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On calculation of eigenvalues and eigenfunctions of a Sturm-Liouville type problem with retarded argument which contains a spectral parameter in the boundary condition

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Abstract

In this study, a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the boundary condition and with transmission conditions at the point of discontinuity is investigated. We obtained asymptotic formulas for the eigenvalues and eigenfunctions.

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Keywords: differential equation with retarded argument, transmission conditions, asymptotics of eigenvalues and eigenfunctions

1 Introduction

Boundary-value problems for differential equations of the second order with retarded argument were studied in [1-5], and various physical applications of such problems can be found in [2].

The asymptotic formulas for the eigenvalues and eigenfunctions of boundary problem of Sturm-Liouville type for second order differential equation with retarded argument were obtained in [5].

The asymptotic formulas for the eigenvalues and eigenfunctions of Sturm-Liouville problem with the spectral parameter in the boundary condition were obtained in [6].

In the articles [7-9], the asymptotic formulas for the eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problem with transmission conditions and with the boundary conditions which include spectral parameter were obtained.

In this article, we study the eigenvalues and eigenfunctions of discontinuous boundary-value problem with retarded argument and a spectral parameter in the boundary condition. Namely, we consider the boundary-value problem for the differential equation

$$p(x)y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1}$$

on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, with boundary conditions

$$y(0) = 0, \tag{2}$$

$$y'(\pi) + \lambda y(\pi) = 0, \tag{3}$$

and transmission conditions

$$\gamma_1 y\left(\frac{\pi}{2} - 0\right) = \delta_1 y\left(\frac{\pi}{2} + 0\right), \tag{4}$$

$$\gamma_2 y'\left(\frac{\pi}{2} - 0\right) = \delta_2 y'\left(\frac{\pi}{2} + 0\right), \tag{5}$$

where $p(x) = p_1^2$ if $x \in \left[0, \frac{\pi}{2}\right)$ and $p(x) = p_2^2$ if $x \in \left(\frac{\pi}{2}, \pi\right]$, the real-valued function $q(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $q(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} q(x)$, the real-valued function $\Delta(x) \geq 0$ continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $\Delta(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x), x - \Delta(x) \geq 0$, if $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geq \frac{\pi}{2}$ if $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is a real spectral parameter; p_1, p_2, γ_1, γ_2, δ_1, δ_2 are arbitrary real numbers and $|\gamma_i| + |\delta i| \neq 0$ for $i = 1, 2$. Also, $\gamma_1 \delta_2 p_1 = \gamma_2 \delta_1 p_2$ holds.

It must be noted that some problems with transmission conditions which arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [10].

Let $w_1(x, \lambda)$ be a solution of Equation 1 on $[0, \frac{\pi}{2}]$, satisfying the initial conditions

$$w_1(0, \lambda) = 0, w_1'(0, \lambda) = -1. \tag{6}$$

The conditions (6) define a unique solution of Equation 1 on $[0, \frac{\pi}{2}]$ [2, p. 12].

After defining above solution, we shall define the solution $w_2\ (x, \lambda)$ of Equation 1 on $[\frac{\pi}{2}, \pi]$ by means of the solution $w_1(x, \lambda)$ by the initial conditions

$$w_2\left(\frac{\pi}{2}, \lambda\right) = \gamma_1 \delta_1^{-1} w_1\left(\frac{\pi}{2}, \lambda\right), \quad \omega_2'\left(\frac{\pi}{2}, \lambda\right) = \gamma_2 \delta_2^{-1} \omega_1'\left(\frac{\pi}{2}, \lambda\right). \tag{7}$$

The conditions (7) are defined as a unique solution of Equation 1 on $[\frac{\pi}{2}, \pi]$.

Consequently, the function $w\ (x, \lambda)$ is defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$w(x, \lambda) = \begin{cases} \omega_1(x, \lambda),\ x \in \left[0, \frac{\pi}{2}\right), \\ \omega_2(x, \lambda),\ x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

is a such solution of Equation 1 on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$; which satisfies one of the boundary conditions and both transmission conditions.

