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# Gamma distribution approach in chance-constrained stochastic programming model

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**Abstract**

In this article, a method is developed to transform the chance-constrained programming problem into a deterministic problem. We have considered a chance-constrained programming problem under the assumption that the random variables $a_{ij}$ are independent with Gamma distributions. This new method uses estimation of the distance between distribution of sum of these independent random variables having Gamma distribution and normal distribution, probabilistic constraint obtained via Essen inequality has been made deterministic using the approach suggested by Polya. The model studied on in practice stage has been solved under the assumption of both Gamma and normal distributions and the obtained results have been compared.

**Keywords:** chance-constrained programming, Essen inequality, Gamma distribution

## 1. Introduction

A chance-constrained stochastic programming (CCSP) models is one of the major approaches for dealing with random parameters in the optimization problems. Charnes and Cooper [1] have first modelled CCSP. Here, they have developed a new conceptual and analytic method which contains temporary planning of optimal stochastic decision rules under uncertainty. Symonds [2] has presented deterministic solutions for the class of chance-constraint programming problem. Kolbin [3] has examined the risk and indefiniteness in planning and managing problems and presented chance-constraint programming models. Stancu-Minasian [4] has suggested a minimum-risk approach to multi-objective stochastic linear programming problems. Hulsurkar et al. [5] have studied on a practice of fuzzy programming approach of multi-objective stochastic linear programming problems. They have used fuzzy programming approach for finding a solution after changing the suggested stochastic programming problem into a linear or a nonlinear deterministic problem. Liu and Iwamura [6] have studied on chance-constraint programming with fuzzy parameters. Chance-constraint programming in stochastic is expanded to fuzzy concept by their studies. They have presented certain equations with chance constraint in some fuzzy concept identical to stochastic programming. Furthermore, they have suggested a fuzzy simulation method for chance constraints for which it is usually difficult to be changed into certain equations. Finally, these fuzzy simulations which became basis for genetic algorithm have been suggested for solving problems of this type and discussing numeric examples. Mohammed [7] has studied on chance-constraint fuzzy goal programming containing right-hand side values with uniform random variable coefficients. He presented the main idea related with the stochastic goal programming and chance-constraint linear goal programming. Kampas and White [8] have suggested the programming based on probability for the control of nitrate pollution in their studies and compared this with the approaches of various probabilistic constraints. Yang and Wen [9] presented a chance-constrained programming model for transmission system planning in the competitive electricity market environment. Huang [10] provided two types of credibility-based chance-constrained models for portfolio selection with fuzzy returns. Ağpak and Gökçen [11] developed new mathematical models for stochastic traditional and U-type assembly lines with a chance-constrained 0-1 integer programming technique. Henrion and Strugarek [12] investigated the convexity of chance constraints with independent random variables. Parpas and Rüstem [13] proposed a stochastic algorithm for the global optimization of chance-constrained problems. They assumed that the probability measure used to evaluate the constraints is known only through its moments. Xu et al. [14] developed a robust hybrid stochastic chance-constraint programming model for supporting municipal solid waste management under uncertainty. Abdelaziz and Masri [15] proposed a chance-constrained approach and a compromise programming approach to transform the multi-objective stochastic linear program with partial linear information on the probability distribution into its equivalent uni-objective problem. Goyal and Ravi [16] presented a polynomial time approximation scheme for the chance-constrained knapsack problem when item sizes are normally distributed and independent of other items.

The classical linear programming problem, which is a specific class of mathematical programming problem, is formulated as follows

$$\max z(x) = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m$$

$$x_j \geq 0 \quad j = 1, \ldots, n$$

where all coefficients (technologic coefficients $a_{ij}$, right-hand side values $b_i$ and objective function coefficients $c_j$ ($j$ = 1,..., $n$ $i$ = 1,..., $m$)) are deterministic. However, when at least one coefficient is a random variable, the problem becomes a stochastic programming problem.

In this article, we have assumed that the $a_{ij}$, ($i$ = 1,..., $m$, $j$ = 1,... $n$) which are the elements of, $m \times n$ type technologic matrix $A$, are random variables having Gamma distribution. In case that these coefficients having Gamma distribution are independent, the estimation of the distance between the distribution of sum of them and normal distribution has been obtained. Essen inequality has been used for these and deterministic equality of chance constraints has been found. The model with random variable coefficients has been solved via the suggested method and it has been implemented on a numeric example. The model has been examined again for the case to have coefficients with normal distribution. It has been observed that the case $a_{ij}$ coefficients have Gamma distribution or normal distribution has given similar results for large values of $n$ with regard to objective function.

