PERIODIC SOLUTIONS OF SECOND-ORDER LIÉNARD EQUATIONS WITH p-LAPLACIAN-LIKE OPERATORS

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The existence of periodic solutions for second-order Liénard equations with p-Laplacian-like operator is studied by applying new generalization of polar coordinates.

1. Introduction

In recent years, the existence of periodic solutions for second-order Liénard equations

$$u'' + f(u,u')u' + g(u) = e(t,u,u') \tag{1.1}$$

and its special case have been studied by many researchers, we refer the readers to [1, 3, 4, 6, 7, 9–12] and the references therein.

Let us consider the so-called one-dimensional p-Laplacian operator $(\phi_p(u'))'$, where $p > 1$ and $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0) = 0$. Periodic boundary conditions containing this operator have been considered in [2, 5].

In [8], Manásevich and Mawhin investigated the existence of periodic solutions to some system cases involving the fairly general vector-valued operator ϕ. They considerd the boundary value problem

$$\left(\phi(u')\right)' = f(t,u,u'), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{1.2}$$

where the function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies some monotonicity conditions which ensure that ϕ is a homeomorphism onto $\mathbb{R}^N$.

Recently, in [16] we studied the existence of periodic solutions for the nonlinear differential equation with a p-Laplacian-like operator

$$\left(\phi(u')\right)' + f(t,u,u') = 0. \tag{1.3}$$

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Motivated by the work of [13], in this paper we use new polar coordinates [13] to investigate the existence of periodic solutions for the second-order generalized Liénard equations with p-Laplacian-like operator

$$(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T]. \tag{1.4}$$

Throughout this paper, we always assume that ϕ, $g \in \mathbb{C}(\mathbb{R},\mathbb{R})$, $f \in \mathbb{C}(\mathbb{R}^2,\mathbb{R})$, $e \in \mathbb{C}([0,T]\times\mathbb{R}^2,\mathbb{R})$. And the following conditions also hold.

(H1) ϕ is continuous and strictly increasing, $y\phi(y) > 0$ for $y \neq 0$, and there exist $p > 2$, $m_2 \geq m_1 > 0$, such that

$$m_1|y|^{p-1} \leq |\phi(y)| \leq m_2|y|^{p-1}. \tag{1.5}$$

(H2) $e \in \mathbb{C}([0,T]\times\mathbb{R}^2,\mathbb{R})$, periodic in t with period T, there exist $\alpha_1,\beta_1,\gamma_1 > 0$, and $p > k > 2$ such that

$$|e(t,x,y)| \leq \alpha_1|x|^{p-1} + \beta_1|y|^{k-1} + \gamma_1 \quad \text{for } (t,x,y) \in [0,T]\times\mathbb{R}^2. \tag{1.6}$$

(H3) $f \in \mathbb{C}(\mathbb{R}^2,\mathbb{R})$, there exist $\alpha_2,\beta_2,\gamma_2 > 0$ such that

$$|f(x,y)| \leq \alpha_2|x|^{p-2} + \beta_2|y|^{k-2} + \gamma_2 \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{1.7}$$

(H4) There exist λ,μ, and $n \geq 0$ such that

$$\begin{aligned} &\frac{m_2}{m_1}\left(\frac{p'}{p'-1}\right)^{p-1}\left(\frac{2n\pi_p}{T}\right)^p + \frac{\alpha_1}{m_1} + \frac{p-1}{p}\left(\frac{\alpha_2}{m_1}\right)^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)^2} < \lambda \\ &\quad \leq \frac{g(x)}{\phi(x)} \leq \mu < \frac{m_1}{m_2}\left(\frac{p'}{p'+1}\right)^{p-1}\left(\frac{2(n+1)\pi_p}{T}\right)^p \\ &\quad - \frac{\alpha_1}{m_2} - \frac{p-1}{p}\left(\frac{\alpha_2}{m_2}\right)^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{1/(p-1)}, \end{aligned} \tag{1.8}$$

where

$$p' = p(p-1), \qquad \pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}. \tag{1.9}$$

(H5) Solutions of (1.4) are unique with respect to initial value.

In this paper, we use a new coordinate to estimate the time when a point moves along a trajectory around the origin and then give some sufficient conditions for the existence of periodic solutions of (1.4).

2. Periodic solutions with a Laplacian-like operator

Let $v = \phi(u')$. Then (1.4) is equivalent to the system

$$\begin{aligned} u' &= \phi^{-1}(v), \\ v' &= -g(u) - f(u,\phi^{-1}(v))\phi^{-1}(v) + e(t,u,\phi^{-1}(v)). \end{aligned} \tag{2.1}$$

Let $u(t,\xi,\eta)$ denote the solution of (1.4) which satisfies the initial value condition

$$u(0,\xi,\eta)=\xi, \qquad v(0,\xi,\eta)=\eta, \tag{2.2}$$

then we have the following conclusion.

LEMMA 2.1. *Suppose (H1)–(H5) hold, then for all $c>0$, there exists constant $A>0$ such that if*

$$\frac{1}{p}|\xi|^p+\frac{p-1}{p}|\eta|^{p/(p-1)}=A^2, \tag{2.3}$$

then

$$\frac{1}{p}\left|u(t,\xi,\eta)\right|^p+\frac{p-1}{p}\left|v(t,\xi,\eta)\right|^{p/(p-1)}\geq c^2 \quad \text{for } t\in[0,T]. \tag{2.4}$$

Proof. Let $(u(t),v(t))$, $t\in[0,T]$, be a solution of (2.1) satisfying $u(0,\xi,\eta)=\xi$, $v(0,\xi,\eta)=\eta$.

