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Strong convergence theorems by a hybrid extragradient-like approximation method for asymptotically nonexpansive mappings in the intermediate sense in Hilbert spaces

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Abstract

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be an asymptotically nonexpansive map in the intermediate sense with the fixed point set $F(S)$. Let $A : C \rightarrow H$ be a Lipschitz continuous map, and $VI(C, A)$ be the set of solutions $u \in C$ of the variational inequality

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The purpose of this study is to introduce a hybrid extragradient-like approximation method for finding a common element in $F(S)$ and $VI(C, A)$. We establish some strong convergence theorems for sequences produced by our iterative method.

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1 Introduction

Let H be a real Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C . A mapping $A : C \rightarrow H$ is called *monotone* [1-3] if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C;$$

and A is called *k-Lipschitz continuous* if there exists a positive constant k such that

$$\|Au - Av\| \leq k \|u - v\|, \quad \forall u, v \in C.$$

Let S be a mapping of C into itself. Denote by $F(S)$ the set of fixed points of S ; that is $F(S) = \{u \in C : Su = u\}$. Recall that S is *nonexpansive* if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C;$$

and S is *asymptotically nonexpansive* [4] if there exists a null sequence $\{\gamma_n\}$ in $[0, +\infty)$ such that

$$\|S^n u - S^n v\| \leq (1 + \gamma_n) \|u - v\|, \quad \forall u, v \in C \text{ and } n \geq 1.$$

We call S an *asymptotically nonexpansive mapping in the intermediate sense* [5] if there exists two null sequences $\{\gamma_n\}$ and $\{c_n\}$ in $[0, +\infty)$ such that

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n) \|x - y\|^2 + c_n, \quad \forall x, y \in C, \forall n \geq 1.$$

Let $A : C \rightarrow H$ be a monotone and k -Lipschitz continuous mapping. The variational inequality problem [6] is to find the elements $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. The idea of an extragradient iterative process was first introduced by Korpelevich in [7]. When $S : C \rightarrow C$ is a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense, a hybrid extragradient-like approximation method was proposed by Ceng et al. [8, Theorem 1.1] to ensure the *weak* convergence of some algorithms for finding a member of $F(S) \cap VI(C, A)$. Meanwhile, assuming S is *nonexpansive*, Ceng et al. in [9] introduced an iterative process and proved its strong convergence to a member of $F(S) \cap VI(C, A)$.

It is known that an asymptotically nonexpansive mapping in the intermediate sense is not necessarily nonexpansive. Extending both [8, Theorem 1.1, 9, Theorem 5], the main result, Theorem 1, of this article provides a technical method to show the strong convergence of an iterative scheme to an element of $F(S) \cap VI(C, A)$, under the weaker assumption on the asymptotical nonexpansivity in the intermediate sense of S .

2 Strong convergence theorems

Let C be a nonempty closed convex subset of a real Hilbert space H . For any x in H , there exists a unique element in C , which is denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all y in C . We call P_C the *metric projection* of H onto C . It is well-known that P_C is a nonexpansive mapping from H onto C , and

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \quad \text{for all } x \in H, y \in C; \tag{1}$$

see for example [10]. It is easy to see that (1) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \text{for all } x \in H, y \in C. \tag{2}$$

Let A be a monotone mapping of C into H . In the context of variational inequality problems, the characterization of the metric projection (1) implies that

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0.$$

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a monotone and k -Lipschitz continuous mapping. Let $S : C \rightarrow C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with nonnegative null sequences $\{\gamma_n\}$ and $\{c_n\}$. Suppose that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $F(S) \cap VI(C, A)$ is nonempty and bounded.*

Assume that

(i) $0 < \mu \leq 1$, and $0 < a < b < \frac{3}{8k\mu}$;

(ii) $a \leq \lambda_n \leq b$, $\alpha_n \geq 0$, $\beta_n \geq 0$, $\alpha_n + \beta_n \leq 1$, and $3/4 < \delta_n \leq 1$, for all $n \geq 0$;

- (iii) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
 - (iv) $\liminf_{n \rightarrow \infty} \beta_n > 0$;
 - (v) $\lim_{n \rightarrow \infty} \beta_n = 1$.
- Set, for all $n \geq 0$,

$$\begin{aligned} \Delta_n &= \sup\{\|x_n - u\| : u \in F(S) \cap VI(C, A)\}, \\ d_n &= 2b(1 - \mu)\alpha_n\Delta_n, \\ w_n &= b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1 - \delta_n)(1 + \gamma_n), \\ v_n &= b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2\beta_n(1 - \delta_n)(1 + \gamma_n), \text{ and} \\ \vartheta_n &= \beta_n\gamma_n\Delta_n^2 + \beta_n c_n. \end{aligned}$$