## 2. Chance-constrained stochastic programming

Stochastic programming deals with the case that input data (prices, right hand side vector, technologic coefficients) are random variables. As parameters are random variables, a probability distribution should be determined. Two frequently used approaches for transforming stochastic programming problem into a deterministic programming problem are chance constraint programming and two-staged programming.

"Chance-constrained programming" which is a stochastic programming method contains fixing the certain appropriate levels for random constraints. Therefore, it is generally used for modelling technical or economic systems. The practices include economic planning, input control, structural design, inventory, air and water quality management problems. In chance constraints, each constraint can be realized with a certain probability.

Stochastic linear programming problem with chance constraints is defined as follows

$$\begin{aligned} &\max(\min) z(x) = \sum_{j=1}^{n} c_j x_j \\ &\mathrm{P}\left[\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right] \geq 1 - u_i \\ &\quad x_j \geq 0, \quad j = 1, \ldots, n \\ &\quad u_i \in (0, 1), \quad i = 1, \ldots, m \end{aligned} \tag{2.1}$$

where $c_j$, $a_{ij}$ and $b_i$ are random variables and $u_i$'s are chosen probabilities. $k$th chance constraint given in model (2.1) is obtained as

$$\mathrm{P}\left[\sum_{j=1}^{n} a_{kj} x_j \leq b_k\right] \geq 1 - u_k \tag{2.2}$$

with lower bound $(1 - u_k)$. Where it is assumed that $x_j$ decision variables are deterministic. $c_j$, $a_{kj}$ and $b_k$ are random variables with known variances and means [17,18].

If $b_k$ is the random variable in the model, and its distribution function is $F_b$ then the deterministic equivalent of chance constraint can be calculated as

$$\begin{aligned} P\left[a_{kj} x_j \leq b_k\right] \geq u_k \Leftrightarrow\ & P\left[b_k \geq a_{kj} x_j\right] \geq u_k \\ \Leftrightarrow\ & 1 - F_b\left(a_{kj} x_j\right) \geq u_k \\ \Leftrightarrow\ & a_{kj} x_j \leq F_b^{-1}(1 - u_k) \end{aligned} \tag{2.3}$$

Assume that $a_{kj}$ is a random variable having normal distribution with the mean $E(a_{kj})$ and the variance $Var(a_{kj})$. Furthermore, covariance between the random variables $a_{kj}$ and $a_{kl}$ is zero. Then, random variable $d_k$ is defined as follows

$$d_k = \sum_{j=1}^{n} a_{kj} x_j$$

where $a_{k1}, \ldots, a_{kn}$'s are random variables with normal distribution and $x_1, \ldots, x_n$'s are unknowns, chance constraint given with inequality (2.2) is defined as follows

$$\phi\left[\frac{b_k - E(d_k)}{\sqrt{Var(d_k)}}\right] \geq \phi\left(K_{u_k}\right) \tag{2.4}$$

where $K_{u_k}$ denotes the value of standard normal variable and $\phi\left(K_{u_k}\right) = 1 - u_k$. Therefore, deterministic equivalent of inequality (2.4) is stated as

$$E\left(d_k\right) + K_{u_k}\sqrt{Var\left(d_k\right)} \leq b_k$$

Solution methods for models constituted by dual and triple combinations of $c_j$, $a_{kj}$ and $b_k$ coefficients and also for the case that $c_j$'s are random variable are different. In this article, these are not mentioned [5,19-21].

## 3. Gamma distribution approach for CCSP

Let, $X_1$, $X_2$,..., $X_n$ be independent random variables with a distribution function $F_n(x)$. Let $\Phi(x)$ be a standard normal distribution function. Then, supremum of absolute distance between $F_n(x)$ and $\Phi(x)$ can be found. The theorem related to this, which is known as Essen Inequality, is as follows.

**Theorem 3.1** *Let $X_1$, $X_2$,..., $X_n$ be independent random variables with given*

$$EX_j = 0 \text{ and } E \mid X_j\mid^3 < \infty\, j = 1, \ldots, n$$

*where if it is as follows*

$$\sigma_j^2 = EX_j^2, \ldots, B_n = \sum_{j=1}^{n} \sigma_j^2, \ldots, F_n(x) = P\left[B_n^{-1/2} \sum_{j=1}^{n} X_j < x\right], \ldots, L_n = B_n^{-3/2} \sum_{j=1}^{n} E \mid X_j\mid^3$$

*then*

$$\sup_x \mid F_n(x) - \Phi(x) \mid \leq SL_n \tag{3.1}$$

*is defined. Here, S is an absolute positive constant* [22].

