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On the refinements of the Jensen-Steffensen inequality

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Abstract

In this paper, we extend some old and give some new refinements of the Jensen-Steffensen inequality. Further, we investigate the log-convexity and the exponential convexity of functionals defined via these inequalities and prove monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, we give several examples of the families of functions for which the results can be applied.

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1. Introduction

One of the most important inequalities in mathematics and statistics is the Jensen inequality (see [[1], p.43]).

Theorem 1.1. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\mathbf{x} = (x_1, ..., x_n) \in I^n$ and $\mathbf{p} = (p_1, ..., p_n)$ be a positive n-tuple, that is, such that $p_i > 0$ for $i = 1, ..., n$. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

Where

$$P_k = \sum_{i=1}^{k} p_i,\ k = 1, \ldots, n. \tag{2}$$

If f is strictly convex, then inequality (1) is strict unless $x_1 = ... = x_n$.

The condition "$\mathbf{p}$ is a positive n-tuple" can be replaced by "$\mathbf{p}$ is a non-negative n-tuple and $P_n > 0$". Note that the Jensen inequality (1) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "$\mathbf{p}$ is a non-negative n-tuple" can be relaxed at the expense of restricting $\mathbf{x}$ more severely. An answer to this question was given by Steffensen [2] (see also [[1], p.57]).

Theorem 1.2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, ..., x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{p} = (p_1, ..., p_n)$ a real n-tuple such that*

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$$0 \leq P_k \leq P_n,\ k = 1, \ldots, n-1,\ P_n > 0, \tag{3}$$

is satisfied, where P_k *are as in* (2), *then* (1) *holds. If f is strictly convex, then inequality* (1) *is strict unless* $x_1 = ... = x_n$.

Inequality (1) under conditions from Theorem 1.2 is called the Jensen-Steffensen inequality. A refinement of the Jensen-Steffensen inequality was given in [3] (see also [[1], p.89]).

Theorem 1.3. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two real n-tuples such that* $a \leq x_1 \leq ... \leq x_n \leq b$ *and* (3) *hold. Then for every convex function* $f : [a, b] \to \mathbb{R}$

$$F_n(x_1, \ldots, x_n) \geq F_{n-1}(x_1, \ldots, x_{n-1}) \geq \cdots \geq F_2(x_1, x_2) \geq F_1(x_1) = 0 \tag{4}$$

holds, where

$$F_k(x_1, \ldots, x_k) = G_k(x_1, \ldots, x_k, p_1, \ldots, p_{k-1}, \bar{P}_k), \tag{5}$$

$$G_k(x_1, \ldots, x_k, p_1, \ldots, p_k) = \frac{1}{P_k}\sum_{i=1}^{k} p_i f(x_i) - f\left(\frac{1}{P_k}\sum_{i=1}^{k} p_i x_i\right), \tag{6}$$

P_k *are as in* (2) *and*

$$\bar{P}_k = \sum_{i=k}^{n} p_i,\ k = 1, \ldots, n. \tag{7}$$

Note that the function G_n defined in (6) is in fact the difference of the right-hand and the left-hand side of the Jensen inequality (1).

In this paper, we present a new refinement of the Jensen-Steffensen inequality, related to Theorem 1.3. Further, we investigate the log-convexity and the exponential convexity of functionals defined as differences of the left-hand and the right-hand sides of these inequalities. We also prove monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, we give several examples of the families of functions for which the obtained results can be applied.

In what follows, I is an interval in $\mathbb{R}$, P_k are as in (2) and $\bar{P}_k$ are as in (7). Note that if (3) is valid, since $\bar{P}_k = P_n - P_{k-1}$, it follows that $\bar{P}_k$ satisfy (3) as well.

2. New refinement of the Jensen-Steffensen inequality

The aim of this section is to give a new refinement of the Jensen-Steffensen inequality. In the proof of this refinement, the following result is needed (see [[1], p.2]).

Proposition 2.1. *If f is a convex function on an interval I and if* $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, *then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{8}$$

If the function f is concave, the inequality reverses.

The main result states.