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by the algorithm:

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \lambda_n \mu A x_n - \lambda_n (1 - \mu) A y_n), \\ z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S^n P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + d_n \|A y_n\| + w_n \|A x_n\|^2 + v_n \|A y_n\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0. \end{array} \right. \quad (3)$$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (3) are well-defined and converge strongly to the same point $q = P_{F(S) \cap VI(C, A)}(x_0)$.

Proof. First note that $\lim_{n \rightarrow \infty} \gamma_n = \lim_{n \rightarrow \infty} c_n = 0$. We will see that $\{\Delta_n\}$ is bounded, and thus $\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \vartheta_n = 0$.

We divide the proof into several steps.

Step 1. We claim that the following statements hold:

- (a) C_n is closed and convex for all $n \in \mathbb{N}$;
- (b) $\|z_n - u\|^2 \leq \|x_n - u\|^2 + d_n \|A y_n\| + w_n \|A x_n\|^2 + v_n \|A y_n\|^2 + \vartheta_n$ for all $n \geq 0$ and $u \in F(S) \cap VI(C, A)$;
- (c) $F(S) \cap VI(C, A) \subset C_n$ for all $n \in \mathbb{N}$.

It is obvious that C_n is closed for all $n \in \mathbb{N}$. On the other hand, the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - z_n), z \rangle \leq \|x_n\|^2 - \|z_n\|^2 + d_n \|A y_n\| + w_n \|A x_n\|^2 + v_n \|A y_n\|^2 + \vartheta_n,$$

which is affine in z . Therefore, C_n is convex.

Let $t_n = P_C(x_n - \lambda_n A y_n)$ for all $n \geq 0$. Assume that $u \in F(S) \cap VI(C, A)$ is arbitrary. In view of (3), the monotonicity of A , and the fact $u \in VI(C, A)$, we conclude that

$$\begin{aligned} &\|t_n - u\|^2 \\ &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n [\langle A y_n - A u, u - \gamma_n \rangle + \langle A u, u - \gamma_n \rangle + \langle A y_n, \gamma_n - t_n \rangle] \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, \gamma_n - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - 2\langle x_n - \gamma_n, \gamma_n - t_n \rangle - \|\gamma_n - t_n\|^2 + 2\lambda_n \langle A y_n, \gamma_n - t_n \rangle \\ &= \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - \|\gamma_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - \gamma_n, t_n - \gamma_n \rangle. \end{aligned} \quad (4)$$

Now, using

$$\gamma_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n),$$

we estimate the last term

$$\begin{aligned} & \langle x_n - \lambda_n A y_n - \gamma_n, t_n - \gamma_n \rangle \\ = & \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - \gamma_n, t_n - \gamma_n \rangle + \lambda_n \mu \langle A x_n - A y_n, t_n - \gamma_n \rangle \\ \leq & \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - (1 - \delta_n)x_n - \delta_n P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), t_n - \gamma_n \rangle \\ & + \lambda_n \mu \|A x_n - A y_n\| \|t_n - \gamma_n\| \\ \leq & \delta_n \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), t_n - \gamma_n \rangle \\ & - (1 - \delta_n) \lambda_n \langle \mu A x_n + (1 - \mu)A y_n, t_n - \gamma_n \rangle + \lambda_n \mu k \|x_n - \gamma_n\| \|t_n - \gamma_n\|. \end{aligned} \tag{5}$$

It follows from the properties (1) and (2) of the projection $P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n)$ that

$$\begin{aligned} & \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), t_n - \gamma_n \rangle \\ = & \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), \\ & t_n - (1 - \delta_n)x_n - \delta_n P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n) \rangle \\ = & (1 - \delta_n) \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), t_n - x_n \rangle \\ & + \delta_n \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), \\ & t_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n) \rangle \\ \leq & (1 - \delta_n) \langle x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n), t_n - x_n \rangle \\ \leq & (1 - \delta_n) \|x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n - P_C(x_n - \lambda_n \mu A x_n - \lambda_n(1 - \mu)A y_n)\| \|t_n - x_n\| \\ \leq & (1 - \delta_n) \|\lambda_n \mu A x_n + \lambda_n(1 - \mu)A y_n\| \|t_n - x_n\| \\ \leq & (1 - \delta_n) \lambda_n (\mu \|A x_n\| + (1 - \mu) \|A y_n\|) (\|t_n - \gamma_n\| + \|\gamma_n - x_n\|). \end{aligned} \tag{6}$$

