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Integral operators on new families of meromorphic functions of complex order

Aabed Mohammed and Maslina Darus*

* Correspondence: maslina@ukm.my
School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor Darul Ehsan, Malaysia

Abstract

In this article, we define and investigate new families of certain subclasses of meromorphic functions of complex order. Considering the new subclasses, several properties for certain integral operators are derived.

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1 Introduction

Let Σ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}, \tag{1.2}$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$..

We say that a function $f \in \Sigma$ is meromorphic starlike of order δ $(0 \le \delta < 1)$, and belongs to the class $\Sigma^*(\delta)$, if it satisfies the inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta. \tag{1.3}$$

A function $f \in \Sigma$ is a meromorphic convex function of order δ $(0 \le \delta < 1)$, if f satisfies the following inequality:

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, \tag{1.4}$$

and we denote this class by $\Sigma_k(\delta)$.

For $f \in \Sigma$, Wang et al. [1] and Nehari and Netanyahu [2] introduced and studied the subclass $\Sigma_N(\beta)$ of Σ consisting of functions $f(z)$ satisfying

$$-\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) < \beta \quad (\beta > 1, \ z \in \mathbb{U}). \tag{1.5}$$

Let $\mathcal{A}$ denote the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disk $\mathbb{U}$.

Analogous to several subclasses [3-10] of analytic functions of $\mathcal{A}$, we define the following subclasses of Σ.

Definition 1.1 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_b^\star(\delta)$ if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right)\right\} > \delta, \tag{1.6}$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 \le \delta < 1$.

Definition 1.2 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma K_b(\delta)$ if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 2\right)\right\} > \delta, \tag{1.7}$$

where $b \in C\backslash\{0\}$, $0 \le \delta < 1$. We note that $f \in \Sigma K_b(\delta)$ if, and only if, $-zf' \in \Sigma_b^\star(\delta)$.

Furthermore, the classes

$$\Sigma_1^\star(\delta) \equiv \Sigma^\star(\delta), \quad \Sigma K_1(\delta) \equiv \Sigma_k(\delta)$$

are the classes of meromorphic starlike functions of order δ and meromorphic convex functions of order δ in $\mathbb{U}^*$, respectively. Moreover, the classes

$$\Sigma_1^\star(0) \equiv \Sigma^\star(0), \quad \Sigma K_1(0) \equiv \Sigma_k(0)$$

are familiar classes of starlike and convex functions in $\mathbb{U}^*$, respectively.

Definition 1.3 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma^\star\mathcal{U}(\alpha, \delta, b)$ if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right)\right| + \delta, \tag{1.8}$$

where $\alpha \ge 0$, $\delta \in [-1,1)$, $\alpha + \delta \ge 0$, $b \in \mathbb{C}\backslash\{0\}$.

Definition 1.4 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma K\mathcal{U}(\alpha, \delta, b)$ if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 2\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 2\right)\right| + \delta, \tag{1.9}$$

where $\alpha \ge 0$, $\delta \in [-1,1)$, $\alpha + \delta \ge 0$, $b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma K\mathcal{U}(\alpha, \delta, b)$ if, and only if, $-zf' \in \Sigma^\star\mathcal{U}(\alpha, \delta, b)$.

For $\alpha = 0$, we have

$$\Sigma^\star\mathcal{U}(0, \delta, b) \equiv \Sigma_b^\star(\delta), \quad \Sigma K\mathcal{U}(0, \delta, b) \equiv \Sigma K_b(\delta).$$

Definition 1.5 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma^\star\mathcal{UH}(\alpha, b)$ if, and only if, f satisfies

$$\left|1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1), \tag{1.10}$$

where $\alpha > 0$, $b \in \mathbb{C}\backslash\{0\}$.

Definition 1.6 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{KUH}(\alpha, b)$ if, and only if, f satisfies

$$\left|1 - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 2\right) - 2\alpha(\sqrt{2}-1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} + 2\right)\right)\right\} + 2\alpha(\sqrt{2}-1). \quad (1.11)$$

where $\alpha > 0$, $b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma\mathcal{KUH}(\alpha, b)$ if, and only if, $-zf' \in \Sigma^\star\mathcal{UH}(\alpha, b)$.