Let

$$r^2(t)=\frac{1}{p}\left|u(t)\right|^p+\frac{p-1}{p}\left|v(t)\right|^{p/(p-1)}. \tag{2.5}$$

It is clear that (H1) implies

$$\left(\frac{|v|}{m_2}\right)^{1/(p-1)}\leq\left|\phi^{-1}(v)\right|\leq\left(\frac{|v|}{m_1}\right)^{1/(p-1)}. \tag{2.6}$$

So we have

$$\begin{aligned}
\left|\frac{dr^2(t)}{dt}\right| &= \left|\left|u(t)\right|^{p-2}u(t)u'(t)+\left|v(t)\right|^{(2-p)/(p-1)}v(t)v'(t)\right| \\
&\leq |u|^{p-1}\left|\phi^{-1}(v)\right|+|v|^{1/(p-1)}\left|-g(u)-f\left(u,\phi^{-1}(v)\right)\phi^{-1}(v)+e\left(t,u,\phi^{-1}(v)\right)\right| \\
&\leq |u|^{p-1}\left|\phi^{-1}(v)\right|+\mu|v|^{1/(p-1)}\left|\phi(u)\right| \\
&\quad +|v|^{1/(p-1)}\left(\alpha_2|u|^{p-2}+\beta_2\left|\phi^{-1}(v)\right|^{k-2}+\gamma_2\right)\left|\phi^{-1}(v)\right| \\
&\quad +|v|^{1/(p-1)}\left(\alpha_1|u|^{p-1}+\beta_1\left|\phi^{-1}(v)\right|^{k-1}+\gamma_1\right) \\
&\leq |u|^{p-1}\left(\frac{|v|}{m_1}\right)^{1/(p-1)}+\mu m_2|v|^{1/(p-1)}|u|^{p-1} \\
&\quad +\alpha_2 m_1^{-1/(p-1)}|v|^{2/(p-1)}|u|^{p-2}+\beta_2 m_1^{(1-k)/(p-1)}|v|^{k/(p-1)} \\
&\quad +\gamma_2 m_1^{-1/(p-1)}|v|^{2/(p-1)}+\alpha_1|v|^{1/(p-1)}|u|^{p-1} \\
&\quad +\beta_1 m_1^{(1-k)/(p-1)}|v|^{k/(p-1)}+\gamma_1|v|^{1/(p-1)} \\
&= l_1|u|^{p-1}|v|^{1/(p-1)}+l_2|v|^{k/(p-1)}+l_3|v|^{2/(p-1)}|u|^{p-2}+l_4|v|^{2/(p-1)}+\gamma_1|v|^{1/(p-1)},
\end{aligned} \tag{2.7}$$

where

$$l_1 = m_1^{-1/(p-1)} + \mu m_2 + \alpha_1, \qquad l_2 = \beta_1 m_1^{(1-k)/(p-1)} + \beta_2 m_1^{(1-k)/(p-1)},$$
$$l_3 = \alpha_2 m_1^{-1/(p-1)}, \qquad l_4 = \gamma_2 m_1^{-1/(p-1)}, \tag{2.8}$$

while

$$\begin{aligned}
l_1|u|^{p-1}|v|^{1/(p-1)} &\le l_1\left(\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}|u|^p\right)\\
&\le l_1 \max\left\{p-1, \frac{1}{p-1}\right\}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right)\\
&= l_1 \max\left\{p-1, \frac{1}{p-1}\right\} r^2,\\
l_2|v|^{k/(p-1)} &\le \frac{k}{p}|v|^{p/(p-1)} + \frac{p-k}{p} l_2^{p/(p-k)} \le \frac{k}{p-1}r^2 + \frac{p-k}{p} l_2^{p/(p-k)}\\
l_3|v|^{2/(p-1)}|u|^{p-2} &\le l_3\left(\frac{2}{p}|v|^{p/(p-1)} + \frac{p-2}{p}|u|^p\right) \le l_3\left(\frac{2}{p-1} + p - 2\right) r^2,\\
l_4|v|^{2/(p-1)} &\le \frac{2}{p}|v|^{p/(p-1)} + \frac{p-2}{p} l_4^{p/(p-2)} \le \frac{2}{p-1}r^2 + \frac{p-2}{p} l_4^{p/(p-2)},\\
\gamma_1|v|^{1/(p-1)} &\le \frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\gamma_1^{p/(p-1)} \le \frac{1}{p-1}r^2 + \frac{p-1}{p}\gamma_1^{p/(p-1)}.
\end{aligned} \tag{2.9}$$

So,

$$\left|\frac{dr^2(t)}{dt}\right| \le br^2(t) + a, \tag{2.10}$$

where

$$a = \frac{p-k}{p} l_2^{p/(p-k)} + \frac{p-2}{p} l_4^{p/(p-2)} + \frac{p-1}{p}\gamma_1^{p/(p-1)},$$
$$b = l_1 \max\left\{p-1, \frac{1}{p-1}\right\} + l_3\left(\frac{2}{p-1} + p - 2\right) + \frac{k+3}{p-1}. \tag{2.11}$$

