INVERSES OF NEW HILBERT-PACHPATTE-TYPE INEQUALITIES

CHANG-JIAN ZHAO AND WING-SUM CHEUNG

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We establish a new inequality of Hilbert type for a finite double number of nonnegative sequences of real numbers and some interrelated results, which are inverse and general forms of Pachpatte's and Handley's results. An integral version and some interrelated results are also obtained. These results provide some new estimates on such types of inequalities.

1. Introduction

The various generalizations and sharpenings of Hilbert's double series inequality and its integral version are obtained by Pachpatte, Handley et al., Gao et al. (see [1, 3, 5, 7, 10, 12, 16–18]). An elegant survey on this kind of inequalities was provided by Yang and Rassias (see [13]). Moreover, Pachpatte [9] established a new Hilbert-type inequality and its integral version as follows.

Theorem 1.1. *Let $\{a_m\}$, $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1,2,\ldots,k$ and $n = 1,2,\ldots,r$ with $a_0 = b_0 = 0$, and let $\{p_m\}$, $\{q_n\}$ be two positive sequences of real numbers defined for $m = 1,2,\ldots,k$, $n = 1,2,\ldots,r$, where k, r are natural numbers. Define $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be two real-valued nonnegative, convex, and submultiplicative functions defined on $\mathbb{R}_+ = [0,\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} \le M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{\nabla a_m}{p_m}\right)\right)^2\right)^{1/2} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{\nabla b_n}{q_n}\right)\right)^2\right)^{1/2}, \tag{1.1}$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}\left(\frac{\phi(Q_n)}{Q_n}\right)^2\right)^{1/2}, \tag{1.2}$$

and $\nabla a_m = a_m - a_{m-1}$, $\nabla b_n = b_n - b_{n-1}$.

Theorem 1.2. *Let* $f\in C^1[[0,x),\mathbb{R}^+]$, $g\in C^1[[0,y),\mathbb{R}^+]$ *with* $f(0)=g(0)=0$ *and let* $p(\sigma)$, $q(\tau)$ *be two positive functions defined for* $\sigma\in[0,x)$ *and* $\tau\in[0,y)$. *Let* $P(s)=\int_0^s p(\sigma)d\sigma$ *and* $Q(t)=\int_0^t q(\tau)d\tau$ *for* $s\in[0,x)$ *and* $t\in[0,y)$, *where* x, y *are positive real numbers. Let* ϕ, *and* ψ *be as in Theorem 1.1. Then*

$$\begin{aligned}\int_0^x\int_0^y\frac{\phi(f(s))\psi(g(t))}{s+t}ds\,dt &\le L(x,y)\left(\int_0^x(x-s)\left(p(s)\phi\left(\frac{f'(s)}{p(s)}\right)\right)^2ds\right)^{1/2}\\ &\quad\times\left(\int_0^y(y-t)\left(q(y)\phi\left(\frac{g'(t)}{q(t)}\right)\right)^2dt\right)^{1/2},\end{aligned} \tag{1.3}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\phi(P(s))}{P(s)}\right)^2ds\right)^{1/2}\left(\int_0^y\left(\frac{\phi(Q(t))}{Q(t)}\right)^2dt\right)^{1/2}, \tag{1.4}$$

and $'$ *denotes the derivative of a function.*

In [4], Handley et al. gave general versions of inequalities (1.1) and (1.3) as follows.

Theorem 1.3. *Let* $\{a_{i,m_i}\}$ $(i=1,2,\ldots,n)$ *be n sequences of nonnegative real numbers defined for* $m_i=1,2,\ldots,k_i$ *with* $a_{1,0}=a_{2,0}=\cdots=a_{n,0}=0$, *and let* $\{p_{i,m_i}\}$ *be n sequences of positive real numbers defined for* $m_i=1,2,\ldots,k_i$, *where* k_i $(i=1,2,\ldots,n)$ *are natural numbers. Set* $P_{i,m_i}=\sum_{s_i=1}^{m_i}p_{i,s_i}$. *Let* ϕ_i $(i=1,2,\ldots,n)$ *be n real-valued nonnegative convex and submultiplicative functions defined on* $\mathbb{R}_+$. *Let* $\alpha_i\in(0,1)$, *and set* $\alpha_i'=1-\alpha_i$ $(i=1,2,\ldots,n)$, $\alpha=\sum_{i=1}^n\alpha_i$, *and* $\alpha'=\sum_{i=1}^n\alpha_i'=n-\alpha$. *Then*

