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Some Orlicz norms inequalities for the composite operator $T \circ d \circ H$

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Abstract

In this article, we first establish the local inequality for the composite operator $T \circ d \circ H$ with Orlicz norms. Then, we extend the local result to the global case in the $L^{\phi}(\mu)$-averaging domains.

Keywords: composite operator, Orlicz norms, $L^{\phi?\varphi?}(?\mu?)$-averaging domains

1 Introduction

Recently as generalizations of the functions, differential forms have been widely used in many fields, such as potential theory, partial differential equations, quasiconformal mappings, and nonlinear analysis; see [1-4]. With the development of the theory of quasiconformal mappings and other relevant theories, a series of results about the solutions to different versions of the A-harmonic equation have been found; see [5-9]. Especially, the research on the inequalities of the various operators and their compositions applied to the solutions to different sorts of the A-harmonic equation has made great progress [5]. The inequalities equipped with the L^p-norm for differential forms have been very well studied. However, the inequalities with Orlicz norms have not been fully developed [9,10]. Also, both L^p-norms and Orlicz norms of differential forms depend on the type of the integral domains. Since Staples introduced the L^s-averaging domains in 1989, several kinds of domains have been developed successively, including $L^s(\mu)$-averaging domains, see [11-13]. In 2004, Ding [14] put forward the concept of the $L^{\phi}(\mu)$-averaging domains, which is considered as an extension of the other domains involved above and specified later.

The homotopy operator T, the exterior derivative operator d, and the projection operator H are three important operators in differential forms; for the first two operators play critical roles in the general decomposition of differential forms [15] while the latter in the Hodge decomposition [16]. This article contributes primarily to the Orlicz norm inequalities for the composite operator $T \circ d \circ H$ applied to the solutions of the nonhomogeneous A-harmonic equation.

In this article, we first introduce some essential notation and definitions. Unless otherwise indicated, we always use Θ to denote a bounded convex domain in $\mathbb{R}^n (n \geq 2)$, and let O be a ball in $\mathbb{R}^n$. Let ρO denote the ball with the same center as O and $diam(\rho O) = \rho diam(O)$, $\rho > 0$. We say v is a weight if $v \in L^1_{loc}(\mathbb{R}^n)$ and $v > 0$ a.e; see [17]. $|D|$ is used to denote the Lebesgue measure of a set $D \subset \mathbb{R}^n$, and the measure μ

is defined by $d\mu = v(x)dx$. We use $||f||_{s,O}$ for $(\int_O |f|^s dx)^{\frac{1}{s}}$ and $||f||_{s,O,v}$ for $(\int_O |f|^s v(x)dx)^{\frac{1}{s}}$.

Let [5,15]$\Lambda^\ell = \Lambda^\ell(\mathbb{R}^n)$, $\ell = 0, 1,\ldots, n$, be the linear space of all ℓ-forms $\hbar(x) = \sum_J \hbar_J(x)dx_J = \sum_J \hbar_{j_1 j_2\cdots j_\ell}(x)dx_{j_1}\wedge dx_{j_2}\cdots\wedge dx_{j_\ell}$ in $\mathbb{R}^n$, where $J = (j_1, j_2,\ldots, j_\ell)$, $1 \le j_1 < j_2 < \ldots < j_\ell \le n$, $\ell = 0, 1,\ldots, n$, are the ordered ℓ-tuples. The Grassman algebra Λ^ℓ is a graded algebra with respect to the exterior products. For $\alpha = \Sigma_J\alpha_J dx_J \in \Lambda^\ell(\mathbb{R}^n)$ and $\beta = \Sigma_J\beta_J dx_J \in \Lambda^\ell(\mathbb{R}^n)$, the inner product in $\Lambda^\ell(\mathbb{R}^n)$ is given by $\langle\alpha, \beta\rangle = \Sigma_J\alpha_J\beta_J$ with summation over all ℓ-tuples $J = (j_1, j_2, \ldots, j_\ell)$, $\ell = 0, 1,\ldots, n$. Let $C^\infty(\Theta, \Lambda^\ell)$ be the set of infinitely differentiable ℓ-forms on $\Theta \subset \mathbb{R}^n$, $D'(\Theta, \Lambda^\ell)$ the space of all differential ℓ-forms in Θ and $L^s(\Theta, \Lambda^\ell)$ the set of the ℓ-forms in Θ satisfying $\int_\Theta (\Sigma_J|\omega_J(x)|^2)^{\frac{s}{2}}dx < \infty$ for all ordered ℓ-tuples J. The exterior derivative d: $D'(\Theta, \Lambda^\ell) \to D'(\Theta, \Lambda^{\ell+1})$, $\ell = 0, 1,\ldots, n-1$, is given by

$$d\hbar(x) = \sum_{i=1}^{n}\sum_{J}\frac{\partial\omega_{j_1 j_2\cdots j_\ell}(x)}{\partial x_i}dx_i\wedge dx_{j_1}\wedge dx_{j_2}\cdots\wedge dx_{j_\ell} \tag{1.1}$$

for all $\hbar \in D'(\Theta, \Lambda^\ell)$, and the Hodge codifferential operator $d^\star$ is defined as $d^\star = (-1)^{n\ell+1} \star d\star : D'(\Theta, \Lambda^{\ell+1}) \to D'(\Theta, \Lambda^\ell)$, where $\star$ is the Hodge star operator.