Lemma 1. Let $w\ (x, \lambda)$ be a solution of Equation 1 and $\lambda > 0$. Then, the following integral equations hold:

$$\begin{aligned} w_1(x, \lambda) = &-\frac{p_1}{s} \sin \frac{s}{p_1} x \\ &- \frac{1}{s} \int_0^x \frac{q(\tau)}{p_1} \sin \frac{s}{p_1}(x - \tau) w_1(\tau - \Delta(\tau), \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right), \end{aligned} \tag{8}$$

$$\begin{aligned} w_2(x, \lambda) = &\frac{\gamma_1}{\delta_1} w_1\left(\frac{\pi}{2}, \lambda\right) \cos \frac{s}{p_2}\left(x - \frac{\pi}{2}\right) + \frac{\gamma_2 p_2 w'_1(\frac{\pi}{2}, \lambda)}{s \delta_2} \sin \frac{s}{p_2}\left(x - \frac{\pi}{2}\right) \\ &- \frac{1}{s} \int_{\pi/2}^x \frac{q(\tau)}{p_2} \sin \frac{s}{p_2}(x - \tau) w_2(\tau - \Delta(\tau), \lambda) d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right). \end{aligned} \tag{9}$$

Proof. To prove this, it is enough to substitute $-\frac{s^2}{p_1^2}\omega_1(\tau,\lambda)-\omega_1''(\tau,\lambda)$ and $-\frac{s^2}{p_2^2}\omega_2(\tau,\lambda)-\omega_2''(\tau,\lambda)$ instead of $-\frac{q(\tau)}{p_1^2}\omega_1(\tau-\Delta(\tau),\lambda)$ and $-\frac{q(\tau)}{p_2^2}\omega_2(\tau-\Delta(\tau),\lambda)$ in the integrals in (8) and (9), respectively, and integrate by parts twice.

Theorem 1. The problem (1)-(5) can have only simple eigenvalues.

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the problem (1)-(5) and

$$\tilde{u}(x,\tilde{\lambda})=\begin{cases}\tilde{u}_1(x,\tilde{\lambda}),\ x\in\left[0,\frac{\pi}{2}\right),\\ \tilde{u}_2(x,\tilde{\lambda}),\ x\in\left(\frac{\pi}{2},\pi\right]\end{cases}$$

be a corresponding eigenfunction. Then, from (2) and (6), it follows that the determinant

$$W\left[\tilde{u}_1(0,\tilde{\lambda}),w_1(0,\tilde{\lambda})\right]=\begin{vmatrix}\tilde{u}_1(0,\tilde{\lambda}) & 0\\ \tilde{u}_1'(0,\tilde{\lambda}) & -1\end{vmatrix}=0,$$

and by Theorem 2.2.2 in [2], the functions $\tilde{u}_1(x,\tilde{\lambda})$ and $w_1(x,\tilde{\lambda})$ are linearly dependent on $[0,\frac{\pi}{2}]$. We can also prove that the functions $\tilde{u}_2(x,\tilde{\lambda})$ and $w_2(x,\tilde{\lambda})$ are linearly dependent on $[\frac{\pi}{2},\pi]$. Hence,

$$\tilde{u}_1(x,\tilde{\lambda})=K_iw_i(x,\tilde{\lambda})\quad(i=1,2)\tag{10}$$

for some $K_1\neq 0$ and $K_2\neq 0$. We must show that $K_1=K_2$. Suppose that $K_1\neq K_2$. From the equalities (4) and (10), we have

$$\begin{aligned}\gamma_1\tilde{u}\left(\frac{\pi}{2}-0,\tilde{\lambda}\right)-\delta_1\tilde{u}\left(\frac{\pi}{2}+0,\tilde{\lambda}\right)&=\gamma_1\tilde{u}_1\left(\frac{\pi}{2},\tilde{\lambda}\right)-\delta_1\tilde{u}_2\left(\frac{\pi}{2},\tilde{\lambda}\right)\\ &=\gamma_1K_1w_1\left(\frac{\pi}{2},\tilde{\lambda}\right)-\delta_1K_2w_2\left(\frac{\pi}{2},\tilde{\lambda}\right)\\ &=\gamma_1K_1\delta_1\gamma_1^{-1}w_2\left(\frac{\pi}{2},\tilde{\lambda}\right)-\delta_1K_2w_2\left(\frac{\pi}{2},\tilde{\lambda}\right)\\ &=\delta_1(K_1-K_2)w_2\left(\frac{\pi}{2},\tilde{\lambda}\right)=0.\end{aligned}$$

