PERIODIC SOLUTIONS OF SECOND-ORDER LIÉNARD EQUATIONS WITH \( p \)-LAPLACIAN-LIKE OPERATORS

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The existence of periodic solutions for second-order Liénard equations with \( p \)-Laplacian-like operator is studied by applying new generalization of polar coordinates.

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1. Introduction

In recent years, the existence of periodic solutions for second-order Liénard equations

\[ u'' + f(u,u')u' + g(u) = e(t,u,u') \]  

and its special case have been studied by many researchers, we refer the readers to [1, 3, 4, 6, 7, 9–12] and the references therein.

Let us consider the so-called one-dimensional \( p \)-Laplacian operator \((\phi_p(u'))'\), where \( p > 1 \) and \( \phi_p : \mathbb{R} \to \mathbb{R} \) is given by \( \phi_p(s) = |s|^{p-2}s \) for \( s \neq 0 \) and \( \phi_p(0) = 0 \). Periodic boundary conditions containing this operator have been considered in [2, 5].

In [8], Manásevich and Mawhin investigated the existence of periodic solutions to some system cases involving the fairly general vector-valued operator \( \phi \). They considered the boundary value problem

\[ (\phi(u'))' = f(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \]  

where the function \( \phi : \mathbb{R}^N \to \mathbb{R}^N \) satisfies some monotonicity conditions which ensure that \( \phi \) is a homeomorphism onto \( \mathbb{R}^N \).

Recently, in [16] we studied the existence of periodic solutions for the nonlinear differential equation with a \( p \)-Laplacian-like operator

\[(\phi(u'))' + f(t,u,u') = 0.\]
2 Periodic solutions for Liénard equations

Motivated by the work of [13], in this paper we use new polar coordinates [13] to investigate the existence of periodic solutions for the second-order generalized Liénard equations with \( p \)-Laplacian-like operator

\[
(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T]. \tag{1.4}
\]

Throughout this paper, we always assume that \( \phi, g \in \mathbb{C}(\mathbb{R}, \mathbb{R}), f \in \mathbb{C}(\mathbb{R}^2, \mathbb{R}), e \in \mathbb{C}([0,T] \times \mathbb{R}^2, \mathbb{R}) \). And the following conditions also hold.

(H1) \( \phi \) is continuous and strictly increasing, \( y \phi(y) > 0 \) for \( y \neq 0 \), and there exist \( p > 2, m_2 \geq m_1 > 0 \), such that

\[
m_1 |y|^{p-1} \leq |\phi(y)| \leq m_2 |y|^{p-1}. \tag{1.5}
\]

(H2) \( e \in \mathbb{C}([0,T] \times \mathbb{R}^2, \mathbb{R}) \), periodic in \( t \) with period \( T \), there exist \( \alpha_1, \beta_1, \gamma_1 > 0 \), and \( p > k > 2 \) such that

\[
|e(t,x,y)| \leq \alpha_1 |x|^{p-1} + \beta_1 |y|^{k-1} + \gamma_1 \quad \text{for} \quad (t,x,y) \in [0,T] \times \mathbb{R}^2. \tag{1.6}
\]

(H3) \( f \in \mathbb{C}(\mathbb{R}^2, \mathbb{R}) \), there exist \( \alpha_2, \beta_2, \gamma_2 > 0 \) such that

\[
|f(x,y)| \leq \alpha_2 |x|^{p-2} + \beta_2 |y|^{k-2} + \gamma_2 \quad \text{for} \quad (x,y) \in \mathbb{R}^2. \tag{1.7}
\]

(H4) There exist \( \lambda, \mu, \) and \( n \geq 0 \) such that

\[
\frac{m_2}{m_1} \left( \frac{p'}{p'-1} \right)^{p-1} \left( \frac{2n\pi_p}{T} \right)^{p} + \frac{\alpha_1}{m_1} + \frac{p-1}{p} \left( \frac{\alpha_2}{m_1} \right)^{p/(p-1)} \left( \frac{m_2}{m_1} \right)^{1/(p-1)^2} < \lambda
\]

\[
\leq \frac{g(x)}{\phi(x)} \leq \mu < \frac{m_1}{m_2} \left( \frac{p'}{p'+1} \right)^{p-1} \left( \frac{2(n+1)\pi_p}{T} \right)^{p}
\]

\[
- \frac{\alpha_1}{m_2} - \frac{p-1}{p} \left( \frac{\alpha_2}{m_2} \right)^{p/(p-1)} \left( \frac{m_2}{m_1} \right)^{1/(p-1)}, \tag{1.8}
\]

where

\[
p' = p(p-1), \quad \pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}. \tag{1.9}
\]

(H5) Solutions of (1.4) are unique with respect to initial value.

In this paper, we use a new coordinate to estimate the time when a point moves along a trajectory around the origin and then give some sufficient conditions for the existence of periodic solutions of (1.4).

2. Periodic solutions with a Laplacian-like operator

Let \( \nu = \phi(u') \). Then (1.4) is equivalent to the system

\[
\begin{align*}
u' &= \phi^{-1}(\nu), \\
\nu &= -g(u) - f(u,\phi^{-1}(\nu))\phi^{-1}(\nu) + e(t,u,\phi^{-1}(\nu)). \tag{2.1}
\end{align*}
\]
Let \( u(t, \xi, \eta) \) denote the solution of (1.4) which satisfies the initial value condition

\[
    u(0, \xi, \eta) = \xi, \quad v(0, \xi, \eta) = \eta,
\]

then we have the following conclusion.