Theorem 2.2. *Let* $\mathbf{x} = (x_1, ..., x_n) \in I^n$ *be a monotonic n-tuple and* $\mathbf{p} = (p_1, ..., p_n)$ *a real n-tuple such that* (3) *holds. Then for a convex function* $f : I \to \mathbb{R}$ *we have*

$$\bar{F}_n(x_1, \ldots, x_n) \geq \bar{F}_{n-1}(x_2, \cdots, x_n) \geq \cdots \geq \bar{F}_2(x_{n-1}, x_n) \geq \bar{F}_1(x_n) = 0, \tag{9}$$

where

$$\begin{aligned}\bar{F}_k(x_{n-k+1}, x_{n-k+2}, \ldots, x_n) \\ = \bar{G}_k(x_{n-k+1}, x_{n-k+2}, \ldots, x_n, P_{n-k+1}, p_{n-k+2}, \ldots, p_n),\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{G}_k(x_{n-k+1}, \ldots, x_n, p_{n-k+1}, \ldots, p_n) \\ = \frac{1}{\bar{P}_{n-k+1}} \sum_{i=n-k+1}^{n} p_i f(x_i) - f\left(\frac{1}{\bar{P}_{n-k+1}} \sum_{i=n-k+1}^{n} p_i x_i\right).\end{aligned} \tag{11}$$

For a concave function f, the inequality signs in (9) *reverse.*

Proof. The claim is that for a convex function *f*,

$$\bar{F}_k(x_{n-k+1}, \ldots, x_n) \geq \bar{F}_{k-1}(x_{n-k+2}, \ldots, x_n)$$

holds for every $k = 2, ..., n$. This inequality is equivalent to

$$\frac{P_{n-k+1}}{P_n}(f(x_{n-k+2}) - f(x_{n-k+1})) \leq f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1}), \tag{12}$$

where

$$\bar{x}_{n-k+1} = \frac{1}{P_n}\left(P_{n-k+1}x_{n-k+1} + \sum_{i=n-k+2}^{n} p_i x_i\right).$$

If $\mathbf{x}$ is increasing then $x_{n-k+1} \leq \bar{x}_{n-k+1}$, while if $\mathbf{x}$ is decreasing then $x_{n-k+1} \geq \bar{x}_{n-k+1}$ for every k. Furthermore, without loss of generality, we can assume that $\mathbf{x}$ is strictly monotonic and that $0 < P_k < P_n$ for $k = 1, ..., n - 1$. Now, applying (8) for a convex function f when $\mathbf{x}$ is strictly increasing yields inequality

$$\frac{f(x_{n-k+2}) - f(x_{n-k+1})}{x_{n-k+2} - x_{n-k+1}} \leq \frac{f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1})}{\frac{P_{n-k+1}}{P_n}(x_{n-k+2} - x_{n-k+1})},$$

while if $\mathbf{x}$ is strictly decreasing we get inequality

$$\frac{f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1})}{\frac{P_{n-k+1}}{P_n}(x_{n-k+2} - x_{n-k+1})} \leq \frac{f(x_{n-k+2}) - f(x_{n-k+1})}{x_{n-k+2} - x_{n-k+1}},$$

both of which are equivalent to (12). If f is concave, the inequalities reverse. Thus, the proof is complete. □

Remark 2.3. *A slight extension of the proof of Theorem 1.3 in* [3]*shows that Theorem 1.3 remains valid if the n-tuple* $\mathbf{x}$ *is assumed to be monotonic instead of increasing. The proof is in fact analogous to the proof of Theorem 2.2.*

Let us observe inequalities (4) and (9). Motivated by them, we define two functionals

$$\Phi_1(\mathbf{x}, \mathbf{p}, f) = F_k(x_1, \ldots, x_k) - F_j(x_1, \ldots, x_j), \quad 1 \leq j < k \leq n, \tag{13}$$

$$\Phi_2(\mathbf{x}, \mathbf{p}, f) = \bar{F}_k(x_{n-k+1}, \ldots, x_n) - \bar{F}_j(x_{n-j+1}, \ldots, x_n), \quad 1 \leq j < k \leq n. \tag{14}$$

where functions F_k and $\bar{F}_k$ are as in (5) and (10), respectively, $\mathbf{x} = (x_1, ..., x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{p} = (p_1, ..., p_n)$ is a real n-tuple such that (3) holds. If function

f is convex on I, then Theorems 1.3 and 2.2, joint with Remark 2.3, imply that $\Phi_i(\mathbf{x}, \mathbf{p}, f) \geq 0$, $i = 1, 2$.