Let us also introduce the following families of new subclasses $\Sigma\mathcal{F}_1(\delta, b)$, $\Sigma\mathcal{F}_2(\alpha, \delta, b)$, and $\Sigma\mathcal{F}_3(\alpha, b)$ as follows.

Definition 1.7 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_1(\delta, b)$, if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1\right)\right\} > \delta. \quad (1.12)$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 \le \delta < 1$.

We note that $f \in \Sigma\mathcal{F}_1(\delta, b)$ if, and only if, $zf'(z) + 2f(z) \in \Sigma_b^\star(\delta)$.

Definition 1.8 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_2(\alpha, \delta, b)$ if, and only if, f satisfies

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1\right)\right| + \delta, \quad (1.13)$$

where $\alpha \ge 0$, $\delta \in [-1,1)$, $\alpha + \delta \ge 0$, $b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma\mathcal{F}_2(\alpha, \delta, b)$ if, and only if, $zf'(z) + 2f(z) \in \Sigma^\star\mathcal{U}(\alpha, \delta, b)$.

Definition 1.9 Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_3(\alpha, b)$ if, and only if, f satisfies

$$\left|1 - \frac{1}{b}\left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1\right) - 2\alpha(\sqrt{2}-1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2}-1), \quad (1.14)$$

where $\alpha > 0$, $b \in \mathbb{C}\backslash\{0\}$.

We note that $f \in \Sigma\mathcal{F}_3(\alpha, b)$ if, and only if, $zf'(z) + 2f(z) \in \Sigma^\star\mathcal{UH}(\alpha, b)$.

Recently, many authors introduced and studied various integral operators of analytic and univalent functions in the open unit disk $\mathbb{U}$[11-21].

Most recently, Mohammed and Darus [22,23] introduced the following two general integral operators of meromorphic functions Σ:

$$\mathcal{H}_n(z) = \frac{1}{z^2}\int_0^z \left(uf_1(u)\right)^{\gamma_1} \ldots \left(uf_n(u)\right)^{\gamma_n} du, \quad (1.15)$$

and

$$\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) = \frac{1}{z^2}\int_0^z \left(-u^2 f_1'(u)\right)^{\gamma_1} \ldots \left(-u^2 f_n'(u)\right)^{\gamma_n} du. \quad (1.16)$$

Goyal and Prajapat [24] obtained the following results for $f \in \Sigma$ to be in the class $\Sigma^*(\delta)$, $0 \le \delta < 1$.

Corollary 1.1 *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{(1-\delta)(3-\delta)}{2-\delta}, \quad 0 \le \delta < 1, \tag{1.17}$$

then $f \in \Sigma^*(\delta)$.

Corollary 1.2 *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right| < \frac{1}{2}, \tag{1.18}$$

then $f \in \Sigma^*$.

Corollary 1.3 *If $f \in \Sigma$ satisfies the following inequality*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{2zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1\right)\right\} > -\frac{1}{2}, \tag{1.19}$$

then $f \in \Sigma^*$.

In this article, we derive several properties for the integral operators $\mathcal{H}_n(z)$ and $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ of the subclasses given by (1.5) and Definitions 1.1 to 1.6.

2 Some properties for $\mathcal{H}_n(z)$

In this section, we investigate some properties for the integral operator $\mathcal{H}_n(z)$ defined by (1.15) of the subclasses given by (1.5), Definitions 1.1, 1.3, and 1.5.

Theorem 2.1 *For $i \in \{1,\ldots, n\}$, let $\gamma_i > 0$, $f_i \in \Sigma$ and*

$$\left|\frac{zf_i''(z)}{f_i'(z)} - 2\frac{zf_i'(z)}{f_i(z)}\right| < \frac{(1-\delta)(3-\delta)}{2-\delta}, \quad (0 \le \delta < 1)\,. \tag{2.1}$$

If

$$\sum_{i=1}^{n} \gamma_i > \frac{2}{1-\delta}, \tag{2.2}$$

then $\mathcal{H}_n(z) \in \Sigma_N(\beta)$, where $\beta > 1$.