Proof to Theorem 3.1 can be found in [[22], pp. 109-111]. In case of equality, as a result of Essen inequality we can give the following equation, for large values of $n$

$$P\left[B_n^{-1/2}\left(\sum_{j=1}^{n} X_j - E\left(\sum_{j=1}^{n} X_j\right)\right) < x\right] = \phi\left(x\right) + \frac{\sum_{j=1}^{n} E\left(X_j - E\left(X_j\right)\right)^3 e^{\frac{-x^2}{2}}\left(1 - x^2\right)}{6\sqrt{2\pi}B_n^{\frac{3}{2}}} + o\left(n^{\frac{-1}{2}}\right) \tag{3.2}$$

Equation 3.2 is used for approximation to standard normal distribution [23].

After defining the Essen inequality given in Theorem 3.1, now we explain Gamma distribution approach for CCSP model. In linear programming, the constraints are constructed as follows:

$$Ax \leq b \Leftrightarrow \begin{bmatrix} a_{11}\, a_{12} \ldots a_{1n} \\ \vdots\; \vdots\; \ddots\; \vdots \\ a_{k1}\, a_{k2} \ldots a_{kn} \\ \vdots\; \vdots\; \ddots\; \vdots \\ a_{m1}\, a_{m2} \ldots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_m \end{bmatrix} \tag{3.3}$$

Here, the matrix $A$ indicates a coefficients matrix. Let $d_k = a_k'x$ $k = 1,..., m$ then $k$th row in (3.3) rewritten as

$$d_k \leq b_k \Leftrightarrow [a_{k1}, a_{k2}, \ldots, a_{kn}] \begin{bmatrix} x_1 \\ \cdot \\ x_k \\ \cdot \\ x_n \end{bmatrix} \leq b_k \tag{3.4}$$

If $a_{kj}$'s which are $k$th row of coefficients matrix $A$ are independent gamma random variables, chance constraints given in model (2.1) are as follows

$$P(d_k \leq b_k) \geq 1 - u_k, \quad k = 1, 2, \ldots, m \tag{3.5}$$

Assume that each random variable $a_{kj}$ has Gamma distribution with $(\alpha_{kj}, \beta_{kj})$ parameters in (3.4). For the purpose of using Essen inequality given in Theorem 3.1, the random variable $r_j = a_{kj}x_j - E(a_{kj}x_j)$, $j = 1,\ldots, n$ is taken into account. Expected value and variance of each random variable $a_{kj}$ as follows:

$$E(a_{kj}) = \alpha_{kj}\beta_{kj} \quad Var(a_{kj}) = \alpha_{kj}\beta_{kj}^2$$

Therefore, the expected value of random variable $r_j$ will be as follows:

$$E(r_j) = E(a_{kj}x_j - E(a_{kj}x_j)) = x_j\left[\alpha_{kj}\beta_{kj} - \alpha_{kj}\beta_{kj}\right] = 0$$

and its variance will be as follows:

$$Var(r_j) = E(d_j)^2 - \left[E(d_j)\right]^2 = x_j^2 Var(a_{kj}) = x_j^2 \alpha_{kj}\beta_{kj}^2$$

Absolute third moment of random variable $d_j$ is found in the following equality

$$E|\, r_j \,|^3 = E|\, a_{kj}x_j - E(a_{kj}x_j)\,|^3 = x_j^3 E|\, a_{kj} - \alpha_{kj}\beta_{kj}\,|^3 \tag{3.6}$$

The expected value in equality (3.6) can be written as follows:

$$\begin{aligned} E\left|a_{kj} - \alpha_{kj}\beta_{kj}\right|^3 &= \int_0^\infty \left|a_{kj} - \alpha_{kj}\beta_{kj}\right|^3 f(a_{kj})da_{kj} \\ &= \int_0^{\alpha_{kj}\beta_{kj}} \left|a_{kj} - \alpha_{kj}\beta_{kj}\right|^3 f(a_{kj})da_{kj} + \int_{\alpha_{kj}\beta_{kj}}^\infty \left|a_{kj} - \alpha_{kj}\beta_{kj}\right|^3 f(a_{kj})da_{kj} \end{aligned} = \mathrm{I}_{kj} + \mathrm{II}_{kj} \tag{3.7}$$

Then, $I_{kj}$ is rewritten as follows

$$\begin{aligned} I_{kj} &= \int_0^{\alpha_{kj}\beta_{kj}} \left[-\left(a_{kj} - \alpha_{kj}\beta_{kj}\right)\right]^3 f(a_{kj})da_{kj} \\ &= -\frac{1}{\Gamma(\alpha_{kj})\beta_{kj}^{\alpha_{kj}}} \int_0^{\alpha_{kj}\beta_{kj}} \left(a_{kj}^3 - 3a_{kj}^2\alpha_{kj}\beta_{kj} + 3a_{kj}\alpha_{kj}^2\beta_{kj}^2 - \alpha_{kj}^3\beta_{kj}^3\right) a_{kj}^{\alpha_{kj}-1} e^{-a_{kj}/\beta_{kj}} da_{kj} \end{aligned}$$