Now, we give mean value theorems for the functionals Φ_i, $i = 1, 2$.

Theorem 2.4. *Let* $\mathbf{x} = (x_1, ..., x_n) \in [a, b]^n$ *be a monotonic n-tuple and* $\mathbf{p} = (p_1, ..., p_n)$ *a real n-tuple such that* (3) *holds. Let* $f \in C^2[a, b]$ *and* Φ_1 *and* Φ_2 *be linear functionals defined as in* (13) *and* (14). *Then there exists* $\xi \in [a, b]$ *such that*

$$\Phi_i(\mathbf{x}, \mathbf{p}, f) = \frac{f''(\xi)}{2}\Phi_i(\mathbf{x}, \mathbf{p}, f_0),\ i = 1, 2, \tag{15}$$

where $f_0(x) = x^2$.

Proof. Analogous to the proof of Theorem 2.3 in [4]. □

Theorem 2.5. *Let* $\mathbf{x} = (x_1, ..., x_n) \in [a, b]^n$ *be a monotonic n-tuple and* $\mathbf{p} = (p_1, ..., p_n)$ *a real n-tuple such that* (3) *holds. Let* $f, g \in C^2[a, b]$ *be such that* $g''(x) \neq 0$ *for every* $x \in [a, b]$ *and let* Φ_1 *and* Φ_2 *be linear functionals defined as in* (13) *and* (14). *If* Φ_1 *and* Φ_2 *are positive, then there exists* $\xi \in [a, b]$ *such that*

$$\frac{\Phi_i(\mathbf{x}, \mathbf{p}, f)}{\Phi_i(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi)}{g''(\xi)},\ i = 1, 2. \tag{16}$$

Proof. Analogous to the proof of Theorem 2.4 in [4]. □

Remark 2.6. *If the inverse of the function* f''/g'' *exists, then (16) gives*

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, f)}{\Phi_i(\mathbf{x}, \mathbf{p}, g)}\right),\ i = 1, 2. \tag{17}$$

3. Log-convexity and exponential convexity of the Jensen-Steffensen differences

We begin this section by recollecting definitions of properties which are going to be explored here and also some useful characterizations of these properties (see [[5], p.373]). Again, I is an open interval in $\mathbb{R}$.

Definition 1. *A function* $h : I \to \mathbb{R}$ *is exponentially convex on I if it is continuous and*

$$\sum_{i,j=1}^{n} \alpha_i\alpha_j h(x_i + x_j) \geq 0$$

holds for every $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ *and* x_i *such that* $x_i + x_j \in I$, $i, j = 1, ..., n$.

Proposition 3.1. *Function* $h : I \to \mathbb{R}$ *is exponentially convex if and only if h is continuous and*

$$\sum_{i,j=1}^{n} \alpha_i\alpha_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ *and* $x_i \in I$, $i = 1, ..., n$.

Corollary 3.2. *If h is exponentially convex, then the matrix* $\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly,*

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0 \quad \text{for every } n \in \mathbb{N}, \quad x_i \in I, \quad i = 1, \ldots, n.$$

Corollary 3.3. *If $h : I \to (0, \infty)$ is an exponentially convex function, then h is a log-convex function, that is, for every $x, y \in I$ and every $\lambda \in [0, 1]$ we have*

$$h(\lambda x + (1 - \lambda)y) \leq h^{\lambda}(x)h^{1-\lambda}(y).$$

Lemma 3.4. *A function $h : I \to (0, \infty)$ is* log-*convex in the J-sense on I, that is, for every $x, y \in I$,*

$$h^2\left(\frac{x+y}{2}\right) \leq h(x)\,h(y)$$

holds if and only if the relation

$$\alpha^2 h(x) + 2\alpha\beta h\left(\frac{x+y}{2}\right) + \beta^2 h(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$.