With respect to the nonhomogeneous A-harmonic equation for differential forms, we indicate its general form as follows:

$$d^*A(x, d\hbar) = B(x, d\hbar), \tag{1.2}$$

where A: $\Theta \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^\ell(\mathbb{R}^n)$ and B: $\Theta \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^{\ell-1}(\mathbb{R}^n)$ satisfy the conditions: $|A(x, \eta)| \le a|\eta|^{s-1}$, $A(x, \eta)\cdot\eta \ge |\eta|^s$, and $|B(x, \eta)| \le b|\eta|^{s-1}$ for almost every $x \in \Theta$ and all $\eta \in \Lambda^\ell(\mathbb{R}^n)$. Here $a, b > 0$ are some constants, and $1 < s < \infty$ is a fixed exponent associated with (1.2). A solution to (1.2) is an element of the Sobolev space $W^{1,s}_{loc}(\Theta, \Lambda^{\ell-1})$ such that

$$\int_\Theta A(x, d\hbar)\cdot d\psi + B(x, d\hbar)\cdot\psi = 0 \tag{1.3}$$

for all $\psi \in W^{1,s}_{loc}(\Theta, \Lambda^{\ell-1})$ with compact support, where $W^{1,s}_{loc}(\Theta, \Lambda^{\ell-1})$ is the space of ℓ-forms whose coefficients are in the Sobolev space $W^{1,s}_{loc}(\Theta)$.

If the operator $B = 0$, (1.2) becomes

$$d^*A(x, d\hbar) = 0, \tag{1.4}$$

which is called the (homogeneous) A-harmonic equation.

In [15], Iwaniec and Lutoborski gave the linear operator $K_y : C^\infty(\Theta, \Lambda^\ell) \to C^\infty(\Theta, \Lambda^{\ell-1})$ as $(K_y\hbar)(x;\theta_1,\ldots,\theta_{\ell-1}) = \int_0^1 t^{\ell-1}\hbar(tx + y - ty; x - y, \theta_1,\ldots,\theta_{\ell-1})dt$ for each $y \in \Theta$. Then, the homotopy operator T: $C^\infty(\Theta, \Lambda^\ell) \to C^\infty(\Theta, \Lambda^{\ell-1})$ is denoted by

$$T\hbar = \int_\Theta \upsilon(y)K_y\hbar dy, \tag{1.5}$$

where $\upsilon \in C_0^\infty(\Theta)$ is normalized so that $\int_\Theta \upsilon(y)dy = 1$. The ℓ-form $\hbar_\Theta \in D'(\Theta, \Lambda^\ell)$ is given by $\hbar_\Theta = |\Theta|^{-1}\int_\Theta \hbar(y)dy (\ell = 0)$, $\hbar_\Theta = d(T\hbar)(\ell = 1,\ldots, n)$. In addition, we have the decomposition $\hbar = d(T\hbar) + T(d\hbar)$ for each $\hbar \in L^s(\Theta, \Lambda^\ell)$, $1 \le s < \infty$.

The definition of the H operator appeared in [16]. Let $L^1_{loc}(\Theta, \Lambda^\ell)$ be the space of ℓ-forms whose coefficients are locally integrable, and $\mathcal{W}(\Theta, \Lambda^\ell)$ the space of all $\Theta \in L^1_{loc}(\Theta, \Lambda^\ell)$ that has generalized gradient. We define the harmonic ℓ-fields by $\mathcal{H}(\Theta, \Lambda^\ell) = \{\Theta \in \mathcal{W}(\Theta, \Lambda^\ell) : d\hbar = d^\star\hbar = 0, \hbar \in L^s(\Theta, \Lambda^\ell) \text{ for some } 1 < s < \infty\}$ and the orthogonal complement of $\mathcal{H}(\Theta, \Lambda^\ell)$ in $L^1(\Theta, \Lambda^\ell)$ as $\mathcal{H}^\perp = \{\omega \in L^1(\Omega, \Lambda^\ell) :< \omega, h >= 0 \; for\; all\; h \in \mathcal{H}(\Theta, \Lambda^\ell)\}$. Then, the H operator is defined by

$$H(\hbar) = \hbar - \Delta G(\hbar), \tag{1.6}$$

where $\hbar$ is in $C^\infty(\Theta, \Lambda^\ell)$, $\Delta = dd^\star + d^\star d$ is the Laplace-Beltrami operator, and $G : C^\infty(\Theta, \Lambda^\ell) \to \mathcal{H}^\perp \cap C^\infty(\Theta, \Lambda^\ell)$ is the Green operator.

2 Main results

In this section, we first present some definitions of elementary conceptions, including Orlicz norms, the Young function, and the $A(\alpha, \beta, \gamma; \Theta)$-weight, then propose the local estimate for the composite operator of $T \circ d \circ H$ with the Orlicz norm, and at last extend it to the global version in the $L^\phi(\mu)$-averaging domains. The proof of all the theorems in this section will be left in next section.

The Orlicz norm or Luxemburg norm differs from the traditional L^p-norm, whose definition is given as follows [18].