**Lemma 2.1.** Suppose (H1)-(H5) hold, then for all \( c > 0 \), there exists constant \( A > 0 \) such that if

\[
    \frac{1}{p} |\xi|^p + \frac{p - 1}{p} |\eta|^{p/(p-1)} = A^2,
\]

then

\[
    \frac{1}{p} |u(t, \xi, \eta)|^p + \frac{p - 1}{p} |v(t, \xi, \eta)|^{p/(p-1)} \geq c^2 \quad \text{for } t \in [0, T].
\]

**Proof.** Let \((u(t), v(t)), t \in [0, T]\), be a solution of (2.1) satisfying \( u(0, \xi, \eta) = \xi, \ v(0, \xi, \eta) = \eta \).

Let

\[
    r^2(t) = \frac{1}{p} |u(t)|^p + \frac{p - 1}{p} |v(t)|^{p/(p-1)}.
\]

It is clear that (H1) implies

\[
    \left( \frac{|v|}{m_2} \right)^{1/(p-1)} \leq |\phi^{-1}(v)| \leq \left( \frac{|v|}{m_1} \right)^{1/(p-1)}.
\]

So we have

\[
    \left| \frac{dr^2(t)}{dt} \right| = \left| \frac{u(t)}{p^2} u(t)u'(t) + \frac{v(t)}{(2-p)/(p-1)} \right| v(t)v'(t) \right|
\]

\[
    \leq |u|^{p-1} |\phi^{-1}(v)| + |v|^{(2-p)/(p-1)} \left| -g(u) - f(u, \phi^{-1}(v)) \phi^{-1}(v) + e(t, u, \phi^{-1}(v)) \right|
\]

\[
    \leq |u|^{p-1} |\phi^{-1}(v)| + \mu |v|^{1/(p-1)} |\phi(u)|
\]

\[
    + |v|^{1/(p-1)} \left( \alpha_2 |u|^{p-2} + \beta_2 |\phi^{-1}(v)|^{k-2} + \gamma_2 \right) |\phi^{-1}(v)|
\]

\[
    + |v|^{1/(p-1)} \left( \alpha_1 |u|^{p-1} + \beta_1 |\phi^{-1}(v)|^{k-1} + \gamma_1 \right)
\]

\[
    \leq |u|^{p-1} \left( \frac{|v|}{m_1} \right)^{1/(p-1)} + \mu m_2 |v|^{1/(p-1)} |u|^{p-1}
\]

\[
    + \alpha_2 m_1^{-1/(p-1)} |u|^{2/(p-1)} |u|^{p-2} + \beta_2 m_1^{(1-k)/(p-1)} |v|^{k/(p-1)}
\]

\[
    + \gamma_2 m_1^{-1/(p-1)} |v|^{2/(p-1)} + \alpha_1 |v|^{1/(p-1)} |u|^{p-1}
\]

\[
    + \beta_1 m_1^{(1-k)/(p-1)} |v|^{k/(p-1)} + \gamma_1 |v|^{1/(p-1)}
\]

\[
    = l_1 |u|^{p-1} |v|^{1/(p-1)} + l_2 |v|^{k/(p-1)} + l_3 |v|^{2/(p-1)} |u|^{p-2} + l_4 |v|^{2/(p-1)} + \gamma_1 |v|^{1/(p-1)},
\]
where
\[ l_1 = m_1^{-1/(p-1)} + \mu m_2 + \alpha_1, \quad l_2 = \beta_1 m_1^{(1-k)/(p-1)} + \beta_2 m_1^{(1-k)/(p-1)}, \]
\[ l_3 = \alpha_2 m_1^{-1/(p-1)}, \quad l_4 = \gamma_2 m_1^{-1/(p-1)}, \] (2.8)

while
\[ l_1 |u|^{p-1} |v|^{1/(p-1)} \leq l_1 \left( \frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} |u|^p \right) \]
\[ \leq l_1 \max \left\{ p-1, \frac{1}{p-1} \right\} \left( \frac{1}{p} |u| + \frac{p-1}{p} |v|^{p/(p-1)} \right) \]
\[ = l_1 \max \left\{ p-1, \frac{1}{p-1} \right\} r^2, \]
\[ l_2 |v|^{k/(p-1)} \leq \frac{k}{p} |v|^{p/(p-1)} + \frac{p-k}{p} l_2^{p/(p-k)} \leq \frac{k}{p-1} r^2 + \frac{p-k}{p} \]
\[ l_3 |v|^{2/(p-1)} |u|^{p-2} \leq l_3 \left( \frac{2}{p} |v|^{p/(p-1)} + \frac{p-2}{p} |u|^p \right) \leq l_3 \left( \frac{2}{p} |u| + p-2 \right) r^2, \]
\[ l_4 |v|^{2/(p-1)} \leq \frac{2}{p} |v|^{p/(p-1)} + \frac{p-2}{p} l_4^{p/(p-2)} \leq \frac{2}{p-1} r^2 + \frac{p-2}{p} l_4^{p/(p-2)}, \]
\[ \gamma_1 |v|^{1/(p-1)} \leq \frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} \gamma_1^{p/(p-1)} \leq \frac{1}{p-1} r^2 + \frac{p-1}{p} \gamma_1^{p/(p-1)}. \]

So,
\[ \left| \frac{dr^2(t)}{dt} \right| \leq br^2(t) + a, \] (2.10)

where
\[ a = \frac{p-k}{p} l_2^{p/(p-k)} + \frac{p-2}{p} l_4^{p/(p-2)} + \frac{p-1}{p} \gamma_1^{p/(p-1)}, \]
\[ b = l_1 \max \left\{ p-1, \frac{1}{p-1} \right\} + l_3 \left( \frac{2}{p} |u| + p-2 \right) + \frac{k+3}{p-1}. \] (2.11)