Since $\delta_1\ (K_1-K_2)\neq 0$, it follows that

$$w_2\left(\frac{\pi}{2},\tilde{\lambda}\right)=0.\tag{11}$$

By the same procedure from equality (5), we can derive that

$$w_2'\left(\frac{\pi}{2},\tilde{\lambda}\right)=0.\tag{12}$$

From the fact that $w_2(x,\tilde{\lambda})$ is a solution of the differential equation (1) on $[\frac{\pi}{2},\pi]$ and satisfies the initial conditions (11) and (12) it follows that $w_1(x,\tilde{\lambda})=0$ identically on $[\frac{\pi}{2},\pi]$ (cf. [2, p. 12, Theorem 1.2.1]).

By using we may also find

$$w_1\left(\frac{\pi}{2},\tilde{\lambda}\right)=w_1'\left(\frac{\pi}{2},\tilde{\lambda}\right)=0.$$

From the latter discussions of $w_2(x,\tilde{\lambda})$, it follows that $w_1(x,\tilde{\lambda})=0$ identically on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$. But this contradicts (6), thus completing the proof.

2 An existance theorem

The function $\omega(x, \lambda)$ defined in Section 1 is a nontrivial solution of Equation 1 satisfying conditions (2), (4) and (5). Putting $\omega(x, \lambda)$ into (3), we get the characteristic equation

$$F(\lambda) \equiv w'(\pi, \lambda) + \lambda\omega(\pi, \lambda) = 0. \tag{13}$$

By Theorem 1.1, the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Equation 13. Let $q_1 = \frac{1}{p_1}\int_0^{\pi/2} |q(\tau)|d\tau$ and $q_2 = \frac{1}{p_2}\int_{\pi/2}^{\pi} q(\tau)d\tau$.

Lemma 2. (1) Let $\lambda \geq 4q_1^2$. Then, for the solution $w_1(x, \lambda)$ of Equation 8, the following inequality holds:

$$\left|w_1(x,\lambda)\right| \leq \left|\frac{p_1}{q_1}\right|, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{14}$$

(2) Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$. Then, for the solution $w_2\,(x, \lambda)$ of Equation 9, the following inequality holds:

$$\left|w_2(x,\lambda)\right| \leq \frac{2p_1}{q_1}\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right\}, \quad x \in \left[\frac{\pi}{2}, \pi\right]. \tag{15}$$

Proof. Let $B_{1\lambda} = \max_{\left[0,\frac{\pi}{2}\right]}\left|w_1(x,\lambda)\right|$. Then, from (8), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leq \left|\frac{p_1}{s}\right| + \frac{1}{s}B_{1\lambda}q_1.$$

If $s \geq 2q_1$, we get (14). Differentiating (8) with respect to x, we have

$$w_1'(x,\lambda) = -\cos\frac{s}{p_1}x - \frac{1}{p_1^2}\int_0^x q(\tau)\cos\frac{s}{p_1}(x-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau. \tag{16}$$

From (16) and (14), it follows that, for $s \geq 2q_1$, the following inequality holds:

$$\left|w'_1(x,\lambda)\right| \leq \sqrt{\frac{s^2}{p_1^2}+1} + 1.$$

Hence,

$$\frac{\left|w'_1(x,\lambda)\right|}{s} \leq \frac{1}{q_1}. \tag{17}$$

Let $B_{2\lambda} = \max_{\left[\frac{\pi}{2},\pi\right]}\left|w_2(x,\lambda)\right|$. Then, from (9), (14) and (17), it follows that, for $s \geq 2q_1$, the following inequalities holds:

$$B_{2\lambda} \leq \frac{|p_1|}{q_1}\left|\frac{\gamma_1}{\delta_1}\right| + |p_2|\left|\frac{\gamma_2}{\delta_2}\right|\frac{1}{|q_1|} + \frac{1}{2q_2}B_{2\lambda}q_2,$$

$$B_{2\lambda} \leq \frac{2\,|p_1|}{q_1}\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right\}.$$

Hence, if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$, we get (15).