It follows that

$$\begin{aligned}
\left(r^2(0) + \frac{a}{b}\right)e^{-bT} &\le \left(r^2(0) + \frac{a}{b}\right)e^{-bt} \le \left(r^2(t) + \frac{a}{b}\right)\\
&\le \left(r^2(0) + \frac{a}{b}\right)e^{bt} \le \left(r^2(0) + \frac{a}{b}\right)e^{bT}, \quad 0 \le t \le T.
\end{aligned} \tag{2.12}$$

Let $A = [(c^2 + a/b)e^{bT} - a/b]^{1/2}$, then $r(0) = A$ implies $r(t) \ge c$. □

Lemma 2.2. *Let $(u(t),v(t))$ be a solution of (2.1). Suppose the conditions of (H1)–(H5) are satisfied. Then there is R such that under the generalized polar coordinates, $r(0)\geq R$ implies that*

$$\frac{d\theta(t)}{dt}\leq 0,\quad t\in[0,T]. \tag{2.13}$$

Proof. Applying generalized polar coordinates,

$$\begin{aligned} u &= p^{1/p}r^{2/p}|\cos\theta|^{(2-p)/p}\cos\theta,\\ v &= \left(\frac{p}{p-1}\right)^{(p-1)/p} r^{2(p-1)/p}|\sin\theta|^{(p-2)/p}\sin\theta, \end{aligned} \tag{2.14}$$

or

$$\begin{aligned} r\cos\theta &= \frac{1}{\sqrt{p}}|u|^{(p-2)/2}u,\\ r\sin\theta &= \sqrt{\frac{p-1}{p}}|v|^{(2-p)/2(p-1)}v. \end{aligned} \tag{2.15}$$

Then $\theta=\tan^{-1}[\sqrt{p-1}(|v|^{((2-p)/2(p-1))}v/|u|^{((p-2)/2)}u)]$. So we have

$$\begin{aligned} \theta' &= \frac{|u|^{((p-2)/2)}|v|^{((2-p)/2(p-1))}}{2\sqrt{p-1}r^2}[uv'-(p-1)u'v]\\ &= -\frac{|u|^{((p-2)/2)}|v|^{((2-p)/2(p-1))}}{2\sqrt{p-1}r^2}\Big[ug(u)+uf(u,\phi^{-1}(v))\phi^{-1}(v)\\ &\qquad +(p-1)v\phi^{-1}(v)-ue(t,u,\phi^{-1}(v))\Big] \end{aligned} \tag{2.16}$$

as

$$\begin{aligned} &ug(u)+uf(u,\phi^{-1}(v))\phi^{-1}(v)+(p-1)v\phi^{-1}(v)-ue(t,u,\phi^{-1}(v))\\ &\quad\geq \lambda u\phi(u)+(p-1)v\phi^{-1}(v)-|u|\left(\alpha_2|u|^{p-2}+\beta_2\left|\phi^{-1}(v)\right|^{k-2}+\gamma_2\right)\left|\phi^{-1}(v)\right|\\ &\qquad -|u|\left(\alpha_1|u|^{p-1}+\beta_1\left|\phi^{-1}(v)\right|^{k-1}+\gamma_1\right)\\ &\quad\geq \lambda m_1|u|^p+(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)}-\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)}\\ &\qquad -\gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)}-\alpha_1|u|^p-(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|\\ &\quad= (\lambda m_1-\alpha_1)|u|^p+(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)}-\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)}\\ &\qquad -\gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)}-(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|. \end{aligned} \tag{2.17}$$

Let

$$\tau = \frac{p(p-1)}{4(k-1)} m_2^{-1/(p-1)}, \qquad \beta' = \frac{4(\beta_1+\beta_2)(k-1)}{p(p-1)} m_1^{(1-k)/(p-1)} m_2^{1/(p-1)}, \tag{2.18}$$

so we have

$$\begin{aligned}
&(\beta_1+\beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} \\
&\quad = \tau|u|\left(|v|^{(k-1)/(p-1)}\beta'\right) \le \tau|u|\left(\frac{k-1}{p-1}|v| + \frac{p-k}{p-1}\beta'^{(p-1)/(p-k)}\right) \\
&\quad = \frac{1}{4} p m_2^{-1/(p-1)} |u||v| + \frac{p(p-k)}{4(k-1)} m_2^{-1/(p-1)} \beta'^{(p-1)/(p-k)} |u| \\
&\quad \le \frac{1}{4} p m_2^{-1/(p-1)} \left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) + \frac{p(p-k)}{4(k-1)} m_2^{-1/(p-1)} \beta'^{(p-1)/(p-k)} |u|.
\end{aligned} \tag{2.19}$$

Let

$$\tau_1 = \frac{1}{4} p(p-1) m_2^{-1/(p-1)}, \qquad \beta_1' = \frac{4\gamma_2}{p(p-1)} \left(\frac{m_2}{m_1}\right)^{1/(p-1)}, \tag{2.20}$$

then

$$\begin{aligned}
\gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} &= \tau_1 |u|\left(|v|^{1/(p-1)}\beta_1'\right) \\
&\le \tau_1|u|\left(\frac{1}{p-1}|v| + \frac{p-2}{p-1}\beta_1'^{(p-1)/(p-2)}\right) \\
&= \frac{1}{4} p m_2^{-1/(p-1)} |u||v| + \frac{p(p-2)}{4} m_2^{-1/(p-1)} \beta_1'^{(p-1)/(p-2)} |u| \\
&\le \frac{1}{4} p m_2^{-1/(p-1)} \left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) \\
&\qquad + \frac{p(p-2)}{4} m_2^{-1/(p-1)} \beta_1'^{(p-1)/(p-2)} |u|.
\end{aligned} \tag{2.21}$$