It follows that
\[ \left( r^2(0) + \frac{a}{b} \right) e^{-bt} \leq \left( r^2(t) + \frac{a}{b} \right) e^{-bt} \leq \left( r^2(t) + \frac{a}{b} \right) \]
\[ \leq \left( r^2(0) + \frac{a}{b} \right) e^{bt} \leq \left( r^2(0) + \frac{a}{b} \right) e^{bt}, \quad 0 \leq t \leq T. \] (2.12)

Let \( A = [(c^2 + a/b)e^{bT} - a/b]^{1/2} \), then \( r(0) = A \) implies \( r(t) \geq c. \)
**Lemma 2.2.** Let \((u(t), v(t))\) be a solution of (2.1). Suppose the conditions of (H1)–(H5) are satisfied. Then there is \(R\) such that under the generalized polar coordinates, \(r(0) \geq R\) implies that

\[
\frac{d\theta(t)}{dt} \leq 0, \quad t \in [0, T].
\]  

**Proof.** Applying generalized polar coordinates,

\[
u = \left(\frac{p}{p-1}\right)^{(p-1)/p} r^{2(p-1)/p} \sin \theta^{(p-2)/p} \sin \theta,
\]

or

\[
\begin{align*}
    r \cos \theta &= \frac{1}{\sqrt{p}} |u|^{(p-2)/2} u, \\
    r \sin \theta &= \sqrt{\frac{p-1}{p}} |v|^{(2-p)/2(p-1)} v.
\end{align*}
\]

Then \(\theta = \tan^{-1} [\sqrt{p - 1} (|v|^{(2-p)/2(p-1)} v / |u|^{(p-2)/2} u)].\) So we have

\[
\theta' = \frac{|u|^{((p-2)/2)} |v|^{((2-p)/2(p-1))}}{2 \sqrt{p - 1} r^2} [uv' - (p - 1) u' v] 
= -\frac{|u|^{((p-2)/2)} |v|^{((2-p)/2(p-1))}}{2 \sqrt{p - 1} r^2} \left[ u g(u) + u f(u, \phi^{-1}(v)) \phi^{-1}(v) + (p - 1) v \phi^{-1}(v) - u e(t, u, \phi^{-1}(v)) \right]
\]

as

\[
ug(u) + u f(u, \phi^{-1}(v)) \phi^{-1}(v) + (p - 1) v \phi^{-1}(v) - u e(t, u, \phi^{-1}(v))
\]

\[
\geq \lambda u \phi(u) + (p - 1) v \phi^{-1}(v) - |u| \left( \alpha_2 |u|^{p-2} + \beta_2 | \phi^{-1}(v) |^{k-2} + \gamma_2 \right) | \phi^{-1}(v) | \\
- |u| \left( \alpha_1 |u|^{p-1} + \beta_1 | \phi^{-1}(v) |^{k-1} + \gamma_1 \right)
\]

\[
\geq \lambda m_1 |u|^p + (p - 1) m_2^{-1/(p-1)} |v|^{p/(p-1)} - \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} \\
- \gamma_2 m_1^{-1/(p-1)} |u| |v|^{1/(p-1)} - \alpha_1 |u|^p - (\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} - \gamma_1 |u|
\]

\[
= (\lambda m_1 - \alpha_1) |u|^p + (p - 1) m_2^{-1/(p-1)} |v|^{p/(p-1)} - \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} \\
- \gamma_2 m_1^{-1/(p-1)} |u| |v|^{1/(p-1)} - (\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} - \gamma_1 |u|.
\]
6 Periodic solutions for Liénard equations

Let

\[
\tau = \frac{p(p - 1)}{4(k - 1)} m_2^{-1/(p-1)}, \quad \beta' = \frac{4(\beta_1 + \beta_2)(k - 1)}{p(p - 1)} m_1^{(1-k)/(p-1)} m_2^{1/(p-1)},
\]

so we have

\[
(\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u| |v|^{(k-1)/(p-1)}
= \tau |u| \left( |v|^{(k-1)/(p-1)} \beta' \right) \leq \tau |u| \left( \frac{k - 1}{p - 1} |v| + \frac{p - k}{p - 1} \beta'(p-1)/(p-k) \right)
= \frac{1}{4} \frac{p p m_2^{-1/(p-1)} |u| |v| + \frac{p(p - k)}{4(k - 1)} m_2^{-1/(p-1)} \beta'(p-1)/(p-k) |u|}{u}
\leq \frac{1}{4} \frac{p p m_2^{-1/(p-1)} \left( \frac{1}{p} |u|^p + \frac{p - 1}{p} |v|^{p/(p-1)} \right) + \frac{p(p - k)}{4(k - 1)} m_2^{-1/(p-1)} \beta'(p-1)/(p-k) |u|}{u}.
\]

(2.19)

Let

\[
\tau_1 = \frac{1}{4} \frac{p(p - 1)}{4} m_2^{-1/(p-1)}, \quad \beta'_1 = \frac{4 \gamma_2}{p(p - 1)} \left( \frac{m_2}{m_1} \right)^{1/(p-1)},
\]

then

\[
\gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} = \tau_1 |u| \left( |v|^{1/(p-1)} \beta'_1 \right)
\leq \tau_1 |u| \left( \frac{1}{p - 1} |v| + \frac{p - 2}{p - 1} \beta'_1 \right)
= \frac{1}{4} \frac{p p m_2^{-1/(p-1)} |u| |v| + \frac{p(p - 2)}{4} m_2^{-1/(p-1)} \beta'_1 |p-1)/(p-2) |u|}{u}
\leq \frac{1}{4} \frac{p p m_2^{-1/(p-1)} \left( \frac{1}{p} |u|^p + \frac{p - 1}{p} |v|^{p/(p-1)} \right) + \frac{p(p - 2)}{4} m_2^{-1/(p-1)} \beta'_1 |p-1)/(p-2) |u|}{u}.
\]