Definition 2.1. *We call a continuously increasing function* φ: $[0, \infty) \to [0, \infty)$ *with* $\varphi(0) = 0$ *and* $\varphi(\infty) = \infty$ *an Orlicz function, and a convex Orlicz function often denotes a Young function. Suppose that* ϕ *is a Young function,* Θ *is a domain with* $\mu(\Theta) < \infty$, *and* f *is a measurable function in* Θ, *then the Orlicz norm of* f *is denoted by*

$$\| f\|_{\varphi(\Theta,\mu)} = \inf\left\{\chi > 0 : \frac{1}{\mu(\Theta)}\int_\Theta \varphi\left(\frac{|f|}{\chi}\right)d\mu \le 1\right\}. \tag{2.1}$$

The following class $G(p, q, C)$ is introduced in [19], which is a special property of a Young function.

Definition 2.2. *Let* f *and* g *be correspondingly a convex increasing function and a concave increasing function on* $[0, \infty)$. *Then, we call a Young function* ϕ *belongs to the class* $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, *if*

$$\text{(i)}\quad \frac{1}{C} \le \frac{\varphi(t^{\frac{1}{p}})}{f(t)} \le C, \qquad \text{(ii)}\quad \frac{1}{C} \le \frac{\varphi(t^{\frac{1}{q}})}{g(t)} \le C \tag{2.2}$$

for all $t > 0$.

Remark. From [19], we assert that ϕ, f, g in above definition are doubling, namely, $\phi(2t) \le C_1\phi(t)$ for all $t > 0$, and the completely similar property remains valid if ϕ is replaced correspondingly with f, g. Besides, we have

$$\text{(i)}\quad C_2t^q \le g^{-1}(\varphi(t)) \le C_3t^q, \qquad \text{(ii)}\quad C_2t^p \le f^{-1}(\varphi(t)) \le C_3t^p, \tag{2.3}$$

where C_1, C_2, and C_3 are some positive constants.

The following weight class appeared in [9].

Definition 2.3. *Let* $v(x)$ *is a measurable function defined on a subset* $\Theta \subset \mathbb{R}^n$. *Then, we call* $v(x)$ *satisfies the* $A(\alpha, \beta, \gamma; \Theta)$*-condition for some positive constants* α, β, γ, *if*

$v(x) > 0$ *a.e. and*

$$\sup_O \left(\frac{1}{|O|} \int_O v^\alpha dx \right) \left(\frac{1}{|O|} \int_O \left(\frac{1}{v} \right)^\beta dx \right)^{\frac{\gamma}{\beta}} < \infty, \tag{2.4}$$

where the supremum is over all balls O with $O \subset \Theta$. *We write* $v(x) \in A(\alpha, \beta, \gamma; \Theta)$.

Remark. Note that the $A(\alpha, \beta, \gamma; \Theta)$-class is an extension of some existing classes of weights, such as $A_r^\Lambda(\Theta)$-weights, $A_r(\lambda, \Theta)$-weights, and $A_r(\Theta)$-weights. Taking the $A_r^\Lambda(\Theta)$-weights for example, if $\alpha = 1, \beta = \frac{1}{r-1}$, and $\gamma = \lambda$ in the above definition, then the $A(\alpha, \beta, \gamma; \Theta)$-class reduces to the desired weights; see [9] for more details about these weights.

The main objective of this section is Theorem 2.4.

Theorem 2.4. *Let* $v \in C^\infty(\Theta, \Lambda^\ell)$, $\ell = 1, 2, \ldots, n$, *be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded convex domain* Θ, $T : C^\infty(\Theta, \Lambda^\ell) \to C^\infty(\Theta, \Lambda^{\ell-1})$ *be the homotopy operator defined in* (1.5), *d be the exterior derivative defined in* (1.1), *and H be the projection operator defined in* (1.6). *Suppose that* ϕ *is a Young function in the class* $G(p, q, C_0)$, $1 \le p < q < \infty$, $C_0 \ge 1$, $\varphi(|v|) \in L^1_{loc}(\Theta; \mu)$, *and* $d\mu = v(x)dx$, *where* $v(x) \in A(\alpha, \beta, \alpha, \Theta)$ *for* $\alpha > 1$ *and* $\beta > 0$ *with* $v(x) \ge \varepsilon > 0$ *for any* $x \in \Theta$. *Then, there exists a constant C, independent of v, such that*

$$\| T(d(H(v))) - (T(d(H(v))))_O \|_{\varphi(O,\mu)} \le C \| v \|_{\varphi(\rho O,\mu)} \tag{2.5}$$

for all balls O with $\rho O \subset \Theta$, *where* $\rho > 1$ is a constant.

The proof of Theorem 2.4 depends upon the following two arguments, that is, Lemma 2.5 and Theorem 2.6.

In [9], Xing and Ding proved the following lemma, which is a weighted version of weak reverse inequality.