Proof On successive differentiation of $\mathcal{H}_n(z)$, which is defined in (1.5), we get

$$z^2\mathcal{H}_n'(z) + 2z\mathcal{H}_n(z) = \left(zf_1(z)\right)^{\gamma_1}\ldots\left(zf_n(z)\right)^{\gamma_n}, \tag{2.3}$$

and

$$z^2\mathcal{H}_n''(z) + 4z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z) = \sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z) + f_i(z)}{zf_i(z)}\right)\left[\left(zf_1(z)\right)^{\gamma_1}\ldots\left(zf_n(z)\right)^{\gamma_n}\right] \tag{2.4}$$

Then from (2.3) and (2.4), we obtain

$$\frac{z^2\mathcal{H}_n''(z) + 4z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z)}{z^2\mathcal{H}_n'(z) + 2z\mathcal{H}_n(z)} = \sum_{i=1}^{n}\gamma_i\left(\frac{f_i'(z)}{f_i(z)} + \frac{1}{z}\right). \tag{2.5}$$

By multiplying (2.5) with z yield

$$\frac{z^2\mathcal{H}_n''(z) + 4z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z)} = \sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)} + 1\right). \tag{2.6}$$

This is equivalent to

$$\frac{z^2\mathcal{H}_n''(z)+3z\mathcal{H}_n'(z)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1=\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right). \tag{2.7}$$

Therefore, we have

$$\frac{-\left(\dfrac{z\mathcal{H}_n''(z)}{\mathcal{H}_n'(z)}+1\right)-2}{1+\dfrac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}}=-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1. \tag{2.8}$$

Then, we easily get

$$\begin{aligned}-\left(\frac{z\mathcal{H}_n''(z)}{\mathcal{H}_n'(z)}+1\right)&=\left(\frac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1\right]\\&\quad+\sum_{i=1}^{n}\gamma_i\left(-\frac{zf_i'(z)}{f_i(z)}\right)+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{2.9}$$

Taking real parts of both sides of (2.9), we obtain

$$\begin{aligned}-\Re\left(\frac{z\mathcal{H}_n''(z)}{\mathcal{H}_n'(z)}+1\right)&=\Re\left\{\left(\frac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1\right]\right\}\\&\quad+\sum_{i=1}^{n}\gamma_i\Re\left(-\frac{zf_i'(z)}{f_i(z)}\right)+3-\sum_{i=1}^{n}\gamma_i\\&\le\left|\left(\frac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1\right]\right|\\&\quad+\sum_{i=1}^{n}\gamma_i\Re\left(-\frac{zf_i'(z)}{f_i(z)}\right)+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{2.10}$$

Let

$$\begin{aligned}\beta&=\left|\left(\frac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1\right]\right|\\&\quad+\sum_{i=1}^{n}\gamma_i\Re\left(-\frac{zf_i'(z)}{f_i(z)}\right)+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{2.11}$$

Since $\left|\left(\frac{2\mathcal{H}_n(z)}{z\mathcal{H}_n'(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+1\right]\right|>0$, applying Corollary 1.1, we have

$$\beta>\delta\sum_{i=1}^{n}\gamma_i+3-\sum_{i=1}^{n}\gamma_i=3-(1-\delta)\sum_{i=1}^{n}\gamma_i. \tag{2.12}$$

Then, by the hypothesis (2.2), we have $\beta > 1$. Therefore, $\mathcal{H}_n(z)\in\Sigma_N(\beta)$, where $\beta > 1$. This completes the proof. □

Letting $\delta = 0$ in Theorem 2.1, we have

Corollary 2.2 *For $i \in \{1,..., n\}$, let $\gamma_i > 0, f_i \in \Sigma$ and*

$$\left| \frac{zf_i''(z)}{f_i'(z)} - 2\frac{zf_i'(z)}{f_i(z)} \right| < \frac{3}{2}. \tag{2.13}$$

If

$$\sum_{i=1}^{n} \gamma_i > 2, \tag{2.14}$$

then $\mathcal{H}_n(z) \in \Sigma_N(\beta)$, where $\beta > 1$.