Theorem 2. The problem (1)-(5) has an infinite set of positive eigenvalues.

Proof. Differentiating (9) with respect to x, we get

$$w'_2(x,\lambda) = -\frac{s\gamma_1}{p_2\delta_1} w_1\left(\frac{\pi}{2},\lambda\right)\sin\frac{s}{p_2}\left(x-\frac{\pi}{2}\right) + \frac{\gamma_2 w'_1(\frac{\pi}{2},\lambda)}{\delta_2}\cos\frac{s}{p_2}\left(x-\frac{\pi}{2}\right) - \frac{1}{p_2^2}\int_{\pi/2}^{x} q(\tau)\cos\frac{s}{p_2}(x-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau. \tag{18}$$

From (8), (9), (13), (16) and (18), we get

$$\begin{aligned}
&-\frac{s\gamma_1}{p_2\delta_1}\left(-\frac{p_1}{s}\sin\frac{s\pi}{2p_1} - \frac{1}{sp_1}\int_0^{\frac{\pi}{2}} q(\tau)\sin\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right) \times \sin\frac{s\pi}{2p_2}\\
&+\frac{\gamma_2}{\delta_2}\left(-\cos\frac{s\pi}{2p_1} - \frac{1}{p_1^2}\int_0^{\frac{\pi}{2}} q(\tau)\cos\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)\\
&\times\cos\frac{s\pi}{2p_2} - \frac{1}{p_2^2}\int_{\pi/2}^{\pi} q(\tau)\cos\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\\
&+\lambda\left(\frac{\gamma_1}{\delta_1}\left[-\frac{p_1}{s}\sin\frac{s\pi}{2p_1} - \frac{1}{sp_1}\int_0^{\frac{\pi}{2}} q(\tau)\sin\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right] \times\cos\frac{s\pi}{2p_2}\right.\\
&+\frac{\gamma_2 p_2}{\delta_2 s}\left[-\cos\frac{s\pi}{2p_1} - \frac{1}{p_1^2}\int_0^{\frac{\pi}{2}} q(\tau)\cos\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right]\\
&\left.\times\sin\frac{s\pi}{2p_2} - \frac{1}{sp_2}\int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\right) = 0.
\end{aligned} \tag{19}$$

Let λ be sufficiently large. Then, by (14) and (15), Equation 19 may be rewritten in the form

$$s\sin s\pi\frac{p_1+p_2}{2p_1p_2} + O(1) = 0. \tag{20}$$

Obviously, for large s, Equation 20 has an infinite set of roots. Thus, the theorem is proved.

3 Asymptotic formulas for eigenvalues and eigenfunctions

Now, we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following, we shall assume that s is sufficiently large. From (8) and (14), we get

$$\omega_1(x,\lambda) = O(1) \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]. \tag{21}$$

From (9) and (15), we get

$$\omega_2(x,\lambda) = O(1) \quad \text{on} \quad \left[\frac{\pi}{2}, \pi\right]. \tag{22}$$

The existence and continuity of the derivatives $\omega'_{1s}(x,\lambda)$ for $0 \le x \le \frac{\pi}{2}$, $|\lambda| < \infty$, and $\omega'_{2s}(x,\lambda)$ for $\frac{\pi}{2} \le x \le \pi$, $|\lambda| < \infty$, follows from Theorem 1.4.1 in [?].

$$\omega'_{1s}(x,\lambda) = O(1), \quad x \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \omega'_{2s}(x,\lambda) = O(1), \quad x \in \left[\frac{\pi}{2}, \pi\right]. \tag{23}$$

Theorem 3. Let n be a natural number. For each sufficiently large n, there is exactly one eigenvalue of the problem (1)-(5) near $\frac{p_1^2 p_2^2}{(p_1+p_2)^2}(2n+1)^2$.