Let

$$\tau_2 = \frac{1}{4} p(p-1) m_2^{-1/(p-1)}, \qquad \beta_2' = \frac{4\alpha_2}{p(p-1)} \left(\frac{m_2}{m_1}\right)^{1/(p-1)} \tag{2.22}$$

then

$$\begin{aligned} &\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} \\ &\quad = \tau_2\left(|v|^{1/(p-1)}\beta_2'|u|^{p-1}\right) \le \tau_2\left(\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\left(\beta_2'|u|^{p-1}\right)^{p/(p-1)}\right) \\ &\quad \le \frac{1}{4}pm_2^{-1/(p-1)}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{p/(p-1)}\right) + \frac{p-1}{p}\tau_2\beta_2'^{\,p/(p-1)}|u|^p. \end{aligned} \tag{2.23}$$

We select λ large enough such that

$$\delta = \lambda m_1 - \alpha_1 - \frac{p-1}{p}\tau_2\beta_2'^{\,p/(p-1)} - m_2^{-1/(p-1)} > 0, \tag{2.24}$$

Let $d = \gamma_1 + (p(p-k)/4(k-1))m_2^{-1/(p-1)}\beta'^{(p-1)/(p-k)} + (p(p-2)/4)m_2^{-1/(p-1)}\beta_1'^{(p-1)/(p-2)}$, we also have

$$d|u| = \delta p|u|\left(\frac{d}{\delta p}\right) \le \delta|u|^p + (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)}, \tag{2.25}$$

therefore

$$\begin{aligned} &ug(u) + uf\left(u,\phi^{-1}(v)\right)\phi^{-1}(v) + (p-1)v\phi^{-1}(v) - ue\left(t,u,\phi^{-1}(v)\right) \\ &\quad \ge \frac{1}{4}pm_2^{-1/(p-1)}\left[\frac{1}{p}|u|^P p + \frac{p-1}{p}|v|^{p/(p-1)}\right] - (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)} \\ &\quad = \frac{1}{4}pm_2^{-1/(p-1)}r^2(t) - (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)}. \end{aligned} \tag{2.26}$$

Lemma 2.1 implies that there is $\mathbb{R} > 0$, such that

$$\frac{1}{4}pm_2^{-1/(p-1)}r^2(t) > (p-1)\delta\left(\frac{d}{p\delta}\right)^{p/(p-1)} \tag{2.27}$$

when $r(0) > \mathbb{R}$, then our assertion is verified. □

Lemma 2.3. *Assume that (H1)–(H5) hold, and*

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2 \quad (A \gg 1) \tag{2.28}$$

then

$$(u(T,\xi,\eta),v(T,\xi,\eta)) \neq (\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta), \tag{2.29}$$

where λ is an arbitrary positive number.

Proof. It follows from Lemma 2.1 that if

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2, \tag{2.30}$$

then

$$\frac{1}{p}\left|u(t,\xi,\eta)\right|^p + \frac{p-1}{p}\left|v(t,\xi,\eta)\right|^{p/(p-1)} \geq c^2 \quad \text{for } t \in [0,T]. \tag{2.31}$$

According to the generalized polar coordinates (2.14), we have

$$r(t) \geq c \quad \text{for } t \in [0,T] \text{ if } r(0) = A. \tag{2.32}$$

On the other hand, when $r(0) \to \infty$, it holds uniformly from (H1)–(H3) that

$$\begin{aligned}
-\theta' &= \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[ug(u) + uf(u,\phi^{-1}(v))\phi^{-1}(v) \\
&\qquad + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v))\Big] \\
&\geq \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[(\lambda m_1 - \alpha_1)|u|^p + (p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)} \\
&\qquad - \alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} - \gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)} \\
&\qquad - (\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)} - \gamma_1|u|\Big]
\end{aligned} \tag{2.33}$$

as

$$\begin{aligned}
&\alpha_2 m_1^{-1/(p-1)}|u|^{p-1}|v|^{1/(p-1)} \\
&\quad = m_2^{-1/(p-1)}\left(|v|^{1/(p-1)}\right)\left[\alpha_2\left(\frac{m_2}{m_1}\right)^{1/(p-1)}|u|^{p-1}\right] \\
&\quad \leq m_2^{-1/(p-1)}\left[\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)}\left(\frac{m_2}{m_1}\right)^{p/(p-1)^2}|u|^p\right] \\
&\quad = \frac{1}{p}m_2^{-1/(p-1)}|v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-p/(p-1)^2}m_2^{1/(p-1)^2}|u|^p.
\end{aligned} \tag{2.34}$$