Making use of (2.11), Corollary 1.2 and Corollary 1.3, one can prove the following results.

Theorem 2.3 *For $i \in \{1,..., n\}$, let $\gamma_i > 0, f_i \in \Sigma$ and*

$$\left| \frac{zf_i''(z)}{f_i'(z)} - \frac{zf_i'(z)}{f_i(z)} + 1 \right| < \frac{1}{2}. \tag{2.15}$$

If

$$\sum_{i=1}^{n} \gamma_i > 2, \tag{2.16}$$

then $\mathcal{H}_n(z) \in \Sigma_N(\beta)$, where $\beta > 1$.

Theorem 2.4 *For $i \in \{1,..., n\}$, let $\gamma_i > 0, f_i \in \Sigma$ and*

$$\Re\left\{ \frac{zf_i'(z)}{f_i(z)} \left(\frac{2zf_i'(z)}{f_i(z)} - \frac{zf_i''(z)}{f_i'(z)} - 1 \right) \right\} > -\frac{1}{2}. \tag{2.17}$$

If

$$\sum_{i=1}^{n} \gamma_i > 2, \tag{2.18}$$

then $\mathcal{H}_n(z) \in \Sigma_N(\beta)$, where $\beta > 1$.

Theorem 2.5 *For $i \in \{1,..., n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma_b^{\star}(\delta_i)$ $(0 \le \delta < 1$ and $b \in \mathbb{C}\backslash\{0\})$. If*

$$0 < \sum_{i=1}^{n} \gamma_i \,(1 - \delta_i) \le 1, \tag{2.19}$$

then $\mathcal{H}_n(z)$is in the class $\Sigma\mathcal{F}_1(\mu, b)$, $\mu = 1 - \sum_{i=1}^{n} \gamma_i \,(1 - \delta_i)$.

Proof From (2.7), we have

$$\frac{z\left(z\mathcal{H}_n''(z) + 3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z)} + 1 = \sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} + 1 \right). \tag{2.20}$$

Equivalently, (2.20) can be written as

$$1 - \frac{1}{b}\left\{ \frac{z\left(z\mathcal{H}_n''(z) + 3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z) + 2\mathcal{H}_n(z)} + 1 \right\} = \sum_{i=1}^{n} \gamma_i \left\{ 1 - \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \tag{2.21}$$

Taking the real part of both terms of the last expression, we have

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right\}=\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}+1-\sum_{i=1}^{n}\gamma_i. \quad (2.22)$$

Since $f_i \in \Sigma_b^\star(\delta_i)$, hence

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right\}>\sum_{i=1}^{n}\gamma_i\delta_i+1-\sum_{i=1}^{n}\gamma_i. \quad (2.23)$$

so that

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right\}>1-\sum_{i=1}^{n}\gamma_i\left(1-\delta_i\right). \quad (2.24)$$

Then $\mathcal{H}_n(z) \in \Sigma\mathcal{F}_1(\mu, b), \quad \mu = 1-\sum_{i=1}^{n}\gamma_i(1-\delta_i)$.

Now, adopting the techniques used by Breaz et al. [11] and Bulut [21], we prove the following two theorems.

Theorem 2.6 *For $i \in \{1,\ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^\star\mathcal{U}(\alpha,\delta,b)$ ($\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$ and $b \in \mathbb{C}\backslash\{0\}$). If*

$$\sum_{i=1}^{n}\gamma_i \leq 1, \quad (2.25)$$

then $\mathcal{H}_n(z)$is in the class $\Sigma\mathcal{F}_2(\alpha,\delta,b)$.