If it is taken as, $-\dfrac{1}{\Gamma(\alpha_{kj})\beta_{kj}^{\alpha_{kj}}} = \Delta$ in integral then $I_{kj}$ can be written as follows

$$\begin{aligned} I_{kj} &= \Delta \int_0^{\alpha_{kj}\beta_{kj}} a_{kj}^{\alpha_{kj}+2} e^{-a_{kj}/\beta_{kj}} da_{kj} - \Delta\left(3\alpha_{kj}\beta_{kj}\right) \int_0^{\alpha_{kj}\beta_{kj}} a_{kj}^{\alpha_{kj}+1} e^{-a_{kj}/\beta_{kj}} da_{kj} + \Delta\left(3\alpha_{kj}^2\beta_{kj}^2\right) \int_0^{\alpha_{kj}\beta_{kj}} a_{kj}^{\alpha_{kj}} e^{-a_{kj}/\beta_{kj}} da_{kj} \\ &\quad - \Delta\left(\alpha_{kj}^3\beta_{kj}^3\right) \int_0^{\alpha_{kj}\beta_{kj}} a_{kj}^{\alpha_{kj}-1} e^{-a_{kj}/\beta_{kj}} da_{kj} \\ &= \Delta\omega_1 + \Delta\left(3\alpha_{kj}\beta_{kj}\right)\omega_2 + \Delta\left(3\alpha_{kj}^2\beta_{kj}^2\right)\omega_3 + \Delta\left(\alpha_{kj}^3\beta_{kj}^3\right)\omega_4 \end{aligned}$$

Here, by making variable change $\dfrac{a_{kj}}{\beta_{kj}} = t_{kj}$,

$$\omega_1 = \beta_{kj}^{\alpha_{kj}+3} \int_0^{\alpha_{kj}} t_{kj}^{\alpha_{kj}+2} e^{-t_{kj}} dt_{kj}$$

is obtained. Incomplete gamma function is defined as follows

$$I(a,x) = \frac{\gamma(a,x)}{\Gamma(a)}$$

here

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt$$

Therefore, $\omega_1$ can be rearranged as follows:

$$\omega_1 = \beta_{kj}^{\alpha_{kj}+3} \Gamma(\alpha_{kj}+3) I(\alpha_{kj}+3, \alpha_{kj})$$

Similarly, it can be written as follows

$$\begin{aligned}
\omega_2 &= \beta_{kj}^{\alpha_{kj}+2} \Gamma(\alpha_{kj}+2) I(\alpha_{kj}+2, \alpha_{kj}) \\
\omega_3 &= \beta_{kj}^{\alpha_{kj}+1} \Gamma(\alpha_{kj}+1) I(\alpha_{kj}+1, \alpha_{kj}) \\
\omega_4 &= \beta_{kj}^{\alpha_{kj}} \Gamma(\alpha_{kj}) I(\alpha_{kj}, \alpha_{kj})
\end{aligned}$$

The second part of the integral can be written as follows

$$\begin{aligned}
\mathrm{II}_{kj} &= \int_{\alpha_{kj}\beta_{kj}}^{\infty} \left(a_{kj} - \alpha_{kj}\beta_{kj}\right)^3 f(a_{kj}) da_{kj} \\
&= \frac{1}{\Gamma(\alpha_{kj})\beta_{kj}^{\alpha_{kj}}} \int_{\alpha_{kj}\beta_{kj}}^{\infty} \left(a_{kj}^3 - 3a_{kj}^2\alpha_{kj}\beta_{kj} + 3a_{kj}\alpha_{kj}^2\beta_{kj}^2 - \alpha_{kj}^3\beta_{kj}^3\right) a_{kj}^{\alpha_{kj}-1} e^{-a_{kj}/\beta_{kj}} da_{kj}
\end{aligned}$$

If it is taken as $\dfrac{1}{\Gamma(\alpha_{kj})\beta_{kj}^{\alpha_{kj}}} = -\Delta$ in integral then $\mathrm{II}_{kj}$ can be written as follows