$$\begin{aligned}&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n\phi_i(a_{i,m_i})}{\left(\sum_{i=1}^n\alpha_i'm_i\right)^{\alpha'}}\\ &\quad\le M(k_1,\ldots,k_n)\prod_{i=1}^n\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{\nabla a_{i,m_i}}{p_{i,m_i}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i},\end{aligned} \tag{1.5}$$

where

$$M(k_1,\ldots,k_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^n\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{1/\alpha_i'}\right)^{\alpha_i'}, \tag{1.6}$$

and $\nabla a_{i,m_i}=a_{i,m_i}-a_{i,m_i-1}$ $(i=1,2,\ldots,n)$.

Theorem 1.4. *Let* $f_i \in C^1[[0,k_i],\mathbb{R}_+]$, $i = 1,2,\ldots,n$, *with* $f_i(0) = 0$ $(i = 1,2,\ldots,n)$, *let* $p_i(\sigma_i)$ *be n positive functions defined for* $\sigma_i \in [0,x_i]$ $(i = 1,\ldots,n)$. *Set* $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i)d\sigma_i$ *for* $s_i \in [0,x_i]$, *where* x_i *are positive real numbers. Let* ϕ_i, α_i, α_i', α, *and* α' *be as in Theorem 1.3. Then*

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \phi_i(f_i(s_i))}{(\sum_{i=1}^n \alpha_i' s_i)^{\alpha'}}ds_1\cdots ds_n \\
&\quad\le L(x_1,\ldots,x_n)\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\phi_i\left(\frac{f_i'(s_i)}{p_i(s_i)}\right)\right)^{1/\alpha_i}ds_i\right)^{\alpha_i},
\end{aligned}\tag{1.7}$$

where

$$L(x_1,\ldots,x_n) = \frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^n\left(\int_0^{x_i}\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{1/\alpha_i'}ds_i\right)^{\alpha_i'}. \tag{1.8}$$

The main purpose of the present paper is to establish their reverse versions, which are more extensive results for this type of inequalities. Our main results are given in the following theorems.

Theorem 1.5. *Let* $\{a_{s_i,t_i,m_{s_i},m_{t_i}}\}$ $(i = 1,2,\ldots,n)$ *be n sequences of nonnegative numbers defined for* $m_{s_i} = 1,2,\ldots,k_{s_i}$, $m_{t_i} = 1,2,\ldots,k_{t_i}$, *with* $a_{s_i,t_i,0,m_{t_i}} = a_{s_i,t_i,m_{s_i},0} = 0$ $(i = 1,2,\ldots,n)$, *where* k_{s_i} *and* k_{t_i} $(i = 1,2,\ldots,n)$ *are natural numbers. Let* $\{p_{s_i,t_i,m_{s_i},m_{t_i}}\}$ *be n sequences of positive real numbers defined for* m_{s_i}, m_{t_i}. *Set*

$$P_{s_i,t_i,m_{s_i},m_{t_i}} = \sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}} \quad (i = 1,2,\ldots,n). \tag{1.9}$$

Let ϕ_i $(i = 1,2,\ldots,n)$ *be n real-valued nonnegative concave and supermultiplicative functions defined on* $\mathbb{R}_+$. *Let* $\alpha_i \in (1,\infty)$. *Set* $\alpha_i' = 1-\alpha_i$ $(i = 1,2,\ldots,n)$, $\alpha = \sum_{i=1}^n \alpha_i$, *and* $\alpha' = \sum_{i=1}^n \alpha_i' = n-\alpha$. *Define operators*

$$\begin{aligned}
\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}} &= a_{s_i,t_i,m_{s_i},m_{t_i}} - a_{s_i,t_i,m_{s_i}-1,m_{t_i}}, \\
\nabla_2 a_{s_i,t_i,m_{s_i},m_{t_i}} &= a_{s_i,t_i,m_{s_i},m_{t_i}} - a_{s_i,t_i,m_{s_i},m_{t_i}-1}, \\
\nabla_2\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}} &= \nabla_2(\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}).
\end{aligned}\tag{1.10}$$