Proof Since $f_i \in \Sigma^\star\mathcal{U}(\alpha,\delta,b)$, it follows from Definition 1.3 that

$$\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}>\alpha\left|\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right|+\delta. \quad (2.26)$$

Considering Definition 1.8 and with the help of (2.21), we obtain

$$\begin{aligned}
&\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right\}-\alpha\left|\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right|-\delta\\
&=1-\sum_{i=1}^{n}\gamma_i+\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}-\alpha\left|\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right|-\delta\\
&\geq 1-\sum_{i=1}^{n}\gamma_i+\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}-\alpha\sum_{i=1}^{n}\gamma_i\left|\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right|-\delta \quad (2.27)\\
&>1-\sum_{i=1}^{n}\gamma_i+\sum_{i=1}^{n}\gamma_i\left\{\alpha\left|\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right|+\delta\right\}-\alpha\sum_{i=1}^{n}\gamma_i\left|\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right|-\delta\\
&=(1-\delta)\left(1-\sum_{i=1}^{n}\gamma_i\right)\geq 0.
\end{aligned}$$

This completes the proof. □

Theorem 2.7 *For $i \in \{1,\ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^\star\mathcal{UH}(\alpha,b)$ ($\alpha > 0$ and $b \in \mathbb{C}\backslash\{0\}$). If*

$$\sum_{i=1}^{n} \gamma_i \leq 1, \tag{2.28}$$

then $\mathcal{H}_n(z)$*is in the class* $\Sigma\mathcal{F}_3(\alpha, b)$.

Proof Since $f_i \in \Sigma^{\star}\mathcal{UH}(\alpha, b)$, it follows from Definition 1.5 that

$$\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right)\right\}+2\alpha(\sqrt{2}-1)-\left|1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|>0. \tag{2.29}$$

Considering this inequality and (2.21), we obtain

$$\begin{aligned}
&\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right)\right\}+2\alpha\left(\sqrt{2}-1\right)-\left|1-\frac{1}{b}\left(\frac{z\left(z\mathcal{H}_n''(z)+3\mathcal{H}_n'(z)\right)}{z\mathcal{H}_n'(z)+2\mathcal{H}_n(z)}+1\right)\right.\\
&\quad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\Re\left\{\sqrt{2}\left[1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right]\right\}+2\alpha\left(\sqrt{2}-1\right)-\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right.\\
&\quad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)+2\alpha\left(\sqrt{2}-1\right)-\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right.\\
&\quad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sqrt{2}+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}-\sqrt{2}\sum_{i=1}^{n}\gamma_i+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|1+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)-2\alpha\left(\sqrt{2}-1\right)\right]-\sum_{i=1}^{n}\gamma_i+2\alpha\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\gamma_i\right.\\
&\quad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha\left(\sqrt{2}-1\right)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}\\
&\quad-\left|\left[1-2\alpha\left(\sqrt{2}-1\right)\right]\left(1-\sum_{i=1}^{n}\gamma_i\right)+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)-2\alpha\left(\sqrt{2}-1\right)\right]\right|\\
&\geq\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha\left(\sqrt{2}-1\right)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right\}\\
&\quad-\sum_{i=1}^{n}\gamma_i\left|1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)-2\alpha\left(\sqrt{2}-1\right)\right|-\left|1-2\alpha\left(\sqrt{2}-1\right)\right|\left(1-\sum_{i=1}^{n}\gamma_i\right)\\
&=\sum_{i=1}^{n}\gamma_i\left\{\Re\sqrt{2}\left[1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)\right]+2\alpha\left(\sqrt{2}-1\right)-\left|1-\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)}+1\right)-2\alpha\left(\sqrt{2}-1\right)\right|\right\}\\
&\quad+\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha\left(\sqrt{2}-1\right)-2\alpha\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\gamma_i-\left|1-2\alpha\left(\sqrt{2}-1\right)\right|\left(1-\sum_{i=1}^{n}\gamma_i\right)\\
&>\left[\sqrt{2}+2\alpha\left(\sqrt{2}-1\right)-\left|1-2\alpha\left(\sqrt{2}-1\right)\right|\right]\left(1-\sum_{i=1}^{n}\gamma_i\right)\\
&>\left(1-\sum_{i=1}^{n}\gamma_i\right)\min\left\{\left(\sqrt{2}-1\right)(1+4\alpha),\sqrt{2}+1\right\}\geq 0.
\end{aligned}$$

This completes the proof. □

3 Some properties for $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$

In this section, we investigate some properties for the integral operator $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ defined by (1.16) of subclasses given by (1.5), Definitions 1.2, 1.4, and 1.6.