(2.21)

Let

\[
\tau_2 = \frac{1}{4} \frac{p(p - 1)}{4} m_2^{-1/(p-1)}, \quad \beta'_2 = \frac{4 \alpha_2}{p(p - 1)} \left( \frac{m_2}{m_1} \right)^{1/(p-1)}
\]

(2.22)
then
\[ \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} \]
\[ = \tau_2 \left( |v|^{1/(p-1)} \beta_2 |u|^{p-1} \right) \leq \tau_2 \left( \frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} \left( \beta_2 |u|^{p-1} \right)^{p/(p-1)} \right) \]
\[ \leq \frac{1}{4} pm_2^{-1/(p-1)} \left( \frac{1}{p} |u|^p + \frac{p-1}{p} |v|^{p/(p-1)} \right) + \frac{p-1}{p} \tau_2 \beta_2^{p/(p-1)} |u|^p. \]  

(2.23)

We select \( \lambda \) large enough such that
\[ \delta = \lambda m_1 - \alpha_1 - \frac{p-1}{p} \tau_2 \beta_2^{p/(p-1)} - m_2^{-1/(p-1)} > 0, \]  

(2.24)

Let \( d = \gamma_1 + (p(p - k)/4(k - 1)) m_2^{-1/(p-1)} \beta_2^{(p-1)/(p-k)} + (p(p - 2)/4) m_2^{-1/(p-1)} \beta_1^{(p-1)/(p-2)} \), we also have
\[ d|u| = \delta |u| \left( \frac{d}{\delta p} \right) \leq \delta |u|^p + (p - 1) \delta \left( \frac{d}{p \delta} \right)^{p/(p-1)}, \]  

(2.25)

therefore
\[ ug(u) + uf(u, \phi^{-1}(v)) \phi^{-1}(v) + (p - 1) v \phi^{-1}(v) - ue(t, u, \phi^{-1}(v)) \geq \frac{1}{4} pm_2^{-1/(p-1)} \left[ \frac{1}{p} |u|^p + \frac{p-1}{p} |v|^{p/(p-1)} \right] - (p - 1) \delta \left( \frac{d}{p \delta} \right)^{p/(p-1)} \]
\[ = \frac{1}{4} pm_2^{-1/(p-1)} r^2(t) - (p - 1) \delta \left( \frac{d}{p \delta} \right)^{p/(p-1)}. \]  

(2.26)

Lemma 2.1 implies that there is \( R > 0 \), such that
\[ \frac{1}{4} pm_2^{-1/(p-1)} r^2(t) > (p - 1) \delta \left( \frac{d}{p \delta} \right)^{p/(p-1)} \]  

(2.27)

when \( r(0) > R \), then our assertion is verified. \( \square \)

**Lemma 2.3.** Assume that (H1)–(H5) hold, and
\[ \frac{1}{p} |\xi|^p + \frac{p-1}{p} |\eta|^{p/(p-1)} = A^2 \quad (A \gg 1) \]  

(2.28)
8 Periodic solutions for Liénard equations

then

\[ (u(T, \xi, \eta), v(T, \xi, \eta)) \neq (\lambda^{2/p} \xi, \lambda^{2(p-1)/p} \eta), \]  

(2.29)

where \( \lambda \) is an arbitrary positive number.

**Proof.** It follows from Lemma 2.1 that if

\[ \frac{1}{p} |\xi|^{p} + \frac{p-1}{p} |\eta|^{p/(p-1)} = A^{2}, \]  

(2.30)

then

\[ \frac{1}{p} |u(t, \xi, \eta)|^{p} + \frac{p-1}{p} |v(t, \xi, \eta)|^{p/(p-1)} \geq c^{2} \text{ for } t \in [0, T]. \]  

(2.31)

According to the generalized polar coordinates (2.14), we have

\[ r(t) \geq c \text{ for } t \in [0, T] \text{ if } r(0) = A. \]  

(2.32)

On the other hand, when \( r(0) \to \infty \), it holds uniformly from (H1)–(H3) that

\[ -\theta' = \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(2-p-1)}}{2 \sqrt{p - Tr^{2}}} \left[ u g(u) + u f(u, \phi^{-1}(v)) \phi^{-1}(v) \right. \]

\[ + (p - 1)v \phi^{-1}(v) - ue(t, u, \phi^{-1}(v)) \]  

\[ \geq \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(2-p-1)}}{2 \sqrt{p - Tr^{2}}} \left[ (\lambda m_{1} - \alpha_{1}) |u|^{p} + (p - 1)m_{2}^{-1/(p-1)} |v|^{p/(p-1)} \right. \]

\[ - \alpha_{2} m_{1}^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} - \gamma_{2} m_{1}^{-1/(p-1)} |u| |v|^{1/(p-1)} \]

\[ - (\beta_{1} + \beta_{2}) m_{1}^{-1/(p-1)} |u| |v|^{1/(p-1)} - \gamma_{1} |u| \]  