Definition 2. *The second order divided difference of a function $f : [a, b] \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in [a, b]$ is defined recursively by*

$$[y_i; f] = f(y_i), \ i = 0, 1, 2,$$
$$[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1,$$

$$[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \tag{18}$$

Remark 3.5. *The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide (see* [[1], p.16]). *Namely, taking the limit $y_1 \to y_0$ in* (18), *we get*

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \ y_2 \neq y_0,$$

provided that f' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$, in (18), *we get*

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' exists.

Next, we study the log-convexity and the exponential convexity of functionals Φ_i (i = 1, 2) defined in (13) and (14).

Theorem 3.6. *Let $\Upsilon = \{f_s : s \in I\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is log-convex in J-sense on I for every three mutually different points $y_0, y_1, y_2 \in [a, b]$. Let Φ_i (i = 1, 2) be linear functionals defined as in* (13) *and* (14). *Further, assume $\Phi_i(\mathbf{x}, \mathbf{p}, f_s) > 0$ (i = 1, 2) for $f_s \in \Upsilon$. Then the following statements hold.*

(i) *The function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is log-convex in J-sense on I.*
(ii) *If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on I, then it is log-convex on I.*

(iii) *If the function* $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ *is differentiable on I, then for every* $s, q, u, v \in I$ *such that* $s \le u$ *and* $q \le v$, *we have*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Upsilon) \le \mu_{u,v}(\mathbf{x}, \Phi_i, \Upsilon) \quad (i = 1, 2) \tag{19}$$

where

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Xi) = \begin{cases} \left(\dfrac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)} \right), & s = q \end{cases} \tag{20}$$

and Ξ *is the family functions* f_s *belong to.*

Proof. (i) For $\alpha, \beta \in \mathbb{R}$ and $s, q \in I$, we define a function

$$g(y) = \alpha^2 f_s(y) + 2\alpha\beta f_{\frac{s+q}{2}}(y) + \beta^2 f_q(y).$$

Applying Lemma 3.4 for the function $s \mapsto [y_0, y_1, y_2; f_s]$ which is log-convex in J-sense on I by assumption, yields that

$$[y_0, y_1, y_2; g] = \alpha^2 [y_0, y_1, y_2; f_s] + 2\alpha\beta [y_0, y_1, y_2; f_{\frac{s+q}{2}}] + \beta^2 [y_0, y_1, y_2; f_q] \ge 0$$

which in turn implies that g is a convex function on I and therefore we have $\Phi_i(\mathbf{x}, \mathbf{p}, g) \ge 0$ $(i = 1, 2)$. Hence,

$$\alpha^2 \Phi_i(\mathbf{x}, \mathbf{p}, f_s) + 2\alpha\beta \Phi_i(\mathbf{x}, \mathbf{p}, f_{\frac{s+q}{2}}) + \beta^2 \Phi_i(\mathbf{x}, \mathbf{p}, f_q) \ge 0.$$

Now using Lemma 3.4 again, we conclude that the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is log-convex in J-sense on I.

(ii) If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is in addition continuous, from (i) it follows that it is then log-convex on I.

(iii) Since by (ii) the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is log-convex on I, that is, the function $s \mapsto \log \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is convex on I, applying (8) we get

$$\frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_s) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_q)}{s - q} \le \frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_u) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_v)}{u - v} \tag{21}$$

for $s \le u$, $q \le v$, $s \ne q$, $u \ne v$, and therefore conclude that

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Upsilon) \le \mu_{u,v}(\mathbf{x}, \Phi_i, \Upsilon), \; i = 1, 2.$$

If $s = q$, we consider the limit when $q \to s$ in (21) and conclude that

$$\mu_{s,s}(\mathbf{x}, \Phi_i, \Upsilon) \le \mu_{u,v}(\mathbf{x}, \Phi_i, \Upsilon), \; i = 1, 2.$$