Theorem 3.1 *For* $i \in \{1,\ldots, n\}$, *let* $\gamma_i > 0, f_i \in \Sigma$ *and*

$$\sum_{i=1}^{n} \gamma_i > \frac{2}{1-\delta}, \quad (0 \leq \delta < 1). \tag{3.1}$$

If $f_i \in \Sigma_k(\delta)$, *then* $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma_N(\beta)$, $\beta > 1$.

Proof On successive differentiation of $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$, which is defined in (1.16), we have

$$2z\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) + z^2\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) = \left(-z^2 f_1'(z)\right)^{\gamma_1} \ldots \left(-z^2 f_n'(z)\right)^{\gamma_n}, \tag{3.2}$$

and

$$\begin{aligned} &z^2\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z) + 4z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \\ &= \sum_{i=1}^{n} \gamma_i \left(\frac{f_i''(z)}{f_i'(z)} + \frac{2}{z}\right)\left[\left(-z^2 f_1'(z)\right)^{\gamma_1} \ldots \left(-z^2 f_n'(z)\right)^{\gamma_n}\right]. \end{aligned} \tag{3.3}$$

Then from (3.2) and (3.3), we obtain

$$\frac{z^2\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z) + 4z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z^2\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2z\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{f_i''(z)}{f_i'(z)} + \frac{2}{z}\right). \tag{3.4}$$

By multiplying (3.4) with z yields,

$$\frac{z^2\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z) + 4z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right). \tag{3.5}$$

that is equivalent to

$$\frac{z^2\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z) + 3z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)} + 1 = \sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right). \tag{3.6}$$

Therefore, we have

$$\frac{-\left(\dfrac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)} + 1\right) - 2}{1 + \dfrac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}} = -\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right) + 1. \tag{3.7}$$

so that

$$\begin{aligned} -\left(\frac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)} + 1\right) = \left(\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)&\left[-\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right) + 1\right] \\ &+ \sum_{i=1}^{n} \gamma_i \left\{-\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right)\right\} + 3 - \sum_{i=1}^{n} \gamma_i. \end{aligned} \tag{3.8}$$

Taking the real parts of both terms of the last expression, we obtain

$$\begin{aligned} -\Re\left(\frac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)} + 1\right) &= \Re\left\{\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\left[-\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right) + 1\right]\right\} \\ &\quad + \sum_{i=1}^{n} \gamma_i \Re\left\{-\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right)\right\} + 3 - \sum_{i=1}^{n} \gamma_i \\ &\leq \left|\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\left[-\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i''(z)}{f_i'(z)} + 2\right) + 1\right]\right| \\ &\quad + \sum_{i=1}^{n} \gamma_i \Re\left\{-\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right)\right\} + 3 - \sum_{i=1}^{n} \gamma_i. \end{aligned} \tag{3.9}$$

Let

$$\beta = \left|\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+2\right)+1\right]\right|+\sum_{i=1}^{n}\gamma_i\Re\left\{-\left(\frac{zf''_i(z)}{f'_i(z)}+1\right)\right\}+3-\sum_{i=1}^{n}\gamma_i. \tag{3.10}$$

Since $\left|\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+2\right)+1\right]\right| > 0, f_i \in \Sigma_k(\delta)$, we get

$$\beta > 3-(1-\delta)\sum_{i=1}^{n}\gamma_i, \tag{3.11}$$

which, in light of the hypothesis (3.1), yields $\beta > 1$.

Therefore, $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma_N(\beta)$, $\beta > 1$. This completes the proof. □

Theorem 3.2 *For* $i \in \{1,\ldots, n\}$, *let* $\gamma_i > 0, f_i \in \Sigma$ *and*

$$1 < \sum_{i=1}^{n}\gamma_i < 2. \tag{3.12}$$

If $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma^\star(\delta)$, *then* $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma_N(\beta)$, $\beta > 1$.

