RESEARCH   **Open Access**

A study of Pescar's univalence criteria for space of analytic functions

Imran Faisal and Maslina Darus[*]

* Correspondence: maslina@ukm.my
School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan,
Malaysia

Abstract

An attempt has been made to give a criteria to a family of functions defined in the space of analytic functions to be univalent. Such criteria extended earlier univalence criteria of Pescar's-type of analytic functions.

2000 MSC: 30C45.

Keywords: analytic functions, univalent functions, integral operator

1. Introduction and preliminaries

Let A denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$.

We denote by S the subclass of A consisting of functions which are univalent in $\mathbb{U}$.

The results in this communication are motivated by Pescar [1]. In [1], a new criteria for an analytic function to be univalent is introduced which is true only for two fixed natural numbers. Then, Breaz and Breaz [2] introduced a new integral operator using product n-multiply analytic functions and gave another univalence criteria for such analytic integral operators. Using such integral operator, we extend the criteria given by Pescar in 2005 and prove that it is true for any two consecutive natural numbers.

First, we recall the main results of Pescar introduced in 1996 and later 2005 as follow:

Lemma 1.1. [1,3] Let α be a complex number with $Re\ \alpha > 0$ such that $c \in \mathbb{C}$, $|c| \le 1$, $c \ne -1$. If $f \in A$ satisfies the condition

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \le 1, \ \forall_z \in \mathbb{U},$$

then the function $(F_\alpha(z))^\alpha = \alpha \int_0^z t^{\alpha-1} f'(t) dt$ is analytic and univalent in $\mathbb{U}$.

Lemma 1.2. [1] Let the function $f \in A$ satisfies $\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \le 1, \quad \forall z \in \mathbb{U}$. Also, let $\alpha \in \mathbb{R}(\alpha \in [1, \frac{3}{2}])$ and $c \in \mathbb{C}$. If $|c| \le \frac{3 - 2\alpha}{\alpha}(c \ne -1)$ and $|g(z)| \le 1$, then the function $G_a(z)$ defined by $(G_\alpha(z))^\alpha = \alpha \int_0^z [f(t)]^{\alpha-1}$ is in the univalent function class S.

Lemma 1.3. [4] If $f \in A$ satisfies the condition $\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \le 1, \quad \forall z \in \mathbb{U}$, then the function f is univalent in $\mathbb{U}$.

Springer

Lemma 1.4. (Schwarz Lemma) Let the analytic function f be regular in the open unit disk $\mathbb{U}$ and let $f(0) = 0$. If $|f(z)| \leq 1$, $(z \in \mathbb{U})$ then $|f(z)| \leq |z|$ where the equality holds true only if $f(z) = kz$ and $|k| = 1$.

Breaz (cf., [2,5]) introduced a family of integral operators for $f_i \in A$ univalent in $\mathbb{U}$ denoted by $G_{n,\alpha}$ such that

$$G_{n,\alpha}(z) = \left[(n(\alpha - 1) + 1) \int_0^z [f_1(t)]^{\alpha-1} \cdots [f_n(t)]^{\alpha-1} dt \right]^{\frac{1}{n(\alpha - 1) + 1}}, \quad f_i \in A. \tag{1}$$

In the case of $n = 1$, the operator $G_{n,\alpha}$ becomes identical to the operator G_α given in Lemma 1.2 which was introduced by Pescar in 1996.

2. Main univalence criteria for analytic function

In this section, we make a criteria for space of analytic functions to be univalent. We give proof and applications only for the first theorem and for the remaining theorems we use the same techniques.

Theorem 2.1. Let $f_i \in A$, $\left| \dfrac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \leq 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \leq 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{4M(3^n - 2^n)}{2^n} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$|f_i(z)| \leq M, \quad \forall i \text{ and } M \geq 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Theorem 2.2. Let $f_i \in A$, $\left| \dfrac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \leq 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \leq 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{6M(4^n - 3^n)}{3^n} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$|f_i(z)| \leq M, \quad \forall i \text{ and } M \geq 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Theorem 2.3. Let $f_i \in A$, $\left| \dfrac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \leq 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \leq 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{8M(5^n - 4^n)}{4^n} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$|f_i(z)| \leq M, \quad \forall i \text{ and } M \geq 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Proof of Theorem 2.1. Since for each $f_i \in A$ implies