$$\begin{aligned}
\mathrm{II}_{kj} &= -\Delta \int_{\alpha_{kj}\beta_{kj}}^{\infty} a_{kj}^{\alpha_{kj}+2} e^{-a_{kj}/\beta_{kj}} da_{kj} + \Delta\left(3\alpha_{kj}\beta_{kj}\right) \int_{\alpha_{kj}\beta_{kj}}^{\infty} a_{kj}^{\alpha_{kj}+1} e^{-a_{kj}/\beta_{kj}} da_{kj} - \Delta\left(3\alpha_{kj}^2\beta_{kj}^2\right) \int_{\alpha_{kj}\beta_{kj}}^{\infty} a_{kj}^{\alpha_{kj}} e^{-a_{kj}/\beta_{kj}} da_{kj} \\
&+ \Delta\left(\alpha_{kj}^3\beta_{kj}^3\right) \int_{\alpha_{kj}\beta_{kj}}^{\infty} a_{kj}^{\alpha_{kj}-1} e^{-a_{kj}/\beta_{kj}} da_{kj} \\
&= -\Delta\xi_1 + \Delta\left(3\alpha_{kj}\beta_{kj}\right)\xi_2 - \Delta\left(3\alpha_{kj}^2\beta_{kj}^2\right)\xi_3 + \Delta\left(\alpha_{kj}^3\beta_{kj}^3\right)\xi_4
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \int_0^{\infty} a_{kj}^{\alpha_{kj}+2} e^{-a_{kj}/\beta_{kj}} da_{kj} - \int_0^{\alpha_{kj}\beta_{kj}} a_{kj}^{\alpha_{kj}+2} e^{-a_{kj}/\beta_{kj}} da_{kj} \\
&= \beta_{kj}^{\alpha_{kj}+3} \Gamma(\alpha_{kj}+3)\left[1 - I(\alpha_{kj}+3, \alpha_{kj})\right].
\end{aligned}$$

In the same way it will be

$$\xi_2 = \beta_{kj}^{\alpha_{kj}+2}\Gamma(\alpha_{kj}+2)\left[1 - I(\alpha_{kj}+2, \alpha_{kj})\right]$$
$$\xi_3 = \beta_{kj}^{\alpha_{kj}+1}\Gamma(\alpha_{kj}+1)\left[1 - I(\alpha_{kj}+1, \alpha_{kj})\right]$$
$$\xi_4 = \beta_{kj}^{\alpha_{kj}}\Gamma(\alpha_{kj})\left[1 - I(\alpha_{kj}, \alpha_{kj})\right].$$

Therefore, for any finite $\alpha_{kj}$ and $\beta_{kj}$, it can easily be seen that $Ed_j = 0$ and $E|d_j|^3 < \infty$. Therefore, the conditions in Theorem 3.1 are satisfied, then $\sigma_j^2$ and $B_n$ is obtained as

$$\sigma_j^2 = Er_j^2 = x_j^2 \alpha_{kj}\beta_{kj}^2$$
$$B_n = \sum_{j=1}^{n} \sigma_j^2 = \sum_{j=1}^{n} x_j^2 \alpha_{kj}\beta_{kj}^2$$

The third absolute moment of random variable, $r_j$, in terms of integrals $\mathrm{I}_{kj}$ and $\mathrm{II}_{kj}$ is written as follows

$$E|\ r_j\ |^3 = x_j^3\left(\mathrm{I}_{kj} + \mathrm{II}_{kj}\right)$$

Then, $L_n$ is obtained as follows

$$L_n = B_n^{-3/2}\sum_{j=1}^{n} E\left|r_j\right|^3 = \frac{\sum_{j=1}^{n} x_j^3\left(\mathrm{I}_{kj} + \mathrm{II}_{kj}\right)}{\left[\sum_{j=1}^{n} x_j^2\alpha_{kj}\beta_{kj}^2\right]^{3/2}} \tag{3.8}$$

Even if $L_n$ defined in Theorem 3.1 is maximum it can be a useful upper bound for left side of (3.1). Following lemma is related to this situation.

**Lemma 3.1** *Maximum value of* $L_n$ *in Equation 3.8 is given by*

$$\max L_n = \frac{nL^*}{\left(nx^*\alpha^*(\beta^*)^2\right)^{3/2}} = \frac{nL^*}{n^{3/2}(x^*\alpha^*)^{3/2}(\beta^*)^3} \tag{3.9}$$

*Proof* Maximum value of $L_n$ given in Equation 3.8 is obtained by maximizing nominator while minimizing the denominator, i.e.

$$\max_j \sum_{j=1}^{n} x_j^3\left(\mathrm{I}_{kj} + \mathrm{II}_{kj}\right)$$

and

$$\min_j \left[\sum_{j=1}^{n} x_j^2\alpha_{kj}\beta_{kj}^2\right]^{3/2}$$

Therefore,

$$\max_j \left|x_j^3\left(\mathrm{I}_{kj} + \mathrm{II}_{kj}\right)\right| = L^*$$

and

$$\min_j |\ x_j^2\alpha_{kj}\beta_{kj}^2\ | = x^*\alpha^*(\beta^*)^2$$

equalities are defined. Then maximum value of $L_n$ given in Equation 3.8 is found as Equation 3.9. This completes the proof of Lemma 3.1.