Then

$$\begin{aligned}
&\sum_{m_{s_1}=1}^{k_{s_1}}\sum_{m_t=1}^{k_{t_1}}\cdots\sum_{m_{s_n}=1}^{k_{s_n}}\sum_{m_{t_n}=1}^{k_{t_n}}\frac{\prod_{i=1}^n \phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}})}{((1/\alpha')\sum_{i=1}^n \alpha_i' m_{s_i}m_{t_i})^{\alpha'}} \\
&\quad\ge C(k_{s_1}k_{t_1},\ldots,k_{s_n}k_{t_n}) \\
&\qquad\times\prod_{i=1}^n\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}(k_{s_i}-m_{s_i}+1)(k_{t_i}-m_{t_i}+1)\left(p_{s_i,t_i,m_{s_i},m_{t_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}}{p_{s_i,t_i,m_{s_i},m_{t_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i},
\end{aligned}\tag{1.11}$$

where

$$C(k_{s_1}k_{t_1},\dots,k_{s_n}k_{t_n}) = \prod_{i=1}^{n}\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}\left(\frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\right)^{1/\alpha_i'}\right)^{\alpha_i'}. \tag{1.12}$$

THEOREM 1.6. *Let $f_i(s_i,t_i)$ $(i=1,2,\dots,n)$ be real-valued continuous functions defined on $[0,x_i)\times[0,y_i)$, where $x_i\in(0,\infty)$, $y_i\in(0,\infty)$, and with $f_i(0,t_i)=f_i(s_i,0)=0$ $(i=1,2,\dots,n)$. Let $p_i(\sigma_i)$ and $q_i(\tau_i)$ be positive continuous functions defined for $\sigma_i\in(0,s_i)$, $\tau_i\in(0,t_i)$. Set*

$$P_i(s_i,t_i) = \int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\,d\sigma_i d\tau_i. \tag{1.13}$$

For the function $f_i(s_i,t_i)$, denote the partial derivatives $(\partial/\partial s_i)f_i(s_i,t_i)$, $(\partial/\partial t_i)f_i(s_i,t_i)$, and $(\partial^2/\partial s_i\partial t_i)f_i(s_i,t_i)$ by $D_1 f_i(s_i,t_i)$, $D_2 f_i(s_i,t_i)$, and $D_2D_1 f_i(s_i,t_i) = D_1D_2 f_i(s_i,t_i)$, respectively. Let ϕ_i, α_i, α_i', α, and α' be as in Theorem 1.5. Then

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\prod_{i=1}^{n}\frac{\phi_i(f_i(s_i,t_i))}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i' s_i t_i\right)^{\alpha'}}ds_1dt_1\cdots ds_ndt_n\\
&\quad = G(x_1y_1,\dots,x_ny_n)\\
&\qquad\times\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left(p_i(s_i)q_i(t_i)\cdot\phi_i\left(\frac{D_2D_1 f_i(s_i,t_i)}{p_i(s_i)q_i(t_i)}\right)\right)^{1/\alpha_i}ds_i\,dt_i\right)^{\alpha_i},
\end{aligned} \tag{1.14}$$

where

$$G(x_1y_1,\dots,x_ny_n) = \prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right)^{1/\alpha_i'}ds_i\,dt_i\right)^{\alpha_i'}. \tag{1.15}$$

These results provide new estimates on these types of inequalities.