Proof. We consider the expression which is denoted by $O(1)$ in Equation 20. If formulas (21)-(23) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s the roots of Equation 20 are situated close to entire numbers. We shall show that, for large n, only one root (20) lies near to each $\frac{4n^2 p_1^2 p_2^2}{(p_1+p_2)^2}$. We consider the function $\phi(s) = \sin s\pi \frac{p_1+p_2}{2p_1p_2} + O(1)$. Its derivative, which has the form $\phi'(s) = \sin s\pi \frac{p_1+p_2}{2p_1p_2} + s\pi \frac{p_1+p_2}{2p_1p_2} \cos s\pi \frac{p_1+p_2}{2p_1p_2} + O(1)$, does not vanish for s close to n for sufficiently large n. Thus, our assertion follows by Rolle's Theorem.

Let n be sufficiently large. In what follows, we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem (1)-(5) situated near $\frac{4n^2 p_1^2 p_2^2}{(p_1+p_2)^2}$. We set $s_n = \frac{2np_1p_2}{p_1+p_2} + \delta_n$. From (20), it follows that $\delta_n = O\left(\frac{1}{n}\right)$. Consequently

$$s_n = \frac{2np_1p_2}{p_1+p_2} + O\left(\frac{1}{n}\right). \tag{24}$$

The formula (24) makes it possible to obtain asymptotic expressions for eigenfunction of the problem (1)-(5). From (8), (16) and (21), we get

$$\omega_1(x,\lambda) = O\left(\frac{1}{s}\right), \tag{25}$$

$$\omega'_1(x,\lambda) = O(1). \tag{26}$$

From (9), (22), (25) and (26), we get

$$\omega_2(x,\lambda) = O\left(\frac{1}{s}\right). \tag{27}$$

By putting (24) in (25) and (27), we derive that

$$u_{1n} = w_1(x,\lambda_n) = O\left(\frac{1}{n}\right),$$
$$u_{2n} = w_2(x,\lambda_n) = O\left(\frac{1}{n}\right).$$

Hence, the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = O\left(\tfrac{1}{n}\right) \quad \text{for } x \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right].$$

Under some additional conditions, the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and have finite limits $q'(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} q'(x)$ and $\Delta''(\frac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta''(x)$, respectively.

(b) $\Delta'(x) \le 1$ in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $\Delta(0) = 0$ and $\lim_{x \to \frac{\pi}{2} + 0} \Delta(x) = 0$.

Using (b), we have

$$x - \Delta(x) \geq 0 \quad \text{for } x \in \left[0, \frac{\pi}{2}\right) \quad \text{and} \quad x - \Delta(x) \geq \frac{\pi}{2} \quad \text{for } x \in \left(\frac{\pi}{2}, \pi\right]. \tag{28}$$

From (25), (27) and (28), we have

$$w_1(\tau - \Delta(\tau), \lambda) = O\left(\frac{1}{s}\right), \tag{29}$$

$$w_2(\tau - \Delta(\tau), \lambda) = O\left(\frac{1}{s}\right). \tag{30}$$

Under the conditions (a) and (b), the following formulas

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} q(\tau) \sin \frac{s}{p_1} \left(\frac{\pi}{2} - \tau\right) d\tau = O\left(\frac{1}{s}\right), \\ &\int_0^{\frac{\pi}{2}} q(\tau) \cos \frac{s}{p_1} \left(\frac{\pi}{2} - \tau\right) d\tau = O\left(\frac{1}{s}\right) \end{aligned} \tag{31}$$

can be proved by the same technique in Lemma 3.3.3 in [?]. Putting these expressions into (19), we have

$$\begin{aligned} 0 = {} & \frac{\gamma_1 p_1}{p_2 \delta_1} \sin \frac{s\pi}{2p_1} \sin \frac{s\pi}{2p_2} - \frac{\gamma_2}{\delta_2} \cos \frac{s\pi}{2p_2} - s p_1 \sin \frac{s\pi}{2p_1} \cos \frac{2\pi}{2p_2} \\ & - \frac{s\gamma_2 p_2}{\delta_2} \cos \frac{s\pi}{2p_1} \sin \frac{s\pi}{2p_2} + O\left(\frac{1}{s}\right), \end{aligned}$$

and using $\gamma_1 \delta_2 p_1 = \gamma_2 \delta_1 p_2$ we get

$$0 = \frac{\gamma_2}{\delta_2} \cos s\pi \frac{p_1 + p_2}{2p_1 p_2} - s p_1 \sin s\pi \frac{p_1 + p_2}{2p_1 p_2} + O\left(\frac{1}{s}\right).$$