So

$$
\begin{aligned}
-\theta' \geq{}& \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Bigg[(\lambda m_1-\alpha_1-\tilde{\alpha})|u|^p+\frac{p'-1}{p'}(p-1)m_2^{-1/(p-1)}|v|^{p/(p-1)}\\
&\qquad - \gamma_2 m_1^{-1/(p-1)}|u||v|^{1/(p-1)}\\
&\qquad - (\beta_1+\beta_2)m_1^{(1-k)/(p-1)}|u||v|^{(k-1)/(p-1)}-\gamma_1|u|\Bigg]\\
={}& \frac{p|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}{2(p-1)^{1/p}}\left[(\lambda m_1-\alpha_1-\tilde{\alpha})\cos^2\theta+\frac{p'-1}{p'}m_2^{-1/(p-1)}\sin^2\theta\right]\\
&- \frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&- \frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p}\\
&- \frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p}\\
={}& a_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&- \frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&- \frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p}\\
&- \frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p},
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{gathered}
\tilde{\alpha}=\frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-p/(p-1)^2}m_2^{1/(p-1)^2},\qquad p'=p(p-1),\\
a_1=\frac{p(p'-1)}{2p'(p-1)^{1/p}m_2^{1/(p-1)}},\qquad b_1=\frac{p'}{p'-1}(\lambda m_1-\alpha_1-\tilde{\alpha})m_2^{1/(p-1)}.
\end{gathered}
\tag{2.36}
$$

Denote $\hat{b}=\min\{b_1,1\}$, then we have

$$
\begin{aligned}
-\theta' \geq{}& a_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&- \frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\hat{b}(p-1)^{2/p}r^{2(p-2)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\cos\theta||\sin\theta|^{(4-p)/p}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\left(\beta_1+\beta_2\right)m_1^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&-\frac{\gamma_1 p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
=\;&\widehat{a}_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p},
\end{aligned}
\tag{2.37}
$$

where

$$
\widehat{a}_1=a_1-\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\widehat{b}(p-1)^{2/p}r^{2(p-2)/p}}-\frac{(\beta_1+\beta_2)m_2^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}-\frac{\gamma_1 p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}.
\tag{2.38}
$$

Assume that it takes time Δt for the motion $(r(t),\theta(t))(r(0)=A,\ \theta(0)=\theta_0)$ to complete one cycle around the origin. It follows from the above inequality that

$$
\begin{aligned}
\Delta t&<\int_{\theta_0}^{\theta_0+2\pi}\frac{d\theta}{\widehat{a}_1\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}\\
&=\frac{4}{\widehat{a}_1}\int_0^{\pi/2}\frac{d\theta}{\left(b_1\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}.
\end{aligned}
\tag{2.39}
$$

Let

$$
\eta=\tan^{-1}\frac{1}{\sqrt{b_1}}\tan\theta,
\tag{2.40}
$$

then

$$
\Delta t<\frac{4}{\widehat{a}_1 b_1^{1/p}}\int_0^{\pi/2}\frac{d\eta}{|\tan\eta|^{(2-p)/p}}=\frac{2}{\widehat{a}_1 b_1^{1/p}}B\left(\frac{1}{p},\frac{p-1}{p}\right)=\frac{2\pi}{\widehat{a}_1 b_1^{1/p}\sin(\pi/p)},
\tag{2.41}
$$

from (H4), we have

$$
a_1 b_1^{1/p}\sin\frac{\pi}{p}=\frac{\pi}{\pi_p}\left(\frac{p'-1}{p'}\right)^{(p-1)/p}\left(\frac{\lambda m_1-\alpha_1-\widetilde{\alpha}}{m_2}\right)^{1/p}>\frac{2n\pi}{T}.
\tag{2.42}
$$

So there exists $\sigma>0$ such that $(a_1-\sigma)b_1^{1/p}\sin(\pi/p)>2n\pi/T$. For the $\sigma>0$, there exists $\mathbb{R}'>0$ such that

$$
0<\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2\widehat{b}(p-1)^{2/p}r^{2(p-2)/p}}+\frac{(\beta_1+\beta_2)m_2^{(1-k)/(p-1)}p^{k/p}}{2\widehat{b}(p-1)^{k/p}r^{2(p-k)/p}}+\frac{\gamma p^{1/p}}{2\widehat{b}(p-1)^{1/p}r^{2(p-1)/p}}<\sigma
\tag{2.43}
$$

for $A > \mathbb{R}'$ large enough. So we have

$$\hat{a}_1 b_1^{1/p} \sin\frac{\pi}{p} = \left(a_1 - \frac{\gamma_2 m_1^{-1/(p-1)} p^{2/p}}{2\hat{b}(p-1)^{2/p} r^{2(p-2)/p}} - \frac{(\beta_1+\beta_2) m_2^{(1-k)/(p-1)} p^{k/p}}{2\hat{b}(p-1)^{k/p} r^{2(p-k)/p}} \right.$$
$$\left. - \frac{\gamma p^{1/p}}{2\hat{b}(p-1)^{1/p} r^{2(p-1)/p}}\right) b_1^{1/p} \sin\frac{\pi}{p} > (a_1 - \sigma) b_1^{1/p} \sin\frac{\pi}{p} > \frac{2n\pi}{T}. \tag{2.44}$$

Therefore

$$\frac{T}{\Delta t} > n \tag{2.45}$$

as

$$\begin{aligned}\alpha_2 m_1^{-1/(p-1)} |u|^{p-1} |v|^{1/(p-1)} &= m_1^{-1/(p-1)} \left(|v|^{1/(p-1)}\right)\left(\alpha_2 |u|^{p-1}\right) \\ &\leq m_1^{-1/(p-1)} \left[\frac{1}{p}|v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)} |u|^p\right] \\ &= \frac{1}{p} m_1^{-1/(p-1)} |v|^{p/(p-1)} + \frac{p-1}{p}\alpha_2^{p/(p-1)} m_1^{-1/(p-1)} |u|^p.\end{aligned} \tag{2.46}$$