2. Proofs and remarks

Proof of Theorem 1.5. From the operators defined in Theorem 1.5, we obtain

$$\begin{aligned}
a_{s_i,t_i,m_{s_i},m_{t_i}} &= \sum_{m_{\xi_i}=1}^{m_{s_i}}\left\{a_{s_i,t_i,m_{\xi_i},m_{t_i}} - a_{s_i,t_i,m_{\xi_i}-1,m_{t_i}}\right\}\\
&= \sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\left\{a_{s_i,t_i,m_{\xi_i},m_{\eta_i}} - a_{s_i,t_i,m_{\xi_i}-1,m_{\eta_i}} - a_{s_i,t_i,m_{\xi_i},m_{\eta_i}-1} + a_{s_i,t_i,m_{\xi_i}-1,m_{\eta_i}-1}\right\}\\
&= \sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}.
\end{aligned} \tag{2.1}$$

From (2.1) and the hypotheses of Theorem 1.5, and in view of Jensen's inequality and inverse Hölder's inequality [6], we obtain

$$\begin{aligned}
&\phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}}) \\
&\quad = \phi_i\left(\frac{P_{s_i,t_i,m_{s_i},m_{t_i}}\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\left(\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}/p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\right)}{\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right) \\
&\quad \geq \phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})\cdot\phi_i\left(\frac{\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\left(\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}/p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\right)}{\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right) \\
&\quad \geq \frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\left(\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}} p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right) \\
&\quad \geq \frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}(m_{s_i}m_{t_i})^{\alpha_i'}\times\left(\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\left(p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}.
\end{aligned} \tag{2.2}$$

On the other hand, noticing $\alpha_i' < 0$ $(i = 1,2,\dots,n)$, $\lambda_i > 0$ $(i = 1,2,\dots,n)$, and applying the well-known means inequality, we have

$$\prod_{i=1}^{n}\lambda_i^{\alpha_i'} \geq \left(\frac{1}{\alpha'}\sum_{i=1}^{n}\alpha_i'\lambda_i\right)^{\alpha'}. \tag{2.3}$$

Hence

$$\begin{aligned}
&\frac{\prod_{i=1}^{n}\phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}})}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i' m_{s_i}m_{t_i}\right)^{\alpha'}} \\
&\quad \geq \prod_{i=1}^{n}\frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\left(\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\left(p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}.
\end{aligned} \tag{2.4}$$

Taking the sum of both sides of (2.4) over m_{s_i}, m_{t_i} $(i = 1,2,\dots,n)$ from 1 to k_{s_i}, k_{s_i}, respectively, and in view of inverse Hölder's inequality, we get

$$\begin{aligned}
&\sum_{m_{s_1}=1}^{k_{s_1}}\sum_{m_t=1}^{k_{t_1}}\cdots\sum_{m_{s_n}=1}^{k_{s_n}}\sum_{m_{t_n}=1}^{k_{t_n}}\frac{\prod_{i=1}^{n}\phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}})}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i' m_{s_i}m_{t_i}\right)^{\alpha'}} \\
&\quad \geq \prod_{i=1}^{n}\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}\left(\frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\left(\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\left(p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \prod_{i=1}^{n}\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}\left(\frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\right)^{1/\alpha_i'}\right)^{\alpha_i'}\\
&\quad\times\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}\left(\sum_{m_{\xi_i}=1}^{m_{s_i}}\sum_{m_{\eta_i}=1}^{m_{t_i}}\left(p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)\right)^{\alpha_i}\\
&= C(k_{s_1}k_{t_1},\dots,k_{s_n}k_{t_n})\\
&\quad\times\prod_{i=1}^{n}\left(\sum_{m_{\xi_i}=1}^{k_{s_i}}\sum_{m_{\eta_i}=1}^{k_{t_i}}(k_{s_i}-m_{\xi_i}+1)(k_{t_i}-m_{\eta_i}+1)\left(p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}{p_{s_i,t_i,m_{\xi_i},m_{\eta_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}\\
&= C(k_{s_1}k_{t_1},\dots,k_{s_n}k_{t_n})\\
&\quad\times\prod_{i=1}^{n}\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}(k_{s_i}-m_{s_i}+1)(k_{t_i}-m_{t_i}+1)\left(p_{s_i,t_i,m_{s_i},m_{t_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}}{p_{s_i,t_i,m_{s_i},m_{t_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}.
\end{aligned}
\tag{2.5}
$$