$$\frac{f_i(z)}{z} = 1 + \sum_{2}^{\infty} a_n z^{n-1}, \quad \forall i$$

and

$$\frac{f_i(z)}{z} = 1 \quad \text{at } z = 0, \ \forall i.$$

We can write

$$\prod_{i=1}^{n} \frac{f_i(z)}{z} = 1 \quad \text{at } z = 0.$$

Now suppose that

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha-1} \cdots \left[\frac{f_n(t)}{t}\right]^{\alpha-1} dt$$

and taking logarithmic derivative and doing some mathematics we get

$$F''(z) = (\alpha - 1) \sum_{i=2}^{\infty} \left(\frac{f_i(z)}{z}\right)^{\alpha-2} \left(\frac{zf'_i(z) - f_i(z)}{z^2} \prod_{j=1}^{n} \left(\frac{f_j(z)}{(z)}\right)^{\alpha-1}\right),$$

$$\Rightarrow \frac{zF''(z)}{F'(z)} = (\alpha - 1) \sum_{i=2}^{\infty} \left(\frac{zf'_i(z)}{f_i(z)} - 1\right),$$

$$\Rightarrow \left|\frac{zF''(z)}{F(z)}\right| \leq (\alpha - 1) \sum_{i=2}^{\infty} \left(\left|\frac{z^2 f'_i(z)}{(f_i(z))^2}\right| \left|\frac{f_i(z)}{z}\right| + 1\right).$$

Using hypothesis of Theorem 2.1 such as $\left|\frac{z^2 f'_i(z)}{(f_i(z))^2}\right| \leq 2$, $|f_i(z)| \leq M$, $\forall i$ for $M \geq 1$ and after doing calculation we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1) \sum_{i=1}^{m} (2M + (2M + 1)), \quad \because 2M \geq 1,$$

$$\Rightarrow \left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1) \sum_{i=1}^{m} \left(2M + \left(2M + \frac{2M}{2}\right) + \left(3M + \frac{3M}{2}\right)\right), \quad \because 3M/2 \geq 1$$

$$\Rightarrow \left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1) \sum_{i=1}^{m} \left(2M + 3M + \frac{9M}{2}M + \cdots + n\text{th term}\right).$$

Therefore, by Lemma 1.1, we get

$$\left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)}\right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right|,$$

$$\Rightarrow \left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)}\right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right|$$

and

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq |c| + \left(\frac{\alpha - 1}{\alpha}\right)\sum_{i=1}^{n}\left(2M + 3M + \frac{9}{2}M + \cdots + n\text{th term}\right).$$

Hence, after calculation, we have

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq |c| + \left(\frac{\alpha - 1}{\alpha}\right)\sum_{i=1}^{n}\left(\frac{4M(3^n - 2^n)}{2^n}\right)$$

or $\quad\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq |c| + \left(\frac{\alpha - 1}{\alpha}\right)n\left(\frac{4M(3^n - 2^n)}{2^n}\right),$

and again using the hypothesis of Theorem 2.1 we get

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq 1,$$

and hence proved.

Theorem 2.4. Let $f_i \in A$, $\left|\frac{z^2 f'_i(z)}{(f_i(z))^2} - 1\right| \leq 1,\quad z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \leq 1 + \left(\frac{\alpha - 1}{\alpha}\right)\left(\frac{6M(3^n - 2^n)}{3^n}\right)n,\quad \alpha \in \mathbb{R},\ c \in \mathbb{C}$$

and

$$\left| f_i(z) \right| \leq M,\quad \forall i \text{ and } M \geq 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Proof. Using the proof of Theorem 2.1, we have

$$\left|\frac{zF''(z)}{F(z)}\right| \leq (\alpha - 1)\sum_{i-2}^{\infty}\left(\left|\frac{z^2 f'_i(z)}{(f_i(z))^2}\right|\left|\frac{f_i(z)}{z}\right| + 1\right).$$

Again, using the hypothesis, we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1)\sum_{i=1}^{m}(2M + (2M + 1)),\quad \because 2M \geq 1,$$

$$\Rightarrow \left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1)\sum_{i-1}^{m}\left(2M + \left(M + \frac{M}{2}\right) + \left(M + \frac{2M}{9}\right)\right),$$

$$\Rightarrow \left|\frac{zF''(z)}{F'(z)}\right| \leq (\alpha - 1)\sum_{i=1}^{m}\left(2M + \frac{4}{3}M + \frac{8}{9}M + \cdots + n\text{th term}\right).$$