(2.33)

as

\[ \alpha_{2} m_{1}^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} \]

\[ = m_{2}^{-1/(p-1)} \left[ |v|^{1/(p-1)} \right] \left[ \alpha_{2} \left( \frac{m_{2}}{m_{1}} \right)^{1/(p-1)} \right] |u|^{p-1} \]

\[ \leq m_{2}^{-1/(p-1)} \left[ \frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} \frac{m_{2}}{m_{1}} \right] \right] \left[ \alpha_{2} \left( \frac{m_{2}}{m_{1}} \right)^{p/(p-1)} \right] |u|^{p} \]  

\[ = \frac{1}{p} m_{2}^{-1/(p-1)} |v|^{p/(p-1)} + \frac{p-1}{p} \alpha_{2} \left( \frac{m_{2}}{m_{1}} \right)^{p/(p-1)} m_{1}^{-p/(p-1)^{2}} m_{2}^{1/(p-1)^{2}} |u|^{p}. \]  

(2.34)
So

\[-\theta' \geq \frac{|u|^{(p-2)/2}|v|^{(2-p)/2}(p-1)}{2\sqrt{p-1}r^2} \left[ (\lambda m_1 - \alpha_1 - \tilde{\alpha}) |u|^p + \frac{p'-1}{p'} (p-1) m_2^{-1/(p-1)} |v|^{p/(p-1)} \right. \]

\[\left. - \gamma_2 m_1^{-1/(p-1)} |u| |v|^{1/(p-1)} \right. \]

\[- \left. - (\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u| |v|^{(k-1)/(p-1)} - \gamma_1 |u| \right] \]

\[= \frac{p |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p}}{2(p-1)^{1/p}} \left[ (\lambda m_1 - \alpha_1 - \tilde{\alpha}) \cos^2 \theta + \frac{p'-1}{p'} m_2^{-1/(p-1)} \sin^2 \theta \right] \]

\[- \frac{\gamma m_1^{-1/(p-1)} p^{2/p}}{2(p-1)^{2/p} r^{2(p-2)/p}} |\cos \theta| |\sin \theta|^{(4-p)/p} \]

\[- \frac{(\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} p^{k/p}}{2(p-1)^{k/p} r^{2(k-p)/p}} |\cos \theta| |\sin \theta|^{(2k-p)/p} \]

\[- \frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p} r^{2(p-1)/p}} |\cos \theta| |\sin \theta|^{(2-p)/p} \]

\[= a_1 \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p} \]

\[- \frac{\gamma m_1^{-1/(p-1)} p^{2/p}}{2(p-1)^{2/p} r^{2(p-2)/p}} |\cos \theta| |\sin \theta|^{(4-p)/p} \]

\[- \frac{(\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} p^{k/p}}{2(p-1)^{k/p} r^{2(k-p)/p}} |\cos \theta| |\sin \theta|^{(2k-p)/p} \]

\[- \frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p} r^{2(p-1)/p}} |\cos \theta| |\sin \theta|^{(2-p)/p}, \]

where

\[\tilde{\alpha} = \frac{p-1}{p} \alpha_2^{1/(p-1)} m_1^{-p/(p-1)^2} m_2^{1/(p-1)^2}, \quad p' = p(p-1), \]

\[a_1 = \frac{p(p'-1)}{2p'(p-1)^{1/p} m_2^{1/(p-1)}}, \quad b_1 = \frac{p'}{p'-1} (\lambda m_1 - \alpha_1 - \tilde{\alpha}) m_2^{1/(p-1)}. \]

Denote \(\hat{b} = \min \{b_1, 1\}\), then we have

\[-\theta' \geq a_1 \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p} \]

\[- \frac{\gamma m_1^{-1/(p-1)} p^{2/p}}{2\hat{b}(p-1)^{2/p} r^{2(p-2)/p}} \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\cos \theta| |\sin \theta|^{(4-p)/p} \]
from (H4), we have

\[
\frac{\beta_1 + \beta_2}{2b(p-1)^{(k-1)/p} r^{(p-k)/p}} \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p} = \hat{a}_1 \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p},
\]

where

\[
\hat{a}_1 = a_1 = \frac{\gamma_2 m_1^{-1/(p-1)} p^{2p}}{2b(p-1)^{2(p-2)/p}} - \frac{(\beta_1 + \beta_2) m_2^{(1-k)/(p-1)} p^{k/p}}{2b(p-1)^{k/p} r^{(p-k)/p}} - \frac{\gamma_1 p^{1/p}}{2b(p-1)^{(p-1)/p} r^{(p-k)/p}},
\]

Assume that it takes time \( \Delta t \) for the motion \((r(t), \theta(t))\) \((r(0) = A, \theta(0) = \theta_0)\) to complete one cycle around the origin. It follows from the above inequality that

\[
\Delta t < \int_{\theta_0}^{\theta_0 + 2\pi} \frac{d\theta}{\hat{a}_1 \left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p}}
\]

\[
= \frac{4}{\hat{a}_1} \int_0^{\pi/2} \frac{d\theta}{\left( b_1 \cos^2 \theta + \sin^2 \theta \right) |\sin \theta|^{(2-p)/p} |\cos \theta|^{(p-2)/p}}.
\]

Let

\[
\eta = \tan^{-1} \frac{1}{\sqrt{b_1}} \tan \theta,
\]

then

\[
\Delta t < \frac{4}{\hat{a}_1 b_1^{1/p}} \int_0^{\pi/2} \frac{d\eta}{\tan \eta |\sin \eta|^{(2-p)/p}} = \frac{2}{\hat{a}_1 b_1^{1/p}} B \left( \frac{1}{p}, \frac{p-1}{p} \right) = \frac{2\pi}{\hat{a}_1 b_1^{1/p} \sin(\pi/p)},
\]

from (H4), we have

\[
a_1 b_1^{1/p} \sin \frac{\pi}{p} = \frac{\pi}{\pi_p} \left( \frac{p'}{p} \right)^{(p-1)/p} \left( \frac{\lambda m_1 - \alpha_1 - \tilde{\alpha}}{m_2} \right)^{1/p} > \frac{2n\pi}{T}.
\]