Dividing by s and using $s_n = \frac{2np_1p_2}{p_1+p_2} + \delta_n$, we have

$$\sin\left(n\pi + \frac{\pi(p_1 + p_2)\delta_n}{2p_1 p_2}\right) = O\left(\frac{1}{n_2}\right).$$

Hence,

$$\delta_n = O\left(\frac{1}{n^2}\right),$$

and finally

$$s_n = \frac{2np_1p_2}{p_1 + p_2} + O\left(\frac{1}{n^2}\right). \tag{32}$$

Thus, we have proven the following theorem.

Theorem 4. If conditions (a) and (b) are satisfied, then the positive eigenvalues $\lambda_n = s_n^2$ of the problem (1)-(5) have the (32) asymptotic representation for $n \to \infty$.

We now may obtain a sharper asymptotic formula for the eigenfunctions. From (8) and (29),

$$w_1(x,\lambda) = -\frac{p_1}{s}\sin\frac{s}{p_1}x + O\left(\frac{1}{s^2}\right). \tag{33}$$

Replacing s by s_n and using (32), we have

$$u_{1n}(x) = \frac{p_1 + p_2}{2p_2n}\sin\frac{2p_2n}{p_1 + p_2}x + O\left(\frac{1}{n^2}\right). \tag{34}$$

From (16) and (29), we have

$$\frac{w'_1(x,\lambda)}{s} = -\frac{\cos\frac{s}{p_1}x}{s} + O\left(\frac{1}{s^2}\right), \quad x \in \left(0, \frac{\pi}{2}\right]. \tag{35}$$

From (9), (30), (31), (33) and (35), we have

$$\begin{aligned} w_2(x,\lambda) = &\left\{-\frac{\gamma_1 p_1 \sin\frac{s\pi}{2p_1}}{s\delta_1} + O\left(\frac{1}{s^2}\right)\right\}\cos\frac{2}{p_2}\left(x - \frac{\pi}{2}\right) \\ &- \left\{\frac{\gamma_2 p_2 \cos\frac{s\pi}{2p_1}}{s\delta_2} + O\left(\frac{1}{s^2}\right)\right\}\sin\frac{s}{p_2}\left(x - \frac{\pi}{2}\right) + O\left(\frac{1}{s^2}\right), \\ w_2(x,\lambda) = &-\frac{\gamma_2 p_2}{s\delta_2}\sin s\left(\frac{\pi(p_2 - p_1}{2p_1p_2} + \frac{x}{2p_2}\right) + O\left(\frac{1}{s^2}\right). \end{aligned}$$

Now, replacing s by s_n and using (32), we have

$$u_{2n}(x) = -\frac{\gamma_2(p_1 + p_2)}{2np_1\delta_2}\sin n\left(\frac{\pi(p_2 - p_1)}{p_1 + p_2} + \frac{p_1x}{p_1 + p_2}\right) + O\left(\frac{1}{n^2}\right). \tag{36}$$

Thus, we have proven the following theorem.

Theorem 5. If conditions (a) and (b) are satisfied, then the eigenfunctions $u_n(x)$ of the problem (1)-(5) have the following asymptotic representation for n $\to \infty$:

$$u_n(x) = \begin{cases} u_{1n}(x) & \text{for } x \in \left[0, \frac{\pi}{2}\right), \\ u_{2n}(x) & \text{for } x \in \left(\frac{\pi}{2}, \pi\right], \end{cases}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ defined as in (34) and (36), respectively.

4 Conclusion

In this study, first, we obtain asymptotic formulas for eigenvalues and eigenfunctions for discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the boundary condition. Then, under additional conditions (a) and (b) the more exact asymptotic formulas, which depend upon the retardation obtained.

Authors' contributions
Establishment of the problem belongs to AB (advisor). ES obtained the asymptotic formulas for eigenvalues and eigenfunctions. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no completing interests.

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