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Sharp Cusa and Becker-Stark inequalities

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Abstract

We determine the best possible constants θ,ϑ,α and β such that the inequalities

$$\left(\frac{2+\cos x}{3}\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\vartheta}$$

and

$$\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\alpha} < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\beta}$$

are valid for $0 < x < \pi/2$. Our results sharpen inequalities presented by Cusa, Becker and Stark.

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1. Introduction

For $0 < x < \pi/2$, it is known in the literature that

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}. \tag{1}$$

Inequality (1) was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. A rigorous proof of inequality (1) was given by Huygens [1], who used (1) to estimate the number π. The inequality is now known as Cusa's inequality [2-5]. Further interesting historical facts about the inequality (1) can be found in [2].

It is the first aim of present paper to establish sharp Cusa's inequality.

Theorem 1. *For* $0 < x < \pi/2$,

$$\left(\frac{2+\cos x}{3}\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\vartheta} \tag{2}$$

with the best possible constants

$$\theta = \frac{\ln(\pi/2)}{\ln(3/2)} = 1.11373998\ldots \quad \text{and} \quad \vartheta = 1.$$

Becker and Stark [6] obtained the inequalities

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2} \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{3}$$

The constant 8 and π^2 are the best possible.

Zhu and Hua [7] established a general refinement of the Becker-Stark inequalities by using the power series expansion of the tangent function via Bernoulli numbers and the property of a function involving Riemann's zeta one. Zhu [8] extended the tangent function to Bessel functions.

It is the second aim of present paper to establish sharp Becker-Stark inequality.

Theorem 2. *For* $0 < \times < \pi/2$,

$$\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\alpha} < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\beta} \tag{4}$$

with the best possible constants

$$\alpha = \frac{\pi^2}{12} = 0.822467033\ldots \quad \text{and} \quad \beta = 1.$$

Remark 1. *There is no strict comparison between the two lower bounds* $\frac{8}{\pi^2 - 4x^2}$ *and* $\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\pi^2/12}$ *in* (3) *and* (4).

The following lemma is needed in our present investigation.

Lemma 1 ([9-11]). *Let* $-\infty < a < b < \infty$, *and f, g* : $[a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable in* (a, b). *Suppose* $g' \neq 0$ *on* $(a; b)$. *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a, b), *then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \text{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x) = g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

2. Proofs of Theorems 1 and 2

Proof of Theorem [1]. Consider the function $f(x)$ defined by

$$F(x) = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln\left(\frac{2 + \cos x}{3}\right)}, \qquad 0 < x < \frac{\pi}{2},$$

$$F(0) = 1 \quad \text{and} \quad \mathrm{F}\left(\frac{\pi}{2}\right) = \frac{\ln(\pi/2)}{\ln(3/2)}.$$

For $0 < x < \pi/2$, let

$$F_1(x) = \ln\left(\frac{\sin x}{x}\right) \quad \text{and} \quad F_2(x) = \ln\left(\frac{2 + \cos x}{3}\right).$$

Then,

$$\frac{F'_1(x)}{F'_2(x)} = \frac{-2x\cos x - x\cos^2 x + 2\sin x + \sin x\cos x}{x\sin^2 x} = \frac{F_3(x)}{F_4(x)},$$

where

$$F_3(x) = -2x\cos x - x\cos^2 x + 2\sin x + \sin x\cos x \quad \text{and} \quad F_4(x) = x\sin^2 x.$$

Differentiating with respect to x yields

$$\frac{F'_3(x)}{F'_4(x)} = \frac{2x + 2x\cos x - \sin x}{\sin x + 2x\cos x} \triangleq F_5(x).$$

Elementary calculations reveal that

$$F'_5(x) = \frac{2F_6(x)}{2x\sin(2x) + 4x^2\cos^2 x + \sin^2 x},$$

where

$$F_6(x) = \sin(2x) + (2x^2 + 1)\sin x - 2x - x\cos x.$$

By using the power series expansions of sine and cosine functions, we find that

$$F_6(x) = x^3 - \frac{1}{10}x^5 - \frac{19}{2520}x^7 + 2\sum_{n=4}^{\infty}(-1)^n u_n(x),$$

where

$$u_n(x) = \frac{4^n - 4n^2 - 3n}{(2n+1)!}x^{2n+1}.$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \geq 4$,

$$\begin{aligned}\frac{u_{n+1}(x)}{u_n(x)} &= \frac{x^2}{2}\frac{2^{2n+2} - 4n^2 - 11n - 7}{(n+1)(2n+3)(4^n - 4n^2 - 3n)} \\ &< \frac{1}{2}\left(\frac{\pi}{2}\right)^2\frac{2^{2n+2} - 4n^2 - 11n - 7}{(n+1)(2n+3)(4^n - 4n^2 - 3n)} \\ &= \frac{\pi^2}{8(n+1)}\frac{4^{n+1} - 4n^2 - 11n - 7}{(2n+3)(4^n - 4n^2 - 3n)} \\ &< \frac{\pi^2}{8(n+1)} < 1.\end{aligned}$$

Hence, for fixed $x \in (0, \pi/2)$, the sequence $n \mapsto u_n(x)$ is strictly decreasing with regard to $n \geq 4$. Hence, for $0 < x < \pi/2$,

$$F_6(x) = x^3 - \frac{1}{10}x^5 - \frac{19}{2520}x^7 > 0 \quad \left(0 < x < \frac{\pi}{2}\right),$$

and therefore, the functions $F_5(x)$ and $\frac{F'_3(x)}{F'_4(x)}$ are both strictly increasing on $(0, \pi/2)$.