In Theorem 3.1, using $L_n$ given in (3.8), following inequality is obtained

$$\sup_x |F_n(x) - \Phi(x)| \leq SL_n$$

$$\sup_x |F_n(x) - \Phi(x)| \leq S \frac{\sum_{j=1}^{n} x_j^3 \left(I_{kj} + II_{kj}\right)}{\left[\sum_{j=1}^{n} x_j^2 \alpha_{kj} \beta_{kj}^2\right]^{3/2}} \tag{3.10}$$

If the suggested constant $S = 0.7975$ [22] in inequality (3.10) and if the value max $\mathrm{L}_n$ given with (3.9) is used following inequality is obtained

$$\sup_x |F_n(x) - \Phi(x)| \leq 0.7975 \frac{L^*}{\sqrt{n}(x^*\alpha^*)^{3/2}(\beta^*)^3} \tag{3.11}$$

Here, $F_n(x)$ is Gamma distribution function, $\Phi(x)$ is that of standard normal distribution. Thus, for $d_k$

$$\frac{d_k - \sum_{j=1}^{n} x_j E(a_{kj})}{\sqrt{\sum_{j=1}^{n} x_j^2 Var(a_{kj})}} = \frac{d_k - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}}$$

is defined. Therefore, constraint (3.5) can be written as follows

$$P\left[\frac{\sum_{j=1}^{n} a_{kj} x_j - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}} \leq \frac{b_k - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}}\right] \geq 1 - (u_k + SL_n)$$

Here, the following inequality is written

$$\Phi\left[\frac{b_k - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}}\right] \geq 1 - (u_k + SL_n). \tag{3.12}$$

There are decision variables $x_j$ ($j = 1,..., n$) in $L_n$ which is on the left side of the inequality (3.12). Since these decision variables are the results of the problem solved after model (2.1) is made deterministic, they are unknown here. Therefore, $L_n$ is not a numeric and it cannot be solved using $\Phi^{-1}(1\text{-}(u_k)\text{+}SL_n)$. Therefore, using the approach suggested [24] right side of inequality (3.12) can be written as follows

$$\Phi\left[\frac{b_k - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}}\right] = \frac{1}{2}\left(1 + \left\{1 - \exp\left(-\frac{2}{\pi}\left[\frac{b_k - \sum_{j=1}^{n} x_j \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} x_j^2 \alpha_j \beta_j^2}}\right]^2\right)\right\}^{1/2}\right) \tag{3.13}$$

and deterministic constraint belonging to inequality (3.12) is then written as fallows

$$\frac{1}{2}\left(1+\left\{1-\exp\left(-\frac{2}{\pi}\left[\frac{b_k-\sum_{j=1}^{n}x_j\alpha_j\beta_j}{\sqrt{\sum_{j=1}^{n}x_j^2\alpha_j\beta_j^2}}\right]^2\right)\right\}^{1/2}\right)\geq 1-\left[u_k+0.7975\left(\frac{\sum_{j=1}^{n}x_j^3\left(\mathrm{I}_{kj}+\mathrm{II}_{kj}\right)}{\left[\sum_{j=1}^{n}x_j^2\alpha_{kj}\beta_{kj}^2\right]^{3/2}}\right)\right]. \quad (3.14)$$

Using Equation (3.2) we can construct the following inequality

$$\frac{1}{2}\left(1+\left\{1-\exp\left(-\frac{2}{\pi}\left[\frac{b_k-\sum_{j=1}^{n}x_j\alpha_{kj}\beta_{kj}}{\sqrt{\sum_{j=1}^{n}x_j^2\alpha_{kj}\beta_{kj}^2}}\right]^2\right)\right\}^{1/2}\right)$$

$$\geq 1-\left[u_k+\frac{\sum_{j=1}^{n}x_j^3 2\alpha_{kj}\beta_{kj}^3 e^{\frac{-\left(\frac{b_k-\sum_{j=1}^{n}x_j\alpha_{kj}\beta_{kj}}{\sqrt{\sum_{j=1}^{n}x_j^2\alpha_{kj}\beta_{kj}^2}}\right)^2}{2}}\left(1-\left(\frac{b_k-\sum_{j=1}^{n}x_j\alpha_{kj}\beta_{kj}}{\sqrt{\sum_{j=1}^{n}x_j^2\alpha_{kj}\beta_{kj}^2}}\right)^2\right)}{6\sqrt{2\pi}\left(\sum_{j=1}^{n}x_j^2\alpha_{kj}\beta_{kj}^2\right)^{\frac{3}{2}}}\right]$$

(3.15)

## 4. Numerical experiments

Consider the CCSP model as follows

$$\begin{aligned} \max z &= 7x_1+2x_2+4x_3 \\ P\left[a_{11}x_1+a_{12}x_2+a_{13}x_3\leq 8\right] &\geq 0.95 \\ P\left[5x_1+x_2+6x_3\leq b_2\right] &\geq 0.10 \\ x_j\geq 0 \quad j &= 1,2,3 \end{aligned} \quad (4.1)$$