The case $u = v$ can be treated similarly. □

Theorem 3.7. *Let* $\Omega = \{f_s : s \in I\}$ *be a family of functions defined on* $[a, b]$ *such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is exponentially convex on I for every three mutually different points* $y_0, y_1, y_2 \in [a, b]$. *Let* $\Phi_i (i = 1, 2)$ *be linear functionals defined as in* (13) *and* (14). *Then the following statements hold.*

(i) *If $n \in \mathbb{N}$ and $s_1, ..., s_n \in I$ are arbitrary, then the matrix*

$$\left[\Phi_i\left(\mathbf{x}, \mathbf{p}, f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{n}$$

is a positive semi-definite matrix for $i = 1, 2$. Particularly,

$$\det\left[\Phi_i\left(\mathbf{x}, \mathbf{p}, f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{n} \geq 0. \tag{22}$$

(ii) *If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on I, then it is also exponentially convex function on I.*

(iii) *If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is positive and differentiable on I, then for every s, q, u, $v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) \leq \mu_{u,v}(\mathbf{x}, \Phi_i, \Omega)\ (i = 1, 2) \tag{23}$$

where $\mu_{s,\,q}(\mathbf{x}, \Phi_i, \Omega)$ is defined in (20).

Proof. (i) Let $\alpha_j \in \mathbb{R}$ ($j = 1, ..., n$) and consider the function

$$g(y) = \sum_{j,k=1}^{n} \alpha_j\alpha_k f_{s_{jk}}(y)$$

for $n \in \mathbb{N}$, where $s_{jk} = \frac{s_j + s_k}{2}$, $s_j \in I$, $1 \leq j, k \leq n$ and $f_{s_{jk}} \in \Omega$. Then

$$[y_0, y_1, y_2; g] = \sum_{j,k=1}^{n} \alpha_j\alpha_k [y_0, y_1, y_2; f_{s_{jk}}]$$

and since $[y_0, y_1, y_2; f_{s_{jk}}]$ is exponentially convex by assumption it follows that

$$[y_0, y_1, y_2; g] = \sum_{j,k=1}^{n} \alpha_j\alpha_k [y_0, y_1, y_2; f_{s_{jk}}] \geq 0$$

and so we conclude that g is a convex function. Now we have

$$\Phi_i(\mathbf{x}, \mathbf{p}, g) \geq 0,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \alpha_j\alpha_k \Phi_i\left(\mathbf{x}, \mathbf{p}, f_{s_{jk}}\right) \geq 0,\ i = 1, 2,$$

which in turn shows that the matrix $\left[\Phi_i\left(\mathbf{x}, \mathbf{p}, f_{s_{jk}}\right)\right]_{j,k=1}^{n}$ is positive semi-definite, so (22) is immediate.

(ii) If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on I, then from (i) and Proposition 3.1 it follows that it is exponentially convex on I.

(iii) If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is differentiable on I, then from (ii) it follows that it is exponentially convex on I. If this function is in addition positive, then Corollary 3.3 implies that it is log-convex, so the statement follows from Theorem 3.6 (iii). □

Remark 3.8. *Note that the results from Theorem 3.6 still hold when two of the points* $y_0, y_1, y_2 \in [a, b]$ *coincide, say* $y_1 = y_0$, *for a family of differentiable functions* f_s *such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is log-convex in J-sense on I, and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 3.5 and taking the appropriate limits. The same is valid for the results from Theorem 3.7.*

Remark 3.9. *Related results for the original Jensen-Steffensen inequality regarding exponential convexity, which are a special case of Theorem 3.7, were given in* [6].

4. Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 3.7 (and Remark 3.8) and so the results of this theorem can be applied for them.