Lemma 2.5. *Let v be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a domain* Θ *and* $0 < s, t < \infty$. *Then, there exists a constant C, independent of v, such that*

$$\left(\int_O |v|^s d\mu \right)^{\frac{1}{s}} \le C(\mu(O))^{\frac{t-s}{st}} \left(\int_{\rho O} |v|^t d\mu \right)^{\frac{1}{t}} \tag{2.6}$$

for all balls O with $\rho O \subset \Theta$ for some $\rho > 1$, *where the measure* μ *is defined as the preceding theorem.*

Remark. We call attention to the fact that Lemma 2.5 contains a $A(\alpha, \beta, \alpha; \Theta)$-weight, which makes the inequality be more flexible and more useful. For example, if let $d\mu = dx$ in Lemma 2.5, then it reduces to the common weak reverse inequality:

$$\| v \|_{s,O} \le C|O|^{\frac{t-s}{st}} \| v \|_{t,\rho O}. \tag{2.7}$$

For the composite operator $T \circ d \circ H$, we have the following inequality with $A(\alpha, \beta, \alpha; \Theta)$-weight.

Theorem 2.6. *Let us assume, in addition to the definitions of the homotopy operator T, the exterior derivative d, the projection operator H, and the measure* μ *in Theorem* 2.4, *that q is any integer satisfying* $1 < q < \infty$, $v \in C^\infty(\Theta, \Lambda^\ell)$, $\ell = 1, 2, \ldots, n$, *be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded convex domain* Θ *and*

$\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q d\mu\right)^{\frac{1}{q}} \leq Cdiam(O)|O|\left(\int_{\rho O} |v|^q d\mu\right)^{\frac{1}{q}}$. *Then, there exists a constant C, independent of v, such that*

$$\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q d\mu\right)^{\frac{1}{q}} \leq Cdiam(O)|O|\left(\int_{\rho O} |v|^q d\mu\right)^{\frac{1}{q}} \tag{2.8}$$

for all balls O with $\rho O \subset \Theta$ *for some* $\rho > 1$.

For the purpose of Theorem 2.6, we will need the following Lemmas 2.7 (the general Hölder inequality) and 2.8 that were proved in [5].

Lemma 2.7. *Let f and g are two measurable functions on* $\mathbb{R}^n$, α, β, γ *are any three positive constants with* $\gamma^{-1} = \alpha^{-1} + \beta^{-1}$. *Then, there exists the inequality such that*

$$\| fg\|_{\gamma,\Theta} \leq \| f\|_{\alpha,\Theta} \| g\|_{\beta,\Theta} \tag{2.9}$$

for any $\Theta \subset \mathbb{R}^n$.

Lemma 2.8. *Let us assume, in addition to the definitions of the homotopy operator T, the exterior derivative d, and the projection operator H in Theorem* 2.4, *that* $v \in C^\infty(\Theta, \Lambda^\ell)$, $\ell = 1, 2,..., n$, *be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded convex domain* Θ *and* $|v| \in L^s_{loc}(\Theta)$. *Then, there exists a constant C, independent of v, such that*

$$\| T(d(H(v))) - (T(d(H(v))))_O\|_{s,O} \leq C \mid O \mid diam(O) \| v\|_{s,\rho O} \tag{2.10}$$

for all balls O with $\rho O \subset \Theta$, where $\rho > 1$ is a constant.

Remark. Note that in Theorem 2.4, ϕ may be any Young function, provided it lies in the class $G(p, q, C_0)$, $1 \leq p < q < \infty$, $C_0 \geq 1$. From [19], we know that the function $\varphi(t) = t^p \log_+^\alpha t$ belongs to $G(p_1, p_2, C)$, $1 \leq p_1 < p < p_2$, $t > 0$, and $\alpha \in \mathbb{R}$. Here $\log_+ t$ is a cutoff function such that $\log_+ t = 1$ for $t \leq e$ otherwise $\log_+ t = \log t$. Moreover, if $\alpha = 0$, one verifies easily that $\phi(t) = t^p$ is as well in the class $G(p_1, p_2, C)$, $1 \leq p_1 < p_2 < \infty$. Therefore, fixing the function $\varphi(t) = t^p \log_+^\alpha t$, $\alpha \in \mathbb{R}$ in Theorem 2.4, we get the following result.

Corollary 2.9. *Let us assume, in addition to the definitions of the homotopy operator T, the exterior derivative d, the projection operator H, and the measure μ in Theorem* 2.4, *that* $\varphi(t) = t^p \log_+^\alpha t$, $p > 1$, $t > 0$, $\alpha \in \mathbb{R}$, $v \in C^\infty(\Theta, \Lambda^\ell)$, $\ell = 1, 2,..., n$, *be a solution of the nonhomogeneous A-harmonic equation* (1.2) *in a bounded convex domain* Θ *and* $\varphi(|v|) \in L^1_{loc}(\Theta; \mu)$. *Then, there exists a constant C, independent of v, such that*

$$\begin{aligned} &\int_O |T(d(H(v))) - (T(d(H(v))))_O|^p \log_+^\alpha \left(|T(d(H(v))) - (T(d(H(v))))_O|\right) d\mu \\ &\leq C \int_{\rho O} |v|^p \log_+^\alpha |v| d\mu \end{aligned} \tag{2.11}$$

for all balls O with $\rho O \subset \Theta$ *for some* $\rho > 1$. The following definition of the $L^\phi(\mu)$-averaging domains can be found in [5,14].