Here, assume that $a_{kj}$ $j$ = 1,2,3 are independent random variables distributed as Gamma distribution with the following parameters ($\alpha_{kj}$, $\beta_{kj}$)

$$\alpha_{11}=4,\ \ \beta_{11}=1,\ \ \alpha_{12}=2,\ \ \beta_{12}=2,\ \alpha_{13}=3,\ \ \beta_{13}=2. \quad (4.2)$$

$b_2$ is normal random variable with the following expected value and variance

$$E(b_2)=7,\ \ Var(b_2)=9$$

In the solving stage of the problem, for using of Essen inequality given in Theorem 3.1 can be defined as

$$r_j = a_{kj}x_j - E\left(a_{kj}x_j\right)$$

here,

$$Var\left(r_j\right) = x_j^2\left(\alpha_{kj}\beta_{kj}^2\right)$$

is found and for $k = 1$, $B_n$ is obtained as follows

$$B_n = 4x_1^2 + 8x_2^2 + 12x_3^2$$

Then, $L_n$ is written as follows

$$L_n = \frac{\sum_{j=1}^{3} x_j^3\left(\mathrm{I}_{kj} + \mathrm{II}_{kj}\right)}{\left(4x_1^2 + 8x_2^2 + 12x_3^2\right)^{3/2}}$$

As a result of the solution of the integrals, $\mathrm{I}_{kj}$ ($k = 1$, $j = 1,2,3$) and $\mathrm{II}_{kj}$ ($k = 1$, $j = 1,2,3$) in $L_n$ can be obtained as

$$\mathrm{I}_{11} = 3.2824,\ \ \mathrm{II}_{11} = 11.2824$$
$$\mathrm{I}_{12} = 6.9766,\ \ \mathrm{II}_{12} = 38.9766$$
$$\mathrm{I}_{13} = 15.3291,\ \ \mathrm{II}_{13} = 63.3291$$

Then, $L_n$ is found as

$$L_n = \frac{14.5648x_1^3 + 45.9532x_2^3 + 78.6582x_3^3}{\left(4x_1^2 + 8x_2^2 + 12x_3^2\right)^{3/2}}.$$

Therefore, in the case where $a_{kj}$ is a random variable with Gamma distribution, deterministic equality of the first chance constraint in model (4.1), using inequality (3.14) is obtained as follows

$$\frac{1}{2}\left[1 + \left\{1 - \exp\left(-\frac{2}{\pi}\left[\frac{8 - (4x_1 + 4x_2 + 6x_3)}{\sqrt{4x_1^2 + 8x_2^2 + 12x_3^2}}\right]^2\right)\right\}^{1/2}\right] \geq 1 - \left[0.05 + 0.7975\left(\frac{14.5648x_1^3 + 45.9532x_2^3 + 78.6582x_3^3}{\left(4x_1^2 + 8x_2^2 + 12x_3^2\right)^{3/2}}\right)\right] \tag{4.3}$$

Using inequality (3.15) we can write as:

$$\begin{aligned} &0.5\left(1 + \left\{1 - \exp\left(-0.6366\frac{(8 - x_4)^2}{x_5}\right)\right\}^{1/2}\right) \geq 0.95 - \left[\frac{(8x_1^3 + 32x_2^3 + 48x_3^3)e^{\frac{-x_6}{2}}(1 - x_6)}{6\sqrt{(6.28)}x_5^{\frac{3}{2}}}\right] \\ &x_4 - 4x_1 - 4x_2 - 6x_3 = 0 \\ &x_5 - (4x_1^2 + 8x_2^2 + 12x_3^2) = 0 \\ &x_6x_5 - (8 - x_4)^2 = 0 \end{aligned} \tag{4.4}$$

Using inequality (2.3) for the second chance constraint, deterministic inequality is obtained as

$$5x_1 + x_2 + 6x_3 \leq 10.855$$

Then, deterministic equality of CCSP model given in (4.1), using inequality (4.3), can be found as follows

$$\max z = 7x_1 + 2x_2 + 4x_3$$

$$\begin{aligned}&\frac{1}{2}\left[1+\left\{1-\exp\left(-\frac{2}{\pi}\left[\frac{8-(4x_1+4x_2+6x_3)}{\sqrt{4x_1^2+8x_2^2+12x_3^2}}\right]^2\right)\right\}^{1/2}\right]\geq 1-\left[0.05+0.7975\left(\frac{14.5648x_1^3+45.9532x_2^3+78.6582x_3^3}{\left(4x_1^2+8x_2^2+12x_3^2\right)^{3/2}}\right)\right]\\&0.7975\left(\frac{14.5648x_1^3+45.9532x_2^3+78.6582x_3^3}{\left(4x_1^2+8x_2^2+12x_3^2\right)^{3/2}}\right)\leq 0.95\end{aligned}\tag{4.5}$$

$$5x_1+x_2+6x_3\leq 10.855$$

$$x_j\geq 0\quad j=1,2,3$$

The second constraint is given for controlling of non-negativity on the right side of first constraint. The nonlinear problem given in (4.5) has been solved with condition

$$0\leq x_1,x_2,x_3\leq 2$$

using software *Lingo 9.0* and the results are shown in Table 1.