This completes the proof. □

Remark 2.1. Let $a_{s_i,t_i,m_{s_i},m_{t_i}}$, m_{s_i}, m_{t_i}, k_{s_i}, k_{t_i}, $a_{s_i,t_i,0,m_{t_i}}$, $a_{s_i,t_i,m_{s_i},0}$, $P_{s_i,t_i,m_{s_i},m_{t_i}}$, ∇_1, and ∇_2 be as in Theorem 1.5. Let ϕ_i, α_i, α_i', α, and α' $(i=1,2,\dots,n)$ be as in Theorem 1.3. Similar to the proof of Theorem 1.5, we obtain

$$
\begin{aligned}
&\sum_{m_{s_1}=1}^{k_{s_1}}\sum_{m_t=1}^{k_{t_1}}\cdots\sum_{m_{sn}=1}^{k_{sn}}\sum_{m_{tn}=1}^{k_{tn}}\frac{\prod_{i=1}^{n}\phi_i(a_{s_i,t_i,m_{s_i},m_{t_i}})}{\left(\sum_{i=1}^{n}\alpha_i' m_{s_i}m_{t_i}\right)^{\alpha'}}\\
&\quad\leq \overline{C}(k_{s_1}k_{t_1},\dots,k_{s_n}k_{t_n})\\
&\qquad\times\prod_{i=1}^{n}\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}(k_{s_i}-m_{s_i}+1)(k_{t_i}-m_{t_i}+1)\left(p_{s_i,t_i,m_{s_i},m_{t_i}}\phi_i\left(\frac{\nabla_2\nabla_1 a_{s_i,t_i,m_{s_i},m_{t_i}}}{p_{s_i,t_i,m_{s_i},m_{t_i}}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i},
\end{aligned}
\tag{2.6}
$$

where

$$
\overline{C}(k_{s_1}k_{t_1},\dots,k_{s_n}k_{t_n})=\frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^{n}\left(\sum_{m_{s_i}=1}^{k_{s_i}}\sum_{m_{t_i}=1}^{k_{t_i}}\left(\frac{\phi_i(P_{s_i,t_i,m_{s_i},m_{t_i}})}{P_{s_i,t_i,m_{s_i},m_{t_i}}}\right)^{1/\alpha_i'}\right)^{\alpha_i'}. \tag{2.7}
$$

This is just a *general form* of inequality (1.5) and the *inverse form* of inequality (1.11).

Moreover, let $\{a_{i,m_i}\}$, m_i, k_i, $a_{i,0}$, p_{i,m_i}, ∇, and P_{i,m_i} $(i = 1,2,\dots,n)$ be as in Theorem 1.3. Let ϕ_i, α_i, α_i', α, and α' $(i = 1,2,\dots,n)$ be as in Theorem 1.5. Similar to the proof of Theorem 1.5, we obtain

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(a_{i,m_i})}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i' m_i\right)^{\alpha'}} \geq \overline{M}(k_1,\dots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{\nabla a_{i,m_i}}{p_{i,m_i}}\right)\right)^{1/\alpha_i}\right)^{\alpha_i}, \tag{2.8}$$

where

$$\overline{M}(k_1,\dots,k_n) = \prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{1/\alpha_i'}\right)^{\alpha_i'}. \tag{2.9}$$