Definition 2.10. *Let* ϕ *be a Young function on* $[0, +\infty)$ *with* $\phi(0) = 0$. *We call a proper subdomain* $\Theta \subset \mathbb{R}^n$ *an* $L^\phi(\mu)$*-averaging domains, if* $\mu(\Theta) < \infty$ *and there exists a constant C such that*

$$\int_{\Theta}\varphi(\tau|\hbar-\hbar_{\Theta}|)d\mu \le C\sup_{4O\subset\Theta}\int_{O}\varphi(\sigma|\hbar-\hbar_{O}|)d\mu \tag{2.12}$$

for all Θ *such that* $\varphi(|\Theta|)\in L^1_{loc}(\Theta;\mu)$, *where the measure* μ *is defined by* $d\mu = v(x)dx$, $v(x)$ *is a weight, and* τ, σ *are constants with* $0<\tau, \sigma \le 1$, *and the supremum is over all balls* O *with* $4O\subset\Theta$.

By Definition 2.10, we arrive at the following global case of Theorem 2.4.

Theorem 2.11. *Let us assume, in addition to the definitions of the homotopy operator* T, *the exterior derivative* d, *the projection operator* H, *the measure* μ, *and the Young function* ϕ *in Theorem* 2.4, *that* $v\in C^{\infty}(\Theta,\Lambda^k)$, $k = 1, 2,\ldots, n$, *be a solution of the non-homogeneous A-harmonic equation* (1.2) *in a bounded* $L^{\phi}(\mu)$*-averaging domains* Θ *and* $\phi(|v|)\in L^1(\Theta;\mu)$. *Then, there is a constant* C, *independent of* v, *such that*

$$\| T(d(H(v))) - (T(d(H(v))))_{\Theta}\|_{\varphi(\Theta,\mu)} \le C||v||_{\varphi(\Theta,\mu)}. \tag{2.13}$$

Since John domains are very special $L^{\phi}(\mu)$-averaging domains, the preceding theorem immediately yields the following corollary.

Corollary 2.12. *Let us assume, in addition to the definitions of the homotopy operator* T, *the exterior derivative* d, *the projection operator* H, *the measure* μ, *and the Young function* ϕ *in Theorem* 2.4, *that* $v\in C^{\infty}(\Theta,\Lambda^k)$, $k = 1, 2,\ldots, n$, *be a solution of the non-homogeneous A-harmonic equation* (1.2) *in a bounded John domains* Θ *and* $\phi(|v|\in L^1(\Theta;\mu)$. *Then, there is a constant* C, *independent of* u, *such that*

$$\| T(d(H(v))) - (T(d(H(v))))_{\Theta}\|_{\varphi(\Theta,\mu)} \le C\| v\|_{\varphi(\Theta,\mu)}. \tag{2.14}$$

Remark. Note that the L^s-averaging domains and $L^s(\mu)$-averaging domains are also special $L^{\phi}(\mu)$-averaging domains. Thus, Theorem 2.11 also holds for the L^s-averaging domains and $L^s(\mu)$-averaging domains, respectively.

3 The proof of main results

In this section, we will give the proof of several theorems mentioned in the previous section.

Proof of Theorem 2.6. Let $t = \frac{\alpha q}{\alpha-1}$ and $r = \frac{\beta q}{\beta+1}$, then $r<q<t$. From Lemma 2.7 with $\frac{1}{q} = \frac{1}{t} + \frac{t-q}{tq}$, Lemma 2.8 and (2.6), we have

$$\begin{aligned}
&\left(\int_O \left|T(d(H(v))) - (T(d(H(v))))_O\right|^q v(x)dx\right)^{\frac{1}{q}}\\
&= \left(\int_O \left(\left|T(d(H(v))) - (T(d(H(v))))_O\right| v(x)^{\frac{1}{q}}\right)^q dx\right)^{\frac{1}{q}}\\
&\le \left(\int_O \left|T(d(H(v))) - (T(d(H(v))))_O\right|^t dx\right)^{\frac{1}{t}}\left(\int_O (v(x))^{\frac{t}{t-q}}dx\right)^{\frac{t-q}{tq}}\\
&\le C_1 diam(O)|O|||v||_{t,\rho_1 O}\left(\int_O (v(x))^{\alpha}dx\right)^{\frac{1}{\alpha q}}\\
&\le C_2 diam(O)|O|^{1+\frac{r-t}{rt}}\|v\|_{r,\rho_2 O}\left(\int_O (v(x))^{\alpha}dx\right)^{\frac{1}{\alpha q}},
\end{aligned} \tag{3.1}$$

where ρ_2, ρ_1 are two constants satisfying $\rho_2>\rho_1>1$.

By virtue of Lemma 2.7 with $\frac{1}{r} = \frac{1}{q} + \frac{q-r}{rq}$, we obtain that

$$
\begin{aligned}
&\| v\|_{r,\rho_2 O} \\
&= \left(\int_{\rho_2 O} |v|^r dx \right)^{\frac{1}{r}} \\
&= \left(\int_{\rho_2 O} \left(|v| (\nu(x))^{\frac{1}{q}} \cdot (\nu(x))^{\frac{-1}{q}} \right)^r dx \right)^{\frac{1}{r}} \\
&\le \left(\int_{\rho_2 O} |v|^q \nu(x) dx \right)^{\frac{1}{q}} \left(\int_{\rho_2 O} (\nu(x))^{\frac{-r}{q-r}} dx \right)^{\frac{q-r}{rq}} \\
&= \left(\int_{\rho_2 O} |v|^q d\mu \right)^{\frac{1}{q}} \left(\int_{\rho_2 O} (\nu(x))^{-\beta} dx \right)^{\frac{1}{\beta q}}.
\end{aligned}
\tag{3.2}
$$