Deterministic equality of CCSP model given in (4.1), using inequality (4.4), can be found as follows

$$\max z=7x_1+2x_2+4x_3$$

$$\begin{aligned}&0.5\left(1+\left\{1-\exp\left(-0.6366\frac{(8-x_4)^2}{x_5}\right)\right\}^{1/2}\right)\geq 0.95-\left[\frac{(8x_1^3+32x_2^3+48x_3^3)e^{\frac{-x_6}{2}}(1-x_6)}{6\sqrt{(6.28)}x_5^{\frac{3}{2}}}\right]\\&x_4-4x_1-4x_2-6x_3=0\\&x_5-(4x_1^2+8x_2^2+12x_3^2)=0\\&x_6x_5-(8-x_4)^2=0\end{aligned}\tag{4.6}$$

$$5x_1+x_2+6x_3\leq 10.855$$

$$x_j\geq 0\quad j=1,2,3,4,5,6$$

As a second case, let us assume that $a_{kj}$ coefficients in the first chance constraint in model (4.1) are independent normal random variables with the following expected value $E(a_{kj})$ and variance $Var(a_{kj})$

$$\begin{aligned}&E(a_{11})=4,\quad Var(a_{11})=4\\&E(a_{12})=4,\quad Var(a_{12})=8\\&E(a_{13})=6,\quad Var(a_{13})=12\end{aligned}\tag{4.7}$$

Then, deterministic equality of chance constraint can be arranged as follows

$$4x_1+4x_2+6x_3+1.645\sqrt{4x_1^2+8x_2^2+12x_3^2}\leq 8\tag{4.8}$$

**Table 1 Solutions results of models (4.5), (4.6), (4.9)**

| Model (4.5) | Model (4.6) | Model (4.9) |
|---|---|---|
| $x_1=1.466249$ | $x_1=1.010669$ | $x_1=1.097394$ |
| $x_2=0.9250464$ | $x_2=0.000000$ | $x_2=0.000000$ |
| $x_3=0.4331181$ | $x_3=0.000000$ | $x_3=0.000000$ |
| $\max z=13.84631$ | $\max z=7.074686$ | $\max z=7.681756$ |

Therefore, deterministic equality of CCSP model given in (4.1) can be found as follows:

$$\begin{aligned}
&\max z = 7x_1 + 2x_2 + 4x_3 \\
&4x_1 + 4x_2 + 6x_3 + 1.645\sqrt{4x_1^2 + 8x_2^2 + 12x_3^2} \le 8 \\
&5x_1 + x_2 + 6x_3 \le 10.855 \\
&x_j \ge 0 \quad j = 1, 2, 3, 4
\end{aligned} \tag{4.9}$$

Model (4.9) has been solved by software *Lingo 9.0* and the results are listed in Table 1.

## 5. Conclusion

In this study, a new method is suggested for the solution of the deterministic equivalence of the CCSP. The main purpose of this article is to transform the chance-constrained model into a deterministic model based on the Essen inequality. According to the Essen inequality, the estimation of the distance between the distribution of a sum of independent random variables and the normal distribution is less than or equal to $SL_n$. This study considers a stochastic optimization model with random technology matrix in which the random variables are independent and follow a Gamma distribution. Deterministic equality of these kinds of problems has been obtained via the suggested method. Furthermore, by adding a second constraint having normal distribution in the right-hand side value, a problem with two chance constraints has been obtained. In this problem, both cases that $a_{kj}$ coefficients have gamma and normal distributions have been examined and for the solution of deterministic models *Lingo 9.0* has been used.

As a result, the upper bounds of the chance constrained are derived by the Essen inequality and developed approximate deterministic equivalent of the model.

The solutions obtained by including the supremum distance defined by the Essen inequality in the model are shown clearly in the solutions results (4.5) and (4.6) in Table 1.

For large values of $n$, the solution results of the models having Gamma and normal distributions are closed to each other. This can be observed in Table 1 by examining the solution results (4.6) and (4.9). Here, it can be seen that coefficients of the objective function and decision variables are very similar.

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**Authors' contributions**
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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