Example 4.1. *Consider a family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to [0, \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2}{dx^2} g_s(x) = e^{sx} > 0$*which shows that* g_s *is convex on* $\mathbb{R}$ *for every* $s \in \mathbb{R}$ *and* $s \mapsto \frac{d^2}{dx^2} g_s(x)$*is exponentially convex by Example 1 given in Jakšetić and Pečarić (submitted). From Jakšetić and Pečarić (submitted), we then also have that* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex.*

For this family of functions, $\mu_{s,q}(\mathbf{x}, \Phi_i, \Xi)$ $(i = 1, 2)$ *from* (20) *become*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, g_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, g_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, id \cdot g_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, g_s)} - \frac{2}{s} \right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, id \cdot g_0)}{3\Phi_i(\mathbf{x}, \mathbf{p}, g_0)} \right), & s = q = 0. \end{cases}$$

Example 4.2. *Consider a family of functions*

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2}{dx^2} f_s(x) = x^{s-2} = e^{(s-2)\ln x} > 0$ *which shows that* f_s *is convex for* $x > 0$ *and* $s \mapsto \frac{d^2}{dx^2} f_s(x)$ *is exponentially convex by Example 1 given in Jakšetić and Pečarić (submitted). From Jakšetić and Pečarić (submitted), we have that* $s \mapsto [y_0, y_1, y_2; f_s]$ *is exponentially convex.*

In this case, $\mu_{s,q}(\mathbf{x}, \Phi_i, \Xi)$ $(i = 1, 2)$ *defined in* (20) *for* $x_j > 0$ $(j = 1, ..., n)$ *are*

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s f_0)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}\right), & s = q \neq 0, 1, \\ \exp\left(1 - \frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_0^2)}{2\Phi_i(\mathbf{x}, \mathbf{p}, f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_0 f_1)}{2\Phi_i(\mathbf{x}, \mathbf{p}, f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i *is positive, then Theorem 2.5 applied for* $f = f_s \in \Omega_2$ *and* $g = f_q \in \Omega_2$ *yields that there exists* $\xi \in [\min_{1 \le i \le n} x_i, \max_{1 \le i \le n} x_i]$ *such that*

$$\xi^{s-q} = \frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ *is invertible for* $s \neq q$*, we then have*

$$\min\{x_1, x_n\} = \min_{1 \le i \le n} x_i \le \left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_q)}\right)^{\frac{1}{s-q}} \le \max_{1 \le i \le n} x_i = \max\{x_1, x_n\}, \tag{24}$$

which together with the fact that $\mu_{s,q}(\boldsymbol{x}, \Phi_i, \Omega_2)$ *is continuous, symmetric and monotonous (by* (23)*), shows that* $\mu_{s,q}(\boldsymbol{x}, \Phi_i, \Omega_2)$ *is a mean.*

Now, by substitutions $x_i \to x_i^t$, $s \to \frac{s}{t}, q \to \frac{q}{t}\,(t \neq 0, s \neq q)$ *from (24) we get*

$$\min\{x_1^t, x_n^t\} = \min_{1 \le i \le n} x_i^t \le \left(\frac{\Phi_i(\mathbf{x}^t, \mathbf{p}, f_{s/t})}{\Phi_i(\mathbf{x}^t, \mathbf{p}, f_{q/t})}\right)^{\frac{t}{s-q}} \le \max_{1 \le i \le n} x_i^t = \max\{x_1^t, x_n^t\},$$

where $\mathbf{x}^t = (x_1^t, \ldots, x_n^t)$*.*

We define a new mean (for $i = 1, 2$*) as follows:*

$$\mu_{s,q;t}(\mathbf{x}, \Phi_i, \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{t}, \frac{q}{t}}(\mathbf{x}^t, \Phi_i, \Omega_2)\right)^{1/t}, & t \neq 0, \\ \mu_{s,q}(\log \mathbf{x}, \Phi_i, \Omega_1), & t = 0. \end{cases} \tag{25}$$

These new means are also monotonous. More precisely, for $s, q, u, v \in \mathbb{R}$ *such that* $s \le u$, $q \le v$, $s \neq u$, $q \neq v$*, we have*

$$\mu_{s,q;t}(\mathbf{x}, \Phi_i, \Omega_2) \le \mu_{u,v;t}(\mathbf{x}, \Phi_i, \Omega_2) \ (i = 1, 2) \tag{26}$$

