 \\ 0.05 & 0 & 0 & 0.8 & 0.05 & 0 & 0.1 \\ 0.05 & 0 & 0.8 & 0 & 0.05 & 0 & 0.1 \\ 0.1 & 0 & 0.05 & 0 & 0 & 0.75 & 0.1 \\ 0.1 & 0 & 0.05 & 0 & 0.75 & 0 & 0.1 \\ 0.7 & 0 & 0.05 & 0 & 0.05 & 0 & 0.2 \end{pmatrix}.$$

Example 2 $\Lambda = \{\lambda_1, ..., \lambda_6\} = \{1, 0.2, 0.7e^{2\pi i/5}, 0.7e^{4\pi i/5}, 0.7e^{6\pi i/5}, 0.7e^{8\pi i/5}\}$ satisfies all the conditions of Theorem 3 with $m = 1$, $p_1 = 5$, $p_2 = 1$, $\omega_2 = s = 0.5 < 1 = \lambda_1$, $\omega_1 = 0.7 > \omega_2 > 0.2 = \lambda_2$, $\omega_2 - \lambda_2 = 0.3$, $\omega_1 - \lambda_2 = 0.5$. Therefore, Λ is realizable by the following row stochastic matrix:

$$\begin{pmatrix} 0 & 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0.7 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 \\ 0.7 & 0 & 0 & 0 & 0 & 0.3 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}.$$

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The two authors contributed equally to this work. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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