By Lemma 1, the function

$$\frac{F'_1(x)}{F'_2(x)} = \frac{F_3(x)}{F_4(x)} = \frac{F_3(x) - F_3(0)}{F_4(x) - F_4(0)}$$

is strictly increasing on (0, $\pi/2$). By Lemma 1, the function

$$F(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x) - F_1(0)}{F_2(x) - F(0)}$$

is strictly increasing on (0, $\pi/2$), and we have

$$1 = F(0) < F(x) = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln\left(\frac{2+\cos x}{3}\right)} < F\left(\frac{\pi}{2}\right) = \frac{\ln(\pi/2)}{\ln(3/2)} \quad \forall x \in \left(0, \frac{\pi}{2}\right).$$

By rearranging terms in the last expression, Theorem 1 follows.

Proof of Theorem 2. Consider the function $f(x)$ defined by

$$f(x) = \frac{\ln\left(\frac{\tan x}{x}\right)}{\ln\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)}, \quad 0 < x < \frac{\pi}{2},$$

$$f(0) = \frac{\pi^2}{12} \quad and \quad f\left(\frac{\pi}{2}\right) = 1.$$

For $0 < x < \pi/2$, let

$$f_1(x) = \ln\left(\frac{\tan x}{x}\right) \quad \text{and} \quad f_2(x) = \ln\left(\frac{\pi^2}{\pi^2 - 4x^2}\right).$$

Then,

$$\frac{f'_1(x)}{f'_2(x)} = \frac{(\pi^2 - 4x^2)(2x - \sin(2x))}{8x^2\sin(2x)} \triangleq g(x).$$

Elementary calculations reveal that

$$4x^3\sin^2(2x)g'(x) = -(\pi^2+4x^2)x\sin(2x) - 2(\pi^2-4x^2)x^2\cos(2x) + \pi^2\sin^2(2x) \triangleq h(x).$$

Motivated by the investigations in [12], we are in a position to prove $h(x) > 0$ for $x \in (0, \pi/2)$.Let

$$H(x) = \begin{cases} \lambda, & x = 0, \\ \frac{h(x)}{x^6\left(\frac{\pi}{2} - x\right)^2} & 0 < x < \frac{\pi}{2}, \\ \mu, & x = \frac{\pi}{2}, \end{cases}$$

Where λ and μ are constants determined with limits:

$$\lambda = \lim_{x \to 0^+} \frac{h(x)}{x^6\left(\frac{\pi}{2} - x\right)^2} = \frac{224\pi^2 - 1920}{45\pi^2} = 0.654740609...,$$

$$\mu = \lim_{t \to (\pi/2)^-} \frac{h(x)}{x^6\left(\frac{\pi}{2} - x\right)^2} = \frac{128}{\pi^4} = 1.31404572....$$

Using Maple, we determine Taylor approximation for the function $H(x)$ by the polynomial of the first order:

$$P_1(x) = \frac{32(7\pi^2 - 60)}{45\pi^2} + \frac{128(7\pi^2 - 60)}{45\pi^3}x,$$

which has a bound of absolute error

$$\varepsilon_1 = \frac{-1920 - 1920\pi^2 + 224\pi^4}{15\pi^4} = 0.650176097...$$

for values $x \in [0,\pi/2]$. It is true that

$$H(x) - (P_1(x) - \mathcal{E}_1) \geq 0, \quad P_1(x) - \mathcal{E}_1 = \frac{64(60\pi^2 + 90 - 7\pi^4)}{45\pi^4} + \frac{128(7\pi^2 - 60)}{45\pi^3}x > 0$$

for $x \in [0, \pi/2]$. Hence, for $x \in [0, \pi/2]$, it is true that $H(x) > 0$ and, therefore, $h(x) > 0$ and $g'(x) > 0$ for $x \in [0, \pi/2]$. Therefore, the function $\frac{f'_1(x)}{f'_2(x)}$ is strictly increasing on. $(0, \pi/2)$.By Lemma 1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

is strictly increasing on $(0, \pi/2)$, and we have

$$\frac{\pi^2}{12} = f(0) < f(x) = \frac{\ln\left(\frac{\tan x}{x}\right)}{\ln\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)} < f\left(\frac{\pi}{2}\right) = 1.$$

By rearranging terms in the last expression, Theorem 2 follows.

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Authors' contributions
All authors read and approved the final manuscript

Competing interests
The authors declare that they have no competing interests.

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