Observe that $\nu(x) \in A(\alpha, \beta, \alpha, \Theta)$, hence

$$
\begin{aligned}
&\left(\int_O (\nu(x))^\alpha dx \right)^{\frac{1}{\alpha q}} \left(\int_{\rho_2 O} (\nu(x))^{-\beta} dx \right)^{\frac{1}{\beta q}} \\
&\le \left(\left(\int_{\rho_2 O} (\nu(x))^\alpha dx \right) \left(\int_{\rho_2 O} (\nu(x))^{-\beta} dx \right)^{\frac{\alpha}{\beta}} \right)^{\frac{1}{\alpha q}} \\
&= \left(|\rho_2 O|^{1+\frac{\alpha}{\beta}} \left(\frac{1}{|\rho_2 O|} \int_{\rho_2 O} (\nu(x))^\alpha dx \right) \left(\frac{1}{|\rho_2 O|} \int_{\rho_2 O} (\nu(x))^{-\beta} dx \right)^{\frac{\alpha}{\beta}} \right)^{\frac{1}{\alpha q}} \\
&\le C_3 |\rho_2 O|^{\frac{1}{\alpha q} + \frac{1}{\beta q}}.
\end{aligned}
\tag{3.3}
$$

Combining (3.1)-(3.3), we obtain that

$$
\begin{aligned}
&\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q \nu(x) dx \right)^{\frac{1}{q}} \\
&\le C_4 diam(O) |O|^{1+\frac{r-t}{rt}} |\rho_2 O|^{\frac{1}{\alpha q} + \frac{1}{\beta q}} \left(\int_{\rho_2 O} |v|^q \nu(x) dx \right)^{\frac{1}{q}} \\
&\le C_5 diam(O) |O| \left(\int_{\rho_2 O} |v|^q d\mu \right)^{\frac{1}{q}}.
\end{aligned}
\tag{3.4}
$$

Therefore, we have completed the proof of Theorem 2.6.

By Lemma 2.5 and Theorem 2.6, we obtain the proof of Theorem 2.4.

Proof of Theorem 2.4. First, we observe that $\mu(O) = \int_O \nu(x)dx \ge \int_O \varepsilon dx = C_1|O|$, thereby

$$
\frac{1}{\mu(O)} \le \frac{C_2}{|O|}
\tag{3.5}
$$

for all balls $O \subset \Theta$.

We obtain from Theorem 2.6 and Lemma 2.5 that

$$\begin{aligned}
&\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q d\mu\right)^{\frac{1}{q}} \\
&\leq C_1 diam(O)|O|\left(\int_{\rho_1 O} |v|^q d\mu\right)^{\frac{1}{q}} \\
&\leq C_2 diam(O)|O|(\mu(\rho_1 O))^{\frac{p-q}{pq}}\left(\int_{\rho_2 O} |v|^p d\mu\right)^{\frac{1}{p}},
\end{aligned} \tag{3.6}$$

where ρ_2, ρ_1 with $\rho_2 > \rho_1 > 1$ are two constants. Note that ϕ is an increasing function, and f is an increasing convex function in $[0, \infty)$, by Jensen's inequality for f, we obtain that

$$\begin{aligned}
&\varphi\left(\frac{1}{\chi}\left(\int_O \left|T(d(H(v))) - (T(d(H(v))))_O\right|^q d\mu\right)^{\frac{1}{q}}\right) \\
&\leq \varphi\left(\frac{1}{\chi}C_2\,|O|\,diam(O)(\mu(\rho_1 O))^{\frac{(p-q)}{pq}}\left(\int_{\rho_2 O}|v|^p d\mu\right)^{\frac{1}{p}}\right) \\
&= \varphi\left(\left(\frac{1}{\chi^p}C_2^p|O|^p(diam(O))^p(\mu(\rho_1 O))^{\frac{(p-q)}{q}}\int_{\rho_2 O}|v|^p d\mu\right)^{\frac{1}{p}}\right) \\
&\leq C_3 f\left(\frac{1}{\chi^p}C_2^p|O|^p(diam(O))^p(\mu(\rho_1 O))^{\frac{(p-q)}{q}}\int_{\rho_2 O}|v|^p d\mu\right) \\
&= C_3 f\left(\int_{\rho_2 O}\frac{1}{\chi^p}C_2^p|O|^p(diam(O))^p(\mu(\rho_1 O))^{\frac{(p-q)}{q}}|v|^p d\mu\right) \\
&\leq C_3\int_{\rho_2 O} f\left(\frac{1}{\chi^p}C_2^p|O|^p(diam(O))^p(\mu(\rho_1 O))^{\frac{(p-q)}{q}}|v|^p\right) d\mu.
\end{aligned} \tag{3.7}$$