Thus, we have

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right|,$$

$$\Rightarrow \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zF''(z)}{\alpha F'(z)} \right| \leq |c| + \frac{1}{\alpha}\left|\frac{zF''(z)}{F'(z)}\right|$$

and

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zF''(z)}{\alpha F'(z)} \right| \le |c| + \left(\frac{\alpha - 1}{\alpha} \right) \sum_{i=1}^{n} \left(2M + \frac{4}{3}M + \frac{8}{9}M + \cdots + n\text{th term} \right),$$

which implies that

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zF''(z)}{\alpha F'(z)} \right| \quad \le \quad |c| + (\frac{\alpha - 1}{\alpha}) \sum_{i=1}^{n} \left(\frac{6M(3^n - 2^n)}{3^n} \right)$$

$$\text{or} \quad \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zF''(z)}{\alpha F'(z)} \right| \quad \le \quad |c| + \left(\frac{\alpha - 1}{\alpha} \right) n \left(\frac{6M(3^n - 2^n)}{3^n} \right).$$

Again, using the hypothesis of Theorem 2.1, we get

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zF''(z)}{\alpha F'(z)} \right| \quad \le \quad 1,$$

and hence proved.

Similarly, we proved the following theorems:

Theorem 2.5. Let $f_i \in A$, $\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \le 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \le 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{8M(4^n - 3^n)}{4^n} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$|f_i(z)| \le M, \quad \forall i \text{ and } M \ge 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Theorem 2.5. Let $f_i \in A$, $\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \le 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \le 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{10M(5^n - 4^n)}{5^n} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$|f_i(z)| \le M, \quad \forall i \text{ and } M \ge 1.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

3. Applications of univalence criteria

Considering $n = 1$ in Theorem 2.1, we obtain the following application:

Corollary 3.1. Let $f_i \in A$, $\left| \frac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \le 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \le 1 + \left(\frac{\alpha - 1}{\alpha} \right) (2M), \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$\left| f_i(z) \right| \le 1, \quad \forall i.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Considering $M = n = 11$ in Theorem 2.1, we obtain second application as follow:

Corollary 3.2. Let $f_i \in A$, $\left| \dfrac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \le 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \le \left(\frac{3\alpha - 2}{\alpha} \right), \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$\left| f_i(z) \right| \le 1, \quad \forall i.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

Considering $M = 1$ in Theorem 2.1, we obtain third application such as:

Corollary 3.3. Let $f_i \in A$, $\left| \dfrac{z^2 f'_i(z)}{(f_i(z))^2} - 1 \right| \le 1$, $z \in \mathbb{U}$ for all $i = \{1, 2, ..., n\}$.

If

$$|c| \le 1 + \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{(3^n - 2^n)}{2^{n-2}} \right) n, \quad \alpha \in \mathbb{R}, \ c \in \mathbb{C}$$

and

$$\left| f_i(z) \right| \le 1, \quad \forall i.$$

Then, the family of functions f denoted by $G_{n,\alpha}$ belong to the class S.

If we substitute $n = 1$ and $M = n = 1$ in Theorem 2.4, we get the results of Corollaries 3.1 and 3.2, respectively.

Other work related to integral operators concerning on univalence criteria and properties can be found in [6,7].

Authors' contributions
The first author is currently a PhD student under supervision of the second author and jointly worked on the results. All authors read and approved the final manuscript.

References
1. Pescar, V: On the univalence of some integral operators. J Indian Acad Math. **27**, 239–243 (2005)
2. Breaz, D, Breaz, N: Univalence of an integral operator. Mathematica (Cluj). **47**(70), 35–38 (2005)
3. Pescar, V: A new generalization of Ahlfors's and Becker's criterion of univalence. Bull Malaysian Math Soc. **19**, 53–54 (1996)
4. Ozaki, S, Nunokawa, M: The Schwarzian derivative and univalent func-tions. Proc Am Math Soc. **33**, 392–394 (1972). doi:10.1090/S0002-9939-1972-0299773-3
5. Breaz, D: Integral Operators on Spaces of Univalent Functions. Publishing House of the Romanian Academy of Sciences, Bucharest (in Romanian) (2004)
6. Darus, M, Faisal, I: A study on Becker's univalence criteria. Abstr Appl Anal **2011**, 13 (2011). Article ID759175,

7.   Mohammed, A, Darus, M: Starlikeness properties for a new integral operator for meromorphic functions. J Appl Math **2011**, 8 (2011). Article ID 804150,