So there exists \( \sigma > 0 \) such that \( (a_1 - \sigma) b_1^{1/p} \sin(\pi/p) > 2n\pi/T \). For the \( \sigma > 0 \), there exists \( \tilde{\alpha} > 0 \) such that

\[
0 < \frac{\gamma_2 m_1^{-1/(p-1)} p^{2p}}{2b(p-1)^{2(p-2)/p}} + \frac{(\beta_1 + \beta_2) m_2^{(1-k)/(p-1)} p^{k/p}}{2b(p-1)^{k/p} r^{(p-k)/p}} + \frac{\gamma_1 p^{1/p}}{2b(p-1)^{(p-1)/p} r^{(p-k)/p}} < \sigma
\]
for \( A > \mathbb{R} \) large enough. So we have

\[
\hat{a}_1b_1^{1/p} \sin \frac{\pi}{p} = \left( a_1 - \frac{\gamma_2 m_1^{-1/(p-1)} p^{2/p}}{2\hat{b}(p - 1)^{2/p} r^{2/(p-2)/p}} - \frac{(\beta_1 + \beta_2) m_2^{(1-k)/(p-1)} p^{k/p}}{2\hat{b}(p - 1)^{k/p} r^{2/(p-k)/p}} \right. \\
\left. - \frac{\gamma p^{1/p}}{2\hat{b}(p - 1)^{1/p} r^{2/(p-1)/p}} \right) b_1^{1/p} \sin \frac{\pi}{p} > (a_1 - \sigma)b_1^{1/p} \sin \frac{\pi}{p} > \frac{2n\pi}{T}.
\]

(2.44)

Therefore

\[
\frac{T}{\Delta t} > n
\]

(2.45)

as

\[
\alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} = m_1^{-1/(p-1)} \left( |v|^{1/(p-1)} \right) (\alpha_2 |u|^{p-1}) \\
\leq m_1^{-1/(p-1)} \left[ \frac{1}{p} |v|^{p/(p-1)} + \frac{p-1}{p} \alpha_2^{p/(p-1)} |u|^p \right] \\
= \frac{1}{p} m_1^{-1/(p-1)} |v|^{p/(p-1)} + \frac{p-1}{p} \alpha_2^{p/(p-1)} m_1^{-1/(p-1)} |u|^p.
\]

Similarly, we have

\[
0 < -\theta' = \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p - 1} r^2} \left[ u g(u) + u f(u, \phi^{-1}(v)) \phi^{-1}(v) + (p - 1)v \phi^{-1}(v) - u e(t, u, \phi^{-1}(v)) \right] \\
\leq \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p - 1} r^2} \left[ (\mu m_2 + \alpha_1) |u|^p + (p - 1)m_1^{-1/(p-1)} |v|^{p/(p-1)} \\
+ \alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} + \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\
+ (\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u| |v|^{(k-1)/(p-1)} + \gamma_1 |u| \right] \\
\leq \frac{|u|^{(p-2)/2} |v|^{(2-p)/2(p-1)}}{2\sqrt{p - 1} r^2} \left[ (\mu m_2 + \alpha_1 + \tilde{\alpha}') |u|^p + \frac{p' + 1}{p'} (p - 1)m_1^{-1/(p-1)} |v|^{p/(p-1)} \\
+ \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\
+ (\beta_1 + \beta_2) m_1^{(1-k)/(p-1)} |u| |v|^{(k-1)/(p-1)} + \gamma_1 |u| \right]
\]
where

\[ \tilde{\alpha}' = \frac{p - 1}{p} \alpha_2 \frac{p'(p-1)}{m_1}, \quad a_2 = \frac{p(p' + 1)}{2p'(p - 1)} m_1^{(1/p - 1)}, \]

\[ b_2 = \frac{p'}{p' + 1} (\mu m_2 + \alpha_1 + \tilde{\alpha}') m_1^{(1/p - 1)}, \]

with the similar argument, we also get

\[ \frac{T}{\Delta t} < n + 1. \]

Then it holds that

\[ n < \frac{T}{\Delta t} < n + 1. \]

To finish the proof, we claim that If \( n < T/\Delta t < n + 1 \), then \((u(T, \xi, \eta), v(T, \xi, \eta)) \neq (\lambda^{2/p} \xi, \lambda^{2(p-1)/p} \eta)\). If there is \( \lambda > 0 \) such that \((u(T, \xi, \eta), v(T, \xi, \eta)) = (\lambda^{2/p} \xi, \lambda^{2(p-1)/p} \eta)\),
then
\[
\left( p^{1/p} r(T)^{2/p} \left| \cos \theta(T) \right| \right)^{(2-p)/p} \cos \theta(T), \left( \frac{p}{p-1} \right)^{(p-1)/p} \\
\times r(T)^{2(p-1)/p} \left| \sin \theta(T) \right|^{(p-2)/p} \sin \theta(T) \right) \\
= \left( \lambda^{2/p} p^{1/p} r(0)^{2/p} \left| \cos \theta(0) \right| \right)^{(2-p)/p} \cos \theta(0), \lambda^{2(p-1)/p} \left( \frac{p}{p-1} \right)^{(p-1)/p} \\
\times r(0)^{2(p-1)/p} \left| \sin \theta(0) \right|^{(p-2)/p} \sin \theta(0) \right) .
\]