Since $1 \leq p < q < \infty$, we have $1 + \frac{p-q}{pq} = 1 + \frac{1}{q} - \frac{1}{p} > 0$, which yields

$$\begin{aligned}
&diam(O)|O|\mu(\rho_1 O)^{\frac{p-q}{pq}} \\
&\leq C_4 diam(\Theta)|O||\rho_1 O|^{\frac{p-q}{pq}} \\
&\leq C_5 diam(\Theta)|O|^{1+\frac{p-q}{pq}} \\
&\leq C_6 diam(\Theta)|\Theta|^{1+\frac{p-q}{pq}} \leq C_7.
\end{aligned} \tag{3.8}$$

It follows from (i) in Definition 2.2 that $f(t) \leq C_8\varphi(t^{\frac{1}{p}})$. Thus,

$$\begin{aligned}
&\int_{\rho_2 O} f\left(\frac{1}{\chi^p}C_2^p|O|^p(diam(O))^p(\mu(\rho_1 O))^{\frac{p-q}{q}}|v|^p\right) d\mu \\
&\leq C_8\int_{\rho_2 O}\varphi\left(\frac{1}{\chi}C_2|O|(diam(O))(\mu(\rho_1 O))^{\frac{p-q}{q}}|v|\right) d\mu \\
&\leq C_8\int_{\rho_2 O}\varphi\left(\frac{1}{\chi}C_9|v|\right) d\mu \\
&\leq C_{10}\int_{\rho_2 O}\varphi\left(\frac{1}{\chi}|v|\right) d\mu.
\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we obtain that

$$\begin{aligned}
&\varphi\left(\frac{1}{\chi}\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q d\mu\right)^{\frac{1}{q}}\right)\\
&\le C_3 \int_{\rho_2 O} f\left(\frac{1}{\chi^p} C_2^p |O|^p (diam(O))^p (\mu(\rho_1 O))^{\frac{(p-q)}{q}} |v|^p\right) d\mu \\
&\le C_{11} \int_{\rho_2 O} \varphi\left(\frac{1}{\chi}|v|\right) d\mu.
\end{aligned} \tag{3.10}$$

Applying Jensen's inequality to g^{-1} and considering that ϕ and g are doubling, we obtain that

$$\begin{aligned}
&\int_O \varphi\left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right) d\mu\\
&= g\left(g^{-1}\left(\int_O \varphi\left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right) d\mu\right)\right)\\
&\le g\left(\int_O g^{-1}\left(\varphi\left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right)\right) d\mu\right)\\
&\le g\left(C_{12}\int_O \left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right)^q d\mu\right)\\
&\le C_{13}\varphi\left(\left(C_{12}\int_O \left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right)^q d\mu\right)^{\frac{1}{q}}\right)\\
&\le C_{14}\varphi\left(\frac{1}{\chi}\left(\int_O |T(d(H(v))) - (T(d(H(v))))_O|^q d\mu\right)^{\frac{1}{q}}\right)\\
&\le C_{15}\int_{\rho_2 O} \varphi\left(\frac{|v|}{\chi}\right) d\mu.
\end{aligned} \tag{3.11}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mu(O)}\int_O \varphi\left(\frac{|T(d(H(v))) - (T(d(H(v))))_O|}{\chi}\right) d\mu\\
&\le \frac{1}{\mu(O)} C_{15}\int_{\rho_2 O} \varphi\left(\frac{|v|}{\chi}\right) d\mu\\
&\le \frac{1}{\mu(\rho_2 O)} C_{16}\int_{\rho_2 O} \varphi\left(\frac{|v|}{\chi}\right) d\mu.
\end{aligned} \tag{3.12}$$

By Definition 2.1 and (3.12), we achieve the desired result

$$||T(d(H(v))) - (T(d(H(v))))_O||_{\varphi(O,\mu)} \le C||v||_{\varphi(\rho O,\mu)}. \tag{3.13}$$

With the aid of Definition 2.10, We proceed now to derive Theorem 2.11.

Proof of Theorem 2.11. Note that Θ is a $L^{\phi}(\mu)$-averaging domains, and ϕ is doubling, from Definition 2.10 and (3.12), we have

$$\begin{aligned}
&\frac{1}{\mu(\Theta)}\int_\Theta \varphi\left(\frac{\left|T(d(H(v))) - (T(d(H(v))))_\Theta\right|}{\chi}\right) d\mu\\
&\le C_1 \frac{1}{\mu(\Theta)} \sup_{4O\subset\Theta}\int_O \varphi\left(\frac{\left|T(d(H(v))) - (T(d(H(v))))_O\right|}{\chi}\right) d\mu\\
&\le C_1 \frac{1}{\mu(\Theta)} \sup_{4O\subset\Theta}\left(C_2\int_{\rho O} \varphi\left(\frac{|v|}{\chi}\right) d\mu\right)\\
&\le C_3 \frac{1}{\mu(\Theta)} \sup_{4O\subset\Theta}\int_\Theta \varphi\left(\frac{|v|}{\chi}\right) d\mu\\
&\le C_3 \frac{1}{\mu(\Theta)} \int_\Theta \varphi\left(\frac{|v|}{\chi}\right) d\mu.
\end{aligned} \tag{3.14}$$

By Definition 2.1 and (3.14), we conclude that

$$\| T(d(H(v))) - (T(d(H(v))))_{\Theta} \|_{\varphi(\Theta,\mu)} \leq C \| v\|_{\varphi(\Theta,\mu)}. \tag{3.15}$$

4 Applications

If we choose A to be a special operator, for example, $A(x, d\hbar) = d\hbar|d\hbar|^{s-2}$, then (1.4) reduces to the following s-harmonic equation:

$$d^{\star}(d\hbar|d\hbar|^{s-2}) = 0. \tag{4.1}$$

In particular, we may let $s = 2$, if $\hbar$ is a function (0-form), then Equation 4.1 is equivalent to the well-known Laplace's equation $\Delta\hbar = 0$. The function $\hbar$ satisfying Laplace's equation is referred to as the harmonic function as well as one of the solutions of Equation 4.1. Therefore, all the results in Section 2 still hold for the $\hbar$. As to the harmonic function, one finds broaden applications in the elliptic partial differential equations, see [20] for more related information.