In view of (4)-(6), $\lambda_n \leq b$, and the inequalities $2\alpha\beta \leq \alpha^2 + \beta^2$ and $(\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2$, we conclude that

$$\begin{aligned} \|t_n - u\|^2 & \leq \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - \|\gamma_n - t_n\|^2 + 2 \langle x_n - \lambda_n A y_n - \gamma_n, t_n - \gamma_n \rangle \\ & \leq \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - \|\gamma_n - t_n\|^2 \\ & \quad + 2 \lambda_n [\delta_n(1 - \delta_n)(\mu \|A x_n\| + (1 - \mu) \|A y_n\|) (\|t_n - \gamma_n\| + \|\gamma_n - x_n\|) \\ & \quad - 2(1 - \delta_n) \lambda_n \langle \mu A x_n + (1 - \mu)A y_n, t_n - \gamma_n \rangle + 2 \lambda_n \mu k \|x_n - \gamma_n\| \|t_n - \gamma_n\|] \\ & \leq \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - \|\gamma_n - t_n\|^2 \\ & \quad + 2 \delta_n(1 - \delta_n) b (\mu \|A x_n\| + (1 - \mu) \|A y_n\|) (\|t_n - \gamma_n\| + \|\gamma_n - x_n\|) \\ & \quad + 2(1 - \delta_n) b (\mu \|A x_n\| + (1 - \mu) \|A y_n\|) \|t_n - \gamma_n\| + 2 b \mu k \|x_n - \gamma_n\| \|t_n - \gamma_n\| \\ = & \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 - \|\gamma_n - t_n\|^2 \\ & \quad + 2 \delta_n(1 - \delta_n) (b^2 \mu^2 \|A x_n\|^2 + b^2 (1 - \mu)^2 \|A y_n\|^2 + \|t_n - \gamma_n\|^2 + \|\gamma_n - x_n\|^2) \\ & \quad + (1 - \delta_n) (b^2 \mu^2 \|A x_n\|^2 + b^2 (1 - \mu)^2 \|A y_n\|^2 + 2 \|t_n - \gamma_n\|^2) \\ & \quad + b \mu k (\|x_n - \gamma_n\|^2 + \|t_n - \gamma_n\|^2) \\ = & \|x_n - u\|^2 - \|x_n - \gamma_n\|^2 (1 - 2 \delta_n(1 - \delta_n) - b k \mu) \\ & \quad - \|t_n - \gamma_n\|^2 (2 \delta_n^2 - \delta_n - b k \mu) \\ & \quad + 2(1 - \delta_n^2) b^2 \mu^2 \|A x_n\|^2 + 2(1 - \delta_n^2) b^2 (1 - \mu)^2 \|A y_n\|^2. \end{aligned} \tag{7}$$

Since $\frac{3}{4} < \delta_n \leq 1$ and $b < \frac{3}{8k\mu}$, we have from (7) for all $n \in \mathbb{N}$,

$$\|t_n - u\|^2 \leq \|x_n - u\|^2 + 4(1 - \delta_n) b^2 \mu^2 \|A x_n\|^2 + 4(1 - \delta_n) b^2 (1 - \mu)^2 \|A y_n\|^2. \tag{8}$$