Similarly, we have

$$\begin{aligned}0 < -\theta' &= \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[ug(u) + uf\left(u,\phi^{-1}(v)\right)\phi^{-1}(v) + (p-1)v\phi^{-1}(v) \\ &\qquad - ue\left(t,u,\phi^{-1}(v)\right)\Big] \\ &\leq \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[(\mu m_2 + \alpha_1)|u|^p + (p-1) m_1^{-1/(p-1)} |v|^{p/(p-1)} \\ &\qquad + \alpha_2 m_1^{-1/(p-1)} |u|^{p-1}|v|^{1/(p-1)} + \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\ &\qquad + (\beta_1+\beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} + \gamma_1|u|\Big] \\ &\leq \frac{|u|^{(p-2)/2}|v|^{(2-p)/2(p-1)}}{2\sqrt{p-1}r^2}\Big[(\mu m_2 + \alpha_1 + \tilde{\alpha}')|u|^p + \frac{p'+1}{p'}(p-1) m_1^{-1/(p-1)} |v|^{p/(p-1)} \\ &\qquad + \gamma_2 m_1^{-1/(p-1)} |u||v|^{1/(p-1)} \\ &\qquad + (\beta_1+\beta_2) m_1^{(1-k)/(p-1)} |u||v|^{(k-1)/(p-1)} + \gamma_1|u|\Big]\end{aligned}$$

$$
\begin{aligned}
&= \frac{p|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}}{2(p-1)^{1/p}}\left[(\mu m_2+\alpha_1+\widetilde{\alpha}')\cos^2\theta+\frac{p'+1}{p'}m_1^{-1/(p-1)}\sin^2\theta\right]\\
&\quad+\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&\quad+\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-P)/p}\\
&\quad+\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p}\\
&= a_2\left(b_2\cos^2\theta+\sin^2\theta\right)|\sin\theta|^{(2-p)/p}|\cos\theta|^{(p-2)/p}\\
&\quad+\frac{\gamma_2 m_1^{-1/(p-1)}p^{2/p}}{2(p-1)^{2/p}r^{2(p-2)/p}}|\cos\theta||\sin\theta|^{(4-p)/p}\\
&\quad+\frac{(\beta_1+\beta_2)m_1^{(1-k)/(p-1)}p^{k/p}}{2(p-1)^{k/p}r^{2(p-k)/p}}|\cos\theta||\sin\theta|^{(2k-p)/p}\\
&\quad+\frac{\gamma_1 p^{1/p}}{2(p-1)^{1/p}r^{2(p-1)/p}}|\cos\theta||\sin\theta|^{(2-p)/p},
\end{aligned}
\tag{2.47}
$$

where

$$
\begin{gathered}
\widetilde{\alpha}'=\frac{p-1}{p}\alpha_2^{p/(p-1)}m_1^{-1/(p-1)},\qquad a_2=\frac{p(p'+1)}{2p'(p-1)^{1/p}m_1^{1/(p-1)}},\\
b_2=\frac{p'}{p'+1}(\mu m_2+\alpha_1+\widetilde{\alpha}')m_1^{1/(p-1)},
\end{gathered}
\tag{2.48}
$$

with the similar argument, we also get

$$\frac{T}{\Delta t}<n+1. \tag{2.49}$$

Then it holds that

$$n<\frac{T}{\Delta t}<n+1. \tag{2.50}$$

To finish the proof, we claim that If $n<T/\Delta t<n+1$, then $(u(T,\xi,\eta),v(T,\xi,\eta))\neq(\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta)$. If there is $\lambda>0$ such that $(u(T,\xi,\eta),v(T,\xi,\eta))=(\lambda^{2/p}\xi,\lambda^{2(p-1)/p}\eta)$, then

$$\begin{aligned}&\left(p^{1/p}r(T)^{2/p}\,|\cos\theta(T)|^{(2-p)/p}\cos\theta(T),\left(\frac{p}{p-1}\right)^{(p-1)/p}\right.\\&\left.\times r(T)^{2(p-1)/p}\,|\sin\theta(T)|^{(p-2)/p}\sin\theta(T)\right)\\&\quad=\left(\lambda^{2/p}p^{1/p}r(0)^{2/p}\,|\cos\theta(0)|^{(2-p)/p}\cos\theta(0),\lambda^{2(p-1)/p}\left(\frac{p}{p-1}\right)^{(p-1)/p}\right.\\&\qquad\left.\times r(0)^{2(p-1)/p}\,|\sin\theta(0)|^{(p-2)/p}\sin\theta(0)\right).\end{aligned}\tag{2.51}$$

So

$$r(T)^{2/p}\,|\cos\theta(T)|^{(2-p)/p}\cos\theta(T)=\lambda^{2/p}r(0)^{2/p}\,|\cos\theta(0)|^{(2-p)/p}\cos\theta(0),\tag{2.52}$$

$$r(T)^{2(p-1)/p}\,|\sin\theta(T)|^{(p-2/p}\sin\theta(T)=\lambda^{2(p-1)/p}r(0)^{2(p-1)/p}\,|\sin\theta(0)|^{(p-2/p}\sin\theta(0).\tag{2.53}$$