Proof It follows from (3.7) that

$$\begin{aligned}-\left(\frac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}+1\right) &= \left(1+\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+1\right)\right]\\ &+2\left(\sum_{i=1}^{n}\gamma_i-1\right)\left(-\frac{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{3.13}$$

Taking the real parts of both terms of the last expression, we obtain

$$\begin{aligned}-\Re\left(\frac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}+1\right) &= \Re\left\{\left(1+\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+1\right)\right]\right\}\\ &+2\left(\sum_{i=1}^{n}\gamma_i-1\right)\Re\left(-\frac{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{3.14}$$

Thus, we have

$$\begin{aligned}-\Re\left(\frac{z\mathcal{H}''_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}+1\right) &= \Re\left\{\left(1+\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+1\right)\right]\right\}\\ &+2\left(\sum_{i=1}^{n}\gamma_i-1\right)\Re\left(-\frac{1}{\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}}\right)+3-\sum_{i=1}^{n}\gamma_i\\ &\le \left|\left(1+\frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf''_i(z)}{f'_i(z)}+1\right)\right]\right|\\ &+2\left(\sum_{i=1}^{n}\gamma_i-1\right)\frac{\Re\left(-\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}\right)}{\left|\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{H_{\gamma_1,\ldots,\gamma_n}(z)}\right|^2}+3-\sum_{i=1}^{n}\gamma_i.\end{aligned} \tag{3.15}$$

Let

$$\beta = \left|\left(1 + \frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right)\right]\right| + 2\left(\sum_{i=1}^{n}\gamma_i - 1\right)\frac{\Re\left(-\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}\right)}{\left|\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}\right|^2} + 3 - \sum_{i=1}^{n}\gamma_i. \tag{3.16}$$

Since $\left|\left(1 + \frac{2\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}\right)\left[-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right)\right]\right| > 0$ and $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma^\star(\delta)$, we have

$$\beta > 2\left(\sum_{i=1}^{n}\gamma_i - 1\right)\frac{\delta}{\left|\frac{z\mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)}\right|^2} + 3 - \sum_{i=1}^{n}\gamma_i. > 3 - \sum_{i=1}^{n}\gamma_i. \tag{3.17}$$

Then, by the hypothesis (3.12), we see that $\beta > 1$. Therefore, $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) \in \Sigma_N(\beta)$, $\beta > 1$. This completes the proof. □

Now, using the method given in the proofs of Theorems 2.5, 2.6, and 2.7, one can prove the following results:

Theorem 3.3 *For* $i \in \{1,\ldots, n\}$, *let* $\gamma_i > 0$ $f_i \in \Sigma K_b(\delta_i)$ $(0 \le \delta < 1$ *and* $b \in \mathbb{C}\backslash\{0\})$. *If*

$$0 < \sum_{i=1}^{n}\gamma_i(1 - \delta_i) \le 1, \tag{3.18}$$

then $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ *is in the class* $\Sigma\mathcal{F}_1(\mu, b)$, $\mu = 1 - \sum_{i=1}^{n}\gamma_i(1 - \delta_i)$.

Theorem 3.4 *For* $i \in \{1,\ldots, n\}$, *let* $\gamma_i > 0$ *and* $f_i \in \Sigma K\mathcal{U}(\alpha, \delta, b)$ $(\alpha \ge 0$, $\delta \in [-1,1)$, $\alpha + \delta \ge 0$ *and* $b \in \mathbb{C}\backslash\{0\})$. *If*

$$\sum_{i=1}^{n}\gamma_i \le 1, \tag{3.19}$$

then $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ *is in the class* $\Sigma\mathcal{F}_2(\alpha, \delta, b)$.

Theorem 3.5 *For* $i \in \{1,\ldots, n\}$, *let* $\gamma_i > 0$ *and* $f_i \in \Sigma K\mathcal{U}\mathcal{H}(\alpha, b)$ $(\alpha \ge 0$, *and* $b \in \mathbb{C}\backslash\{0\})$. *If*

$$\sum_{i=1}^{n}\gamma_i \le 1, \tag{3.20}$$

then $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ *is in the class* $\Sigma\mathcal{F}_3(\alpha, \delta, b)$.

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Competing interests
The authors declare that they have no competing interests.

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