In view of the fact that $u \in VI(A, C)$ and properties of P_C , we obtain

$$\begin{aligned}
 \|y_n - u\|^2 &= \|(1 - \delta_n)(x_n - u) + \delta_n(P_C(x_n - \lambda_n \mu Ax_n - \lambda_n(1 - \mu)Ay_n) - u)\|^2 \\
 &\leq (1 - \delta_n)\|x_n - u\|^2 + \delta_n\|P_C(x_n - \lambda_n \mu Ax_n - \lambda_n(1 - \mu)Ay_n) - P_C(u)\|^2 \\
 &\leq (1 - \delta_n)\|x_n - u\|^2 + \delta_n\|x_n - \lambda_n \mu Ax_n - \lambda_n(1 - \mu)Ay_n - u\|^2 \\
 &= (1 - \delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 - 2\langle \lambda_n \mu Ax_n + \lambda_n(1 - \mu)Ay_n, x_n - u \rangle \\
 &\quad + \|\lambda_n \mu Ax_n + \lambda_n(1 - \mu)Ay_n\|^2] \\
 &= (1 - \delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 - 2\lambda_n \mu \langle Ax_n, x_n - u \rangle - 2\lambda_n(1 - \mu)\langle Ay_n, x_n - u \rangle \\
 &\quad + \|\lambda_n \mu Ax_n + \lambda_n(1 - \mu)Ay_n\|^2] \tag{9} \\
 &\leq (1 - \delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 + 2\lambda_n(1 - \mu)\|Ay_n\|\|x_n - u\| \\
 &\quad + \lambda_n^2 \mu \|Ax_n\|^2 + \lambda_n^2(1 - \mu)\|Ay_n\|^2] \\
 &\leq (1 - \delta_n)\|x_n - u\|^2 + \delta_n[\|x_n - u\|^2 + 2b(1 - \mu)\Delta_n\|Ay_n\| \\
 &\quad + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2] \\
 &\leq \|x_n - u\|^2 + \delta_n[2b(1 - \mu)\Delta_n\|Ay_n\| + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2] \\
 &\leq \|x_n - u\|^2 + 2b(1 - \mu)\Delta_n\|Ay_n\| + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2.
 \end{aligned}$$

Since S is asymptotically nonexpansive in the intermediate sense, in view of $S^n u = u$, we conclude that

$$\begin{aligned}
 \|z_n - u\|^2 &= \|(1 - \alpha_n - \beta_n)x_n + \alpha_n \gamma_n + \beta_n S^n t_n - u\|^2 \\
 &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 + \alpha_n\|\gamma_n - u\|^2 + \beta_n\|S^n t_n - u\|^2 \\
 &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 \\
 &\quad + \alpha_n[\|x_n - u\|^2 + 2b(1 - \mu)\Delta_n\|Ay_n\| + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2] \\
 &\quad + \beta_n[(1 + \gamma_n)\|t_n - u\|^2 + c_n] \\
 &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 \\
 &\quad + \alpha_n[\|x_n - u\|^2 + 2b(1 - \mu)\Delta_n\|Ay_n\| + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2] \tag{10} \\
 &\quad + \beta_n(1 + \gamma_n)[\|x_n - u\|^2 + 2(1 - \delta_n)b^2 \mu \|Ax_n\|^2 + 2(1 - \delta_n)b^2(1 - \mu)\|Ay_n\|^2] \\
 &\quad + \beta_n c_n \\
 &\leq \|x_n - u\|^2 + \beta_n \gamma_n \Delta_n^2 + 2b(1 - \mu)\alpha_n \Delta_n\|Ay_n\| \\
 &\quad + (b^2 \mu \alpha_n + 2b^2 \mu^2 \beta_n(1 - \delta_n)(1 + \gamma_n))\|Ax_n\|^2 \\
 &\quad + (b^2(1 - \mu)\alpha_n + 2b^2(1 - \mu)^2 \beta_n(1 - \delta_n)(1 + \gamma_n))\|Ay_n\|^2 \\
 &\quad + \beta_n c_n.
 \end{aligned}$$

This implies that $u \in C_n$. Therefore, $F(S) \cap VI(C, A) \subset C_n$.

Step 2. We prove that the sequence $\{x_n\}$ is well-defined and $F(S) \cap VI(C, A) \subset C_n \cap Q_n$ for all $n \geq 0$.

We prove this assertion by mathematical induction. For $n = 0$ we get $Q_0 = C$. Hence, by step 1, we deduce that $F(S) \cap VI(C, A) \subset C_1 \cap Q_1$. Assume that x_k is defined and $F(S) \cap VI(C, A) \subset C_k \cap Q_k$ for some $k \geq 1$. Then, y_k, z_k are well-defined elements of C . We notice that C_k is a closed convex subset of C since

$$C_k = \{z \in C : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z \rangle \leq d_n \|Ay_n\| + w_n \|Ax_n\|^2 + v_n \|Ay_n\|^2 + \vartheta_n\}.$$

It is easy to see that Q_k is closed and convex. Therefore, $C_k \cap Q_k$ is a closed and convex subset of C , since by the assumption we have $F(S) \cap VI(C, A) \subset C_k \cap Q_k$. This means that $P_{C_k \cap Q_k} x_0$ is well-defined.