From (2.52) we have

$$r(T)^{2/p}\,|\cos\theta(T)|^{2/p}\operatorname{sgn}\cos\theta(T)=(\lambda r(0))^{2/p}\,|\cos\theta(0)|^{2/p}\operatorname{sgn}\cos\theta(0),\tag{2.54}$$

so, $\operatorname{sgn}\cos\theta(T)=\operatorname{sgn}\cos\theta(0)$, therefore, $r(T)^{2/p}|\cos\theta(T)|^{2/p}=(\lambda r(0))^{2/p}|\cos\theta(0)|^{2/p}$, moreover,

$$r(T)\cos\theta(T)=\lambda r(0)\cos\theta(0).\tag{2.55}$$

Similarly from (2.53) one has

$$r(T)\sin\theta(T)=\lambda r(0)\sin\theta(0).\tag{2.56}$$

So, from (2.55) and (2.56), we have

$$r(T)=\lambda r(0),\qquad(\cos\theta(T),\sin\theta(T))=(\cos\theta(0),\sin\theta(0)).\tag{2.57}$$

Therefore,

$$\theta(T)=\theta(0)+2k\pi\qquad\text{or}\qquad\theta(T)-\theta(0)=2k\pi.\tag{2.58}$$

However, from $n\Delta t<T<(n+1)\Delta t$, we have

$$\theta(T)-\theta(0)<\theta(n\Delta t)-\theta(0)=-2n\pi,\tag{2.59}$$

$$\theta(T)-\theta(0)>\theta((n+1)\Delta t)-\theta(0)=-2(n+1)\pi,\tag{2.60}$$

since $\theta'<0$. So there is no integer k such that $\theta(T)-\theta(0)=2k\pi$.

Therefore, the conclusion follows. □

Theorem 2.4. *Suppose (H1)–(H5) hold. Then (1.4) has at least one T-periodic solution $u(t)$.*

Proof. By Lemma 2.3, we know that there exists $A > 0$ ($A \gg 1$) such that if

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2, \tag{2.61}$$

then

$$(u(T,\xi,\eta), v(T,\xi,\eta)) \neq (\lambda^{2/p}\xi, \lambda^{2(p-1)/p}\eta) \quad \text{for } \lambda > 0. \tag{2.62}$$

Assume that

$$\xi_1 = u(T,\xi,\eta), \qquad \eta_1 = v(T,\xi,\eta). \tag{2.63}$$

Consider a two-dimensional open region D_A bounded by

$$D_A = \left\{(\xi,\eta) : \frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{p/(p-1)} = A^2\right\}, \tag{2.64}$$

then we define a topological mapping

$$H : D_A \longmapsto \mathbb{R}^2, \qquad (\xi,\eta) \longmapsto (\xi_1,\eta_1). \tag{2.65}$$

It follows from Lemma 2.3 that

$$(\xi_1,\eta_1) \neq (\lambda^{2/p}\xi, \lambda^{2(p-1)/p}\eta), \quad (\xi,\eta) \in \partial D_A. \tag{2.66}$$

Now we define a homotopy $h : \overline{D}_A \times [0,1] \to R^2$ by

$$\begin{aligned} h(\xi,\eta,\mu) &= -(\mu^{2/p}\xi, \mu^{2(p-1)/p}\eta) + ((1-\mu)^{2/p}\xi_1, (1-\mu)^{2(p-1)/p}\eta_1) \\ &= -\begin{pmatrix} \mu^{2/p} & 0 \\ 0 & \mu^{2(p-1)/p} \end{pmatrix} I(\xi,\eta) + \begin{pmatrix} (1-\mu)^{2/p} & 0 \\ 0 & (1-\mu)^{2(p-1)/p} \end{pmatrix} H(\xi,\eta), \end{aligned} \tag{2.67}$$

for $\mu \in [0,1]$. It is easy to see that $h(\xi,\eta,0), h(\xi,\eta,1) \neq 0$ for $(\xi,\eta) \in \partial D_A$. Then we show that $h(\xi,\eta,\mu) \neq 0$ for $(\xi,\eta) \in \partial D_A$, where $\mu \in (0,1)$. If not, there is $\mu_0 \in (0,1)$, $(\xi,\eta) \in \partial D_A$ such that $h(\xi,\eta,\mu_0) = 0$, that is,

$$(\xi_1,\eta_1) = \left(\left(\frac{\mu}{1-\mu}\right)^{2/p}\xi, \left(\frac{\mu}{1-\mu}\right)^{2(p-1)/p}\eta\right), \tag{2.68}$$

which is impossible. So $h(\xi,\eta,\mu) \neq 0$ for $\mu \in [0,1]$.

Then, $\deg\{h(\xi,\eta,0), D_A, 0\} = \deg\{h(\xi,\eta,1), D_A, 0\}$, that is,

$$\deg\{H, D_A, 0\} = \deg\{-I, D_A, 0\} \neq 0. \tag{2.69}$$

Therefore, H has at least one fixed point $(\xi^*,\eta^*) \in D_A$. It is easy to see that $u(t) = u(t,\xi^*,\eta^*)$ is a T-periodic solution of (1.4). □

If we let $\phi(u) = \varphi_p(u) = |u|^{p-2}u$, $p > 2$, then we have the following special cases of (1.4):

$$(\varphi_p(u'))' + f(u,u')u' + g(u) = p(t,u,u') \quad t \in [0,T], \tag{2.70}$$

so we can easy get the following results.