We know that

$$\mu_{\frac{s}{t},\frac{q}{t}}(\mathbf{x}^t,\Phi_i,\Omega_2)=\left(\frac{\Phi_i(\mathbf{x}^t,\mathbf{p},f_{s/t})}{\Phi_i(\mathbf{x}^t,\mathbf{p},f_{q/t})}\right)^{\frac{t}{s-q}}\le\mu_{\frac{u}{t},\frac{v}{t}}(\mathbf{x}^t,\Phi_i,\Omega_2)=\left(\frac{\Phi_i(\mathbf{x}^t,\mathbf{p},f_{u/t})}{\Phi_i(\mathbf{x}^t,\mathbf{p},f_{v/t})}\right)^{\frac{t}{u-v}},$$

for $s, q, u, v \in I$ *such that* $s/t \le u/t$, $q/t \le v/t$ *and* $t \ne 0$, *since* $\mu_{s,q}(\boldsymbol{x}, \Phi_i, \Omega_2)$ *are monotonous in both parameters, so the claim follows. For* $t = 0$, *we obtain the required result by taking the limit* $t \to 0$.

Example 4.3. *Consider a family of functions*

$$\Omega_3=\{h_s:(0,\infty)\to(0,\infty):s\in(0,\infty)\}$$

defined by

$$h_s(x)=\begin{cases}\frac{s^{-x}}{\ln^2 s}, & s\ne 1,\\ \frac{x^2}{2}, & s=1.\end{cases}$$

Exponential convexity of $s \mapsto \frac{d^2}{dx^2}h_s(x) = s^{-x}$ *on* $(0,\infty)$ *is given by Example 2 in Jakšetić and Pečarić (submitted).*

$\mu_{s,q}(\boldsymbol{x}, \Phi_i, \Xi)$ $(i = 1, 2)$ *defined in* (20) *in this case for* $x_j > 0$ $(j = 1, \ldots, n)$ *are*

$$\mu_{s,q}(\mathbf{x},\Phi_i,\Omega_3)=\begin{cases}\left(\frac{\Phi_i(\mathbf{x},\mathbf{p},h_s)}{\Phi_i(\mathbf{x},\mathbf{p},h_q)}\right)^{\frac{1}{s-q}}, & s\ne q,\\ \exp\left(-\frac{\Phi_i(\mathbf{x},\mathbf{p},id\cdot h_s)}{s\Phi_i(\mathbf{x},\mathbf{p},h_s)}-\frac{2}{s\ln s}\right), & s=q\ne 1,\\ \exp\left(-\frac{2\Phi_i(\mathbf{x},\mathbf{p},id\cdot h_1)}{3\Phi_i(\mathbf{x},\mathbf{p},h_1)}\right), & s=q=1.\end{cases}$$

Example 4.4. *Consider a family of functions*

$$\Omega_4=\{k_s:(0,\infty)\to(0,\infty):s\in(0,\infty)\}$$

defined by

$$k_s(x)=\frac{e^{-x\sqrt{s}}}{s}$$

Exponential convexity of $s \mapsto \frac{d^2}{dx^2}k_s(x) = e^{-x\sqrt{s}}$ *on* $(0, \infty)$ *is given by Example 3 in Jakšetić and Pečarić (submitted).*

In this case, $\mu_{s,q}(\boldsymbol{x}, \Phi_i, \Xi)$ $(i = 1, 2)$ *defined in* (20) *for* $x_j > 0$ $(j = 1, \ldots, n)$ *are*

$$\mu_{s,q}(\mathbf{x},\Phi_i,\Omega_4)=\begin{cases}\left(\frac{\Phi_i(\mathbf{x},\mathbf{p},k_s)}{\Phi_i(\mathbf{x},\mathbf{p},k_q)}\right)^{\frac{1}{s-q}}, & s\ne q,\\ \exp\left(-\frac{\Phi_i(\mathbf{x},\mathbf{p},id\cdot k_s)}{2\sqrt{s}\Phi_i(\mathbf{x},\mathbf{p},k_s)}-\frac{1}{s}\right), & s=q.\end{cases}$$

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Authors' contributions

JP made the main contribution in conceiving the presented research. IF and JP worked on the results from Section 2, while SK and JP worked jointly on the results of Sections 3 and 4. IF and SK drafted the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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