By the definition of x_{k+1} and of Q_{k+1} , we deduce that $C_k \cap Q_k \subset Q_{k+1}$. Hence, $F(S) \cap VI(C, A) \subset Q_{k+1}$. Exploiting Step 1 we conclude that $F(S) \cap VI(C, A) \subset C_{k+1} \cap Q_{k+1}$.

Step 3. We claim that the following assertions hold:

(d) $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists and hence $\{x_n\}$, as well as $\{\Delta_n\}$, is bounded.

(e) $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

(f) $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$.

Let $u \in F(S) \cap VI(C, A)$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $u \in F(S) \cap VI(C, A) \subset C_n \cap Q_n$, we conclude that

$$\|x_{n+1} - x_0\| \leq \|u - x_0\|, \quad \forall n \geq 0. \tag{11}$$

This means that $\{x_n\}$ is bounded, and so are $\{y_n\}$, Ax_n and $\{Ay_n\}$, because of the Lipschitz-continuity of A . On the other hand, we have $x_n = P_{Q_n} x_0$ and $x_{n+1} \in C_n \cap Q_n \subset Q_n$. This implies that

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \quad \forall n \geq 0. \tag{12}$$

In particular, $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ hence $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists. It follows from (12) that

$$\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = 0. \tag{13}$$

Since $x_{n+1} \in C_n$ we obtain

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + d_n \|Ay_n\|^2 + w_n \|Ax_n\|^2 + v_n \|Ay_n\|^2 + \vartheta_n.$$

In view of $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \delta_n = 1$ and from the boundedness of $\{Ax_n\}$ and $\{Ay_n\}$ we infer that $\lim_{n \rightarrow \infty} (x_{n+1} - z_n) = 0$. Combining with (13) we deduce that $\lim_{n \rightarrow \infty} (x_n - z_n) = 0$.

Step 4. We claim that the following assertions hold:

(g) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

(h) $\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0$.

In view of (3), $z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S^n t_n$, and $S^n u = u$, we obtain from (9) and (8) that

$$\begin{aligned} & \|z_n - u\|^2 \\ &= \|(1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S^n t_n - u\|^2 \\ &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 + \alpha_n \|y_n - u\|^2 + \beta_n \|S^n t_n - u\|^2 \\ &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 \\ &\quad + \alpha_n \left[\|x_n - u\|^2 + 2b(1 - \mu)\Delta_n \|Ay_n\| + b^2 \mu \|Ax_n\|^2 + b^2(1 - \mu)\|Ay_n\|^2 \right] \\ &\quad + \beta_n \left[(1 + \gamma_n)\|t_n - u\|^2 + c_n \right] \\ &\leq (1 - \alpha_n - \beta_n)\|x_n - u\|^2 \\ &\quad + \alpha_n \left[\|x_n - u\|^2 + 2b(1 - \mu)\Delta_n \|Ay_n\| + b^2 \mu \|Ax_n\|^2(1 - \mu)\|Ay_n\|^2 \right] \\ &\quad + \beta_n(1 + \gamma_n) \left[\|x_n - u\|^2 - (1 - 2\delta_n(1 - \delta_n))\|x_n - \gamma_n\|^2 - bk\mu \right. \\ &\quad \quad \left. - (2\delta_n^2 - 1 - bk\mu)\|t_n - \gamma_n\|^2 + 4(1 - \delta_n)b^2 \mu^2 \|Ax_n\|^2 \right. \\ &\quad \quad \left. + 4(1 - \delta_n)b^2(1 - \mu)^2 \|Ay_n\|^2 + \beta_n c_n \right] \\ &\leq \|x_n - u\|^2 + \beta_n \gamma_n \Delta_n^2 + \beta_n c_n \\ &\quad + 2b(1 - \mu)\alpha_n \Delta_n \|Ay_n\| + [b^2 \mu \alpha_n + 4b^2 \mu^2 \beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ax_n\|^2 \\ &\quad + [b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2 \beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ay_n\|^2 \\ &\quad - [\beta_n(1 + \gamma_n)(1 - 2\delta_n(1 - \delta_n) - bk\mu)] \|x_n - \gamma_n\|^2 \\ &\quad - [\beta_n(1 + \gamma_n)(2\delta_n^2 - \delta_n - bk\mu)] \|t_n - \gamma_n\|^2. \end{aligned} \tag{14}$$