(2.51)

So
\[
\begin{align*}
\left| r(T) \right|^{2/p} \left| \cos \theta(T) \right|^{(2-p)/p} \cos \theta(T) &= \lambda^{2/p} r(0)^{2/p} \left| \cos \theta(0) \right| \cos \theta(0) ,
\end{align*}
\]

(2.52)

\[
\begin{align*}
\left| r(T) \right|^{2(p-1)/p} \left| \sin \theta(T) \right|^{(p-2)/p} \sin \theta(T) &= \lambda^{2(p-1)/p} r(0)^{2(p-1)/p} \left| \sin \theta(0) \right| \sin \theta(0) .
\end{align*}
\]

(2.53)

From (2.52) we have
\[
\begin{align*}
\left| r(T) \right|^{2/p} \left| \cos \theta(T) \right|^{2/p} \text{sgn} \cos \theta(T) &= \left( \lambda r(0) \right)^{2/p} \left| \cos \theta(0) \right|^{2/p} \text{sgn} \cos \theta(0) ,
\end{align*}
\]

(2.54)

so, \( \text{sgn} \cos \theta(T) = \text{sgn} \cos \theta(0) \), therefore,
\[
\begin{align*}
\left| r(T) \right|^{2/p} \cos \theta(T) &= \left( \lambda r(0) \right)^{2/p} \cos \theta(0) ,
\end{align*}
\]

(2.55)

Similarly from (2.53) one has
\[
\begin{align*}
\left| r(T) \right|^{2/p} \sin \theta(T) &= \lambda r(0) \sin \theta(0) .
\end{align*}
\]

(2.56)

So, from (2.55) and (2.56), we have
\[
\begin{align*}
r(T) = \lambda r(0), \quad (\cos \theta(T), \sin \theta(T)) &= (\cos \theta(0), \sin \theta(0)).
\end{align*}
\]

(2.57)

Therefore,
\[
\theta(T) = \theta(0) + 2k\pi \quad \text{or} \quad \theta(T) - \theta(0) = 2k\pi .
\]

(2.58)

However, from \( n\Delta t < T < (n+1)\Delta t \), we have
\[
\begin{align*}
\theta(T) - \theta(0) < \theta(n\Delta t) - \theta(0) &= -2n\pi ,
\end{align*}
\]

(2.59)

\[
\begin{align*}
\theta(T) - \theta(0) > \theta((n+1)\Delta t) - \theta(0) &= -2(n+1)\pi ,
\end{align*}
\]

(2.60)

since \( \theta' < 0 \). So there is no integer \( k \) such that \( \theta(T) - \theta(0) = 2k\pi \).

Therefore, the conclusion follows.
Theorem 2.4. Suppose (H1)-(H5) hold. Then (1.4) has at least one $T$-periodic solution $u(t)$.

Proof. By Lemma 2.3, we know that there exists $A > 0$ ($A \gg 1$) such that if

$$\frac{1}{p} |\xi|^p + \frac{p-1}{p} |\eta|^{p(p-1)} = A^2,$$

then

$$(u(T, \xi, \eta), v(T, \bar{T}, \eta)) \neq (\lambda^{2/p} \xi, \lambda^{2(p-1)/p} \eta) \quad \text{for} \quad \lambda > 0.$$  (2.62)

Assume that

$$\xi_1 = u(T, \xi, \eta), \quad \eta_1 = v(T, \bar{T}, \eta).$$  (2.63)

Consider a two-dimensional open region $D_A$ bounded by

$$D_A = \left\{ (\xi, \eta) : \frac{1}{p} |\xi|^p + \frac{p-1}{p} |\eta|^{p(p-1)} = A^2 \right\},$$  (2.64)

then we define a topological mapping

$$H : D_A \rightarrow \mathbb{R}^2, \quad (\xi, \eta) \mapsto (\xi_1, \eta_1).$$  (2.65)

It follows from Lemma 2.3 that

$$(\xi_1, \eta_1) \neq (\lambda^{2/p} \xi, \lambda^{2(p-1)/p} \eta), \quad (\xi, \eta) \in \partial D_A.$$  (2.66)

Now we define a homotopy $h : \overline{D_A} \times [0,1] \rightarrow \mathbb{R}^2$ by

$$h(\xi, \eta, \mu) = -\left( \frac{\mu^{2/p}}{\mu^{2(p-1)/p}} \right) I(\xi, \eta) + \left( \begin{array}{c} 0 \\ \mu^{2(p-1)/p} \end{array} \right) H(\xi, \eta),$$  (2.67)

for $\mu \in [0,1]$. It is easy to see that $h(\xi, \eta, 0), h(\xi, \eta, 1) \neq 0$ for $(\xi, \eta) \in \partial D_A$. Then we show that $h(\xi, \eta, \mu) \neq 0$ for $(\xi, \eta) \in \partial D_A$, where $\mu \in (0,1)$. If not, there is $\mu_0 \in (0,1), (\xi, \eta) \in \partial D_A$ such that $h(\xi, \eta, \mu_0) = 0$, that is,

$$(\xi_1, \eta_1) = \left( \left( \frac{\mu}{1-\mu} \right)^{2/p} \xi, \left( \frac{\mu}{1-\mu} \right)^{2(p-1)/p} \eta \right),$$  (2.68)

which is impossible. So $h(\xi, \eta, \mu) \neq 0$ for $\mu \in [0,1]$.