Theorem 2.5. *Assume (H2) and (H3) hold and solutions of (2.70) are unique with respect to initial value, moreover suppose that there exist λ, μ, and n such that*

$$\begin{gathered}\left(\frac{p'}{p'-1}\right)^{p-1}\left(\frac{2n\pi_p}{T}\right)^p + \alpha_1 + \frac{p-1}{p}\alpha_2{}^{p/p-1} \\ < \lambda \le \frac{g(x)}{\phi_p(x)} \le \mu < \left(\frac{p'}{p'+1}\right)^{p-1}\left(\frac{2(n+1)\pi_p}{T}\right)^p - \alpha_1 - \frac{p-1}{p}\alpha_2{}^{p/p-1},\end{gathered} \tag{2.71}$$

then (2.70) has at least one T-periodic solution.

3. Example

In this section, we present an example to illustrate our main results. Consider the following differential equation:

$$(\phi(u'))' + f(u,u')u' + g(u) = e(t,u,u'), \quad t \in [0,T], \tag{3.1}$$

where

$$\begin{gathered}\phi(x) = |x|(x+\sin x), \qquad f(x,y) = |y|^{3/4} + a, \quad a > 0, \qquad g(x) = 2\phi(x), \\ e(t,x,y) = -\frac{2}{3}|x|x - |y|^{3/4}y + b\cos 2\pi t, \quad b > 0.\end{gathered} \tag{3.2}$$

We claim that

$$\frac{2}{3}|x|^2 \le |\phi(x)| \le 2|x|^2. \tag{3.3}$$

In fact, if $x \ne 0$, we have

$$|\phi(x)| = |x|^2\left|1 + \frac{\sin x}{x}\right| > |x|^2\left(1 - \frac{1}{\pi}\right) > \frac{2}{3}|x|^2, \tag{3.4}$$

so (3.3) holds. Therefore, $p = 3$, $m_1 = 2/3$, $m_2 = 2$. Also, we can get $\alpha_1 = 2/3$, $\beta_1 = 1$, $\gamma_1 = b$, $\alpha_2 = 0$, $\beta_2 = 1$, $\gamma_2 = a$, $k = 11/4$.

Let $n = 0$ and $T = 1$, then conditions (H1)–(H4) are satisfied.

Now, we check that condition (H5) is satisfied.

Suppose that $x_1(t)$ and $x_2(t)$ are two different solutions to (3.1) satisfying

$$x_1(t_0) = x_2(t_0) = x_0, \qquad x_1'(t_0) = x_2'(t_0) = x_0'. \tag{3.5}$$

Let $y = \phi(x')$, then $(x_i(t), y_i(t)) = (x_i(t), \phi(x_i'(t)))$ $(i = 1,2)$ are two different solutions to the system

$$\begin{aligned} x' &= \phi^{-1}(y), \\ y' &= -g(x) - f(x, \phi^{-1}(y))\phi^{-1}(y) + e(t, x, \phi^{-1}(y)), \end{aligned} \tag{3.6}$$

satisfying $(x_i(t_0), y_i(t_0)) = (x_0, \phi(x'(t_0)))$ $(i = 1,2)$.

Without loss of generality, we assume that there exists $t_1 > t_0$ such that

$$x_2(t) > x_1(t), \quad t \in (t_0, t_1]. \tag{3.7}$$

As $x_1(t_0) = x_2(t_0) = x_0$, $x_1'(t_0) = x_2'(t_0) = x_0'$, and $x_i \in \mathbb{C}^2[t_0, t_1]$, so there exists $t^* \in (t_0, t_1)$ such that

$$x_2'(t) > x_1'(t), \quad t \in (t_0, t^*]. \tag{3.8}$$

Therefore, for $t \in (t_0, t^*]$, we have

$$\begin{aligned} y_2(t) - y_1(t) = &-\int_{t_0}^{t} \Big\{ [g(x_2(s)) - g(x_1(s))] + [f(x_2(s), x_2'(s))x_2'(s) - f(x_1(s), x_1'(s))x_1'(s)] \\ &\quad - [e(s, x_2(s), x_2'(s)) - e(s, x_1(s), x_1'(s))] \Big\} ds \\ = &-\int_{t_0}^{t} \Big\{ 2[\phi(x_2(s)) - \phi(x_1(s))] + 2\Big[|x_2'(s)|^{3/4} x_2'(s) - |x_1'(s)|^{3/4} x_1'(s) \Big] \\ &\quad + a(x_2'(s) - x_1'(s)) + \frac{2}{3}\Big[|x_2'(s)| x_2'(s) - |x_1'(s)| x_1'(s) \Big] \Big\} ds < 0. \end{aligned} \tag{3.9}$$

That is,

$$\phi(x_2'(t)) - \phi(x_1'(t)) < 0, \quad t \in (t_0, t^*]. \tag{3.10}$$

So, $x_2'(t) < x_1'(t), t \in (t_0, t^*]$, this is a contradiction.

Therefore, by Theorem 2.4, we can conclude that (3.1) has at least one 1-periodic solution.

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