Thus, we have

$$\begin{aligned} & \beta_n(1 + \gamma_n)(1 - 2\delta_n(1 - \delta_n) - bk\mu) \|x_n - \gamma_n\|^2 \\ & \leq \|x_n - u\|^2 - \|z_n - u\|^2 + \beta_n\gamma_n\Delta_n^2 + \beta_nc_n \\ & \quad + 2b(1 - \mu)\alpha_n\Delta_n \|Ay_n\| + [b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ax_n\|^2 \\ & \quad + [b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2\beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ay_n\|^2 \\ & \leq (\|x_n - u\| + \|z_n - u\|) \|x_n - z_n\| + \beta_n\gamma_n\Delta_n^2 + \beta_nc_n \\ & \quad + 2b(1 - \mu)\alpha_n\Delta_n \|Ay_n\| + [b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ax_n\|^2 \\ & \quad + [b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2\beta_n(1 + \gamma_n)(1 - \delta_n)] \|Ay_n\|^2. \end{aligned}$$

Since $bk\mu < 3/8$ and $3/4 \leq \delta_n \leq 1$ for all $n \geq 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - \gamma_n\|^2 = 0.$$

In the same manner, from (14), we conclude that

$$\lim_{n \rightarrow \infty} \|t_n - \gamma_n\|^2 = 0.$$

Since A is k -Lipschitz continuous, we obtain $\|Ay_n - Ax_n\| \rightarrow 0$. On the other hand,

$$\|x_n - t_n\| \leq \|x_n - \gamma_n\| + \|\gamma_n - t_n\|,$$

which implies that $\|x_n - t_n\| \rightarrow 0$. Since $z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n\gamma_n + \beta_nS^n t_n$, we have

$$z_n - x_n = -\alpha_n x_n + \alpha_n \gamma_n + \beta_n(S^n t_n - x_n).$$

From $\|z_n - x_n\| \rightarrow 0$, $\alpha_n \rightarrow 0$, $\liminf_{n \rightarrow \infty} \beta_n > 0$ and the boundedness of $\{x_n, \gamma_n\}$ we deduce that $\|S^n t_n - x_n\| \rightarrow 0$. Thus, we get $\|t_n - S^n t_n\| \rightarrow 0$. By the triangle inequality, we obtain

$$\begin{aligned} \|x_n - S^n x_n\| & \leq \|x_n - t_n\| + \|t_n - S^n t_n\| + \|S^n t_n - S^n x_n\| \\ & \leq \|x_n - t_n\| + \|t_n - S^n t_n\| + \sqrt{(1 + \gamma_n) \|t_n - x_n\|} + c_n. \end{aligned}$$

Hence, $\|x_n - S^n x_n\| \rightarrow 0$. Since $\|x_n - x_{n+1}\| \rightarrow 0$, it follows from Lemma 2.7 of Sahu et al. [5] that $\|x_n - Sx_n\| \rightarrow 0$. By the uniform continuity of S , we obtain $\|x_n - S^m x_n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $m \geq 1$.

Step 5. We claim that $\omega_w(x_n) \subset F(S) \cap VI(C, A)$, where

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightarrow x \text{ weakly for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

The proof of this step is similar to that of [8, Theorem 1.1, step 5] and we omit it.

A similar argument as mentioned in [9, Theorem 5, Step 6] proves the following assertion.

Step 6. The sequences $\{x_n\}$, $\{\gamma_n\}$ and $\{z_n\}$ converge strongly to the same point $q = P_{F(S) \cap VI(C, A)}(x_0)$, which completes the proof.

For $\alpha_n = 0$, $\beta_n = 1$ and $\delta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 1, we get the following corollary.

Corollary 2. Let C be a nonempty closed convex subset of a real Hilbert spaces H . Let $A : C \rightarrow H$ be a monotone and k -Lipschitz continuous mapping and let $S : C \rightarrow C$ be a

uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with nonnegative null sequences $\{\gamma_n\}$ and $\{c_n\}$.

Suppose that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $F(S) \cap VI(C, A)$ is nonempty and bounded. Set $\vartheta_n = \gamma_n \Delta_n + c_n$. Let μ be a constant in $(0, 1]$, and let $\{\lambda_n\}$ be a sequence in $[a, b]$ with $a > 0$ and $b < \frac{3}{8k\mu}$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n \mu Ax_n - \lambda_n(1 - \mu)Ay_n), \\ z_n = S^n P_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0. \end{cases} \quad (15)$$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (15) are well-defined and converge strongly to the same point $q = P_{F(S) \cap VI(C, A)}(x_0)$.