We may make use of the following two specific examples to conform the convenience of the main inequality (3.11) in evaluating the upper bound for the L^{ϕ}-norm of $|T(d(H(v))) - (T(d(H(v))))_O|$. Obviously, we may take advantages of (3.11) to make this estimating process easily, without calculating $T(d(H(v)))$ and $(T(d(H(v))))_O$ complicatedly.

Example 4.1. *Let* ε, r *be two distinct constants satisfying* $\frac{1}{e} < \varepsilon < r < 1$, $y = (y_1, y_2, \ldots, y_n)$ *be a fixed point in* $\mathbb{R}^n (n > 2)$, $\phi(t) = t^p \log_+ t$, $p > 1$, $v = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{2-n}{2}}$ *and* $O = \{x = (x_1, \ldots, x_n)| : \varepsilon^2 \leq (x_1 - y_1)^2 + \ldots + (x_n - y_n) \leq r^2\}$.

First, by simple computation, we have

$$v_{x_i} = (2-n)(x_i - y_i)\left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{-n}{2}}, \tag{4.2}$$

$$v_{x_i x_i} = (2-n)\left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{-(n+2)}{2}}\left(\sum_{i=1}^n (x_i - y_i)^2 - n(x_i - y_i)^2\right), \tag{4.3}$$

then we get

$$\Delta v = \sum_{i=1}^n v_{x_i x_i} = 0, \tag{4.4}$$

so the harmonic property of v is confirmed.

Observe that $|O| = \sigma_n r^n$, where σ_n denotes the volume of a unit ball in $\mathbb{R}^n$ $(n > 2)$, and $1 < \frac{1}{r^{n-2}} \leq |v| = |\left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{2-n}{2}}| \leq \frac{1}{\varepsilon^{n-2}}$, applying (3.11) with $\chi = 1$, $d\mu = dx$, we obtain

$$\begin{aligned}
&\int_O \varphi(|T(d(H(v))) - (T(d(H(v))))_O|)dx \\
&= \int_O (|T(d(H(v))) - (T(d(H(v))))_O|)^p \log_+ (|T(d(H(v))) - (T(d(H(v))))_O|)dx \\
&\leq C\left(\int_{\rho O} |v|^p \log_+ |v|\, dx\right) \\
&\leq C\left(\left(\frac{1}{\varepsilon^{(n-2)}}\right)^p \log \frac{1}{\varepsilon^{(n-2)}}\, |\rho O|\right) \\
&= \left(\frac{1}{\varepsilon^{(n-2)p}}(\sigma_n \rho^n r^n)\right) \log \frac{1}{\varepsilon^{(n-2)}} \\
&= \frac{C\rho^n \sigma_n r^n}{\varepsilon^{(n-2)p}} \log \frac{1}{\varepsilon^{(n-2)}}.
\end{aligned} \tag{4.5}$$

Example 4.2. *Let us assume, in addition to the definitions of* ε, r, ϕ *of Example* 4.1, *that* $y = (y_1, y_2)$ *be a fixed point in* $\mathbb{R}^2$, $v = \log\left(\sum_{i=1}^{2} (x_i - y_i)^2\right)^{\frac{1}{2}}$ *and* $O = \{x = (x_1, x_2)| : \varepsilon^2 \leq (x_1 - y_1)^2 + (x_2 - y_2) \leq r^2\}$.

Similarly, we observe to begin with that

$$v_{x_i} = \frac{x_i - y_i}{\sum_{i=1}^{2} (x_i - y_i)^2}, \tag{4.6}$$

$$v_{x_i x_i} = \frac{\sum_{i=1}^{2} (x_i - y_i)^2 - 2(x_i - y_i)^2}{\left(\sum_{i=1}^{2} (x_i - y_i)^2\right)^2}. \tag{4.7}$$

Thus,

$$\Delta v = \sum_{i=1}^{2} u_{x_i x_i} = 0, \tag{4.8}$$

which implies the function v is harmonic.

With respect to the estimation of $\int_O \varphi(|T(d(H(v))) - (T(d(H(v))))_O|)dx$, Example 4.2 proceeds in much the same way after replacing $|O| = \sigma_n r^n$ and $1 < |v| \leq \frac{1}{\varepsilon^{n-2}}$ with $|O| = \pi r^2$ and $|\log \varepsilon| < |v| \leq |\log r| < 1$, respectively. Here we omit the reminder process.

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Authors' contributions
ZD finished the proof and the writing work. YW gave ZD some excellent advices in the proof and writing. GB gave ZD lots of help in selecting the examples as applications. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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