Then, $\deg\{h(\xi, \eta, 0), D_A, 0\} = \deg\{h(\xi, \eta, 1), D_A, 0\}$, that is,

$$\deg\{H, D_A, 0\} = \deg\{-I_D, D_A, 0\} \neq 0.$$  (2.69)

Therefore, $H$ has at least one fixed point $(\xi^*, \eta^*) \in D_A$. It is easy to see that $u(t) = u(t, \xi^*, \eta^*)$ is a $T$-periodic solution of (1.4). \qed
If we let \( \phi(u) = \varphi_p(u) = |u|^{p-2}u, \ p > 2 \), then we have the following special cases of (1.4):
\[
(\varphi_p(u'))' + f(u,u')u' + g(u) = p(t,u,u') \quad t \in [0,T],
\]
(2.70)
so we can easily get the following results.

**Theorem 2.5.** Assume (H2) and (H3) hold and solutions of (2.70) are unique with respect to initial value, moreover suppose that there exist \( \lambda, \mu, a, n \) such that
\[
\left(\frac{p'}{p'-1}\right)^{p-1}\left(\frac{2n\pi p}{T}\right)^p + \alpha_1 + \frac{p-1}{p} \alpha_2^{p/p-1}
\]
\[
< \lambda \leq \frac{g(x)}{\phi_p(x)} \leq \mu < \left(\frac{p'}{p'+1}\right)^{p-1}\left(\frac{2(n+1)\pi p}{T}\right)^p - \alpha_1 - \frac{p-1}{p} \alpha_2^{p/p-1},
\]
(2.71)
then (2.70) has at least one \( T \)-periodic solution.

**3. Example**

In this section, we present an example to illustrate our main results. Consider the following differential equation:
\[
(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T],
\]
(3.1)
where
\[
\phi(x) = |x|(x + \sin x), \quad f(x,y) = |y|^{3/4} + a, \quad a > 0, \quad g(x) = 2\phi(x),
\]
(3.2)
e(t,x,y) = -\frac{2}{3}|x|x - |y|^{3/4}y + b\cos 2\pi t, \quad b > 0.

We claim that
\[
\frac{2}{3}|x|^2 \leq |\phi(x)| \leq 2|x|^2.
\]
(3.3)
In fact, if \( x \neq 0 \), we have
\[
|\phi(x)| = |x|\left|1 + \frac{\sin x}{x}\right| > |x|^2\left(1 - \frac{1}{\pi}\right) > \frac{2}{3}|x|^2,
\]
(3.4)
so (3.3) holds. Therefore, \( p = 3, m_1 = 2/3, m_2 = 2 \). Also, we can get \( \alpha_1 = 2/3, \beta_1 = 1, \gamma_1 = b, \alpha_2 = 0, \beta_2 = 1, \gamma_2 = a, k = 11/4. \)
Let \( n = 0 \) and \( T = 1 \), then conditions (H1)–(H4) are satisfied.
Now, we check that condition (H5) is satisfied.
Suppose that \( x_1(t) \) and \( x_2(t) \) are two different solutions to (3.1) satisfying
\[
x_1(t_0) = x_2(t_0) = x_0, \quad x_1'(t_0) = x_2'(t_0) = x_0'.
\]
(3.5)
16 Periodic solutions for Liénard equations

Let \( y = \phi(x') \), then \((x_i(t), y_i(t)) = (x_i(t), \phi(x'_i(t)))\) \((i = 1, 2)\) are two different solutions to the system

\[
\begin{align*}
  x' &= \phi^{-1}(y), \\
  y' &= -g(x) - f(x, \phi^{-1}(y)) \phi^{-1}(y) + e(t, x, \phi^{-1}(y)),
\end{align*}
\]

satisfying \((x_i(t_0), y_i(t_0)) = (x_0, \phi(x'(t_0)))\) \((i = 1, 2)\).

Without loss of generality, we assume that there exists \( t_1 > t_0 \) such that

\[
x_2(t) > x_1(t), \quad t \in (t_0, t_1].
\]

As \( x_1(t_0) = x_2(t_0) = x_0, x'_1(t_0) = x'_2(t_0) = x'_0 \), and \( x_i \in \mathbb{C}^2[t_0, t_1] \), so there exists \( t^* \in (t_0, t_1) \) such that

\[
x'_2(t) > x'_1(t), \quad t \in (t_0, t^*].
\]

Therefore, for \( t \in (t_0, t^*] \), we have

\[
y_2(t) - y_1(t) = -\int_{t_0}^{t} \left\{ \left[ g(x_2(s)) - g(x_1(s)) \right] + \left[ f(x_2(s), x'_2(s)) x'_2(s) - f(x_1(s), x'_1(s)) x'_1(s) \right] \\
- \left[ e(s, x_2(s), x'_2(s)) - e(s, x_1(s), x'_1(s)) \right] \right\} ds
= -\int_{t_0}^{t} \left\{ 2\left[ \phi(x_2(s)) - \phi(x_1(s)) \right] + 2\left[ \left| x'_2(s) \right|^{3/4} x'_2(s) - \left| x'_1(s) \right|^{3/4} x'_1(s) \right] \\
+ a(x'_2(s) - x'_1(s)) + \frac{2}{3} \left[ \left| x'_2(s) \right| x'_2(s) - \left| x'_1(s) \right| x'_1(s) \right] \right\} ds < 0.
\]

That is,

\[
\phi(x'_2(t)) - \phi(x'_1(t)) < 0, \quad t \in (t_0, t^*].
\]

So, \( x'_2(t) < x'_1(t), t \in (t_0, t^*] \), this is a contradiction.

Therefore, by Theorem 2.4, we can conclude that (3.1) has at least one 1-periodic solution.

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