In Theorem 1, if we set $\alpha_n = 0$ and $\beta_n = 1$ for all $n \in \mathbb{N}$ then the following result concerning variational inequality problems holds.

Corollary 3. Let C be a nonempty closed convex subset of a real Hilbert spaces H . Let $A : C \rightarrow H$ be a monotone and k -Lipschitz continuous mapping and let $S : C \rightarrow C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with null sequences $\{\gamma_n\}$ and $\{c_n\}$.

Suppose that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $F(S) \cap VI(C, A)$ is nonempty and bounded. Let μ be a constant in $(0, 1]$, let $\{\lambda_n\}$ be a sequence in $[a, b]$ with $a > 0$ and $b < \frac{3}{8k\mu}$, and let $\{\delta_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \delta_n = 1$ and $\delta_n > \frac{3}{4}$ for all $n \geq 0$. Set $\Delta_n = \sup \{\|x_n - u\| : u \in F(S) \cap VI(C, A)\}$, $w_n = 4b^2\mu^2(1 + \gamma_n)(1 - \delta_n)$, $\vartheta_n = \gamma_n \Delta_n + c_n$ for all $n \geq 0$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \lambda_n \mu Ax_n - \lambda_n(1 - \mu)Ay_n), \\ z_n = S^n P_C(x_n - \lambda_n Ay_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + w_n \|Ax_n\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0. \end{cases} \quad (16)$$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (16) are well-defined and converge strongly to the same point $q = P_{F(S) \cap VI(C, A)}(x_0)$.

The following theorem is yet another easy consequence of Theorem 1.

Corollary 4. Let H be a real Hilbert space. Let $A : H \rightarrow H$ be a monotone and k -Lipschitz continuous mapping and let $S : H \rightarrow H$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with null sequences $\{\gamma_n\}$ and $\{c_n\}$.

Suppose that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $F(S) \cap A^{-1}(0)$ is nonempty and bounded. Let μ be a constant in $(0, 1]$, let $\{\lambda_n\}$ be a sequence in $[a, 3b/4]$ with $0 < 4a/3 < b < \frac{3}{8k\mu}$, and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n \leq 1, \forall n \geq 0$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (iii) $\liminf_{n \rightarrow \infty} \beta_n > 0$;
- (iv) $\lim_{n \rightarrow \infty} \delta_n = 1$ and $\delta_n > \frac{3}{4}$ for all $n \geq 0$.

Set

$$\begin{aligned} \Delta_n &= \sup\{\|x_n - u\| : u \in F(S) \cap A^{-1}(0)\}, \\ d_n &= 2b(1 - \mu)\alpha_n\Delta_n, \\ w_n &= b^2\mu\alpha_n + 4b^2\mu^2\beta_n(1 - \delta_n)(1 + \gamma_n), \\ v_n &= b^2(1 - \mu)\alpha_n + 4b^2(1 - \mu)^2\beta_n(1 - \delta_n)(1 + \gamma_n), \text{ and} \\ \vartheta_n &= \beta_n\gamma_n\Delta_n^2 + \beta_nc_n \end{aligned}$$

for all $n \geq 0$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ \gamma_n = x_n - \lambda_n\mu Ax_n - \lambda_n(1 - \mu)Ay_n, \\ z_n = (1 - \beta_n)x_n - \alpha_n\mu Ax_n - \alpha_n\lambda_n(1 - \mu)Ay_n + \beta_n S^n(x_n - \frac{\lambda_n}{\delta_n}Ay_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + dn\|Ay_n\| + w_n\|Ax_n\|^2 + v_n\|Ay_n\|^2 + \vartheta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0. \end{array} \right. \quad (17)$$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (17) are well-defined and converge strongly to the same point $q = P_{F(S) \cap A^{-1}(0)}(x_0)$.

Proof. Replace λ_n by $\lambda'_n = \frac{\lambda_n}{\delta_n}$. Then, $a \leq \lambda'_n < \frac{4}{3}\lambda_n < b < \frac{3}{8k\mu}$. For $C = H$, we have $P_C = I$ and $VI(C, A) = A^{-1}(0)$. In view of Theorem 1, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are well-defined and converge strongly to the same point $q = P_{F(S) \cap A^{-1}(0)}(x_0)$.

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Authors' contributions

All authors contribute equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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