This is just an *inverse form* of inequality (1.5) which was given by Handley et al. [4]. Let $n = 2$, $\alpha_1 = \alpha_2 = 2$, then $\alpha_1' = \alpha_2' = -1$, putting them in inequality (2.8), we have

$$\begin{aligned}\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\frac{\phi_1(a_{1,m_1})\phi_2(a_{2,m_2})}{(m_1+m_2)^{-2}} &\geq \overline{M}(k_1,k_2)\left(\sum_{m_1=1}^{k_1}(k_1-m_1+1)\left(p_{1,m_i}\phi_1\left(\frac{\nabla a_{1,m_1}}{p_{1,m_1}}\right)\right)^{1/2}\right)^{2} \\ &\quad\times\left(\sum_{m_2=1}^{k_2}(k_2-m_2+1)\left(p_{2,m_2}\phi_2\left(\frac{\nabla a_{2,m_2}}{p_{2,m_2}}\right)\right)^{1/2}\right)^{2},\end{aligned} \tag{2.10}$$

where

$$\overline{M}(k_1,k_2) = 4\left(\sum_{m_1=1}^{k_1}\left(\frac{\phi_1(P_{1,m_1})}{P_{1,m_1}}\right)^{-1}\right)^{-1}\left(\sum_{m_2=1}^{k_2}\left(\frac{\phi_2(P_{2,m_2})}{P_{2,m_2}}\right)^{-1}\right)^{-1}. \tag{2.11}$$

This is just an *inverse form* of inequality (1.1) which was given by Pachpatte [9].

Proof of Theorem 1.6. From the hypotheses of Theorem 1.6, we obtain

$$f_i(s_i,t_i) = \int_0^{s_i}\int_0^{t_i} D_2D_1 f_i(\sigma_i,\tau_i)\,d\sigma_i\,d\tau_i. \tag{2.12}$$

From (2.12) and using Jensen's integral inequality and Hölder's integral inequality [6], we obtain

$$\begin{aligned}
\phi_i(f_i(s_i,t_i)) &= \phi_i\left(\frac{P_i(s_i,t_i)\int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\left(D_2D_1f_i(\sigma_i,\tau_i)/p_i(\sigma_i)q_i(\tau_i)\right)d\sigma_i\,d\tau_i}{\int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\,d\sigma_i\,d\tau_i}\right)\\
&\geq \phi_i(P_i(s_i,t_i))\cdot\phi_i\left(\frac{\int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\left(D_2D_1f_i(\sigma_i,\tau_i)/p_i(\sigma_i)q_i(\tau_i)\right)d\sigma_i\,d\tau_i}{\int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\,d\sigma_i\,d\tau_i}\right)\\
&\geq \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\int_0^{t_i}\int_0^{s_i} p_i(\sigma_i)q_i(\tau_i)\cdot\phi_i\left(\frac{D_2D_1f_i(\sigma_i,\tau_i)}{p_i(\sigma_i)q_i(\tau_i)}\right)d\sigma_i\,d\tau_i\\
&\geq \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}(s_it_i)^{\alpha_i'}\left(\int_0^{t_i}\int_0^{s_i}\left(p_i(\sigma_i)q_i(\tau_i)\cdot\phi_i\left(\frac{D_2D_1f_i(\sigma_i,\tau_i)}{p_i(\sigma_i)q_i(\tau_i)}\right)\right)^{1/\alpha_i}d\sigma_i\,d\tau_i\right)^{\alpha_i}.
\end{aligned}\tag{2.13}$$

From the well-known inequality for means, we have

$$\begin{aligned}
&\prod_{i=1}^{n}\frac{\phi_i(f_i(s_i,t_i))}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i's_it_i\right)^{\alpha'}}\\
&\quad\geq\prod_{i=1}^{n}\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\left(\int_0^{t_i}\int_0^{s_i}\left(p_i(\sigma_i)q_i(\tau_i)\cdot\phi_i\left(\frac{D_2D_1f_i(\sigma_i,\tau_i)}{p_i(\sigma_i)q_i(\tau_i)}\right)\right)^{1/\alpha_i}d\sigma_i\,d\tau_i\right)^{\alpha_i}.
\end{aligned}\tag{2.14}$$

Integrating both sides of (2.14) over s_i, t_i from 1 to x_i, y_i $(i=1,2,\dots,n)$ and in view of Hölder's integral inequality and Fubini's theorem, we observe that

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\prod_{i=1}^{n}\frac{\phi_i(f_i(s_i,t_i))}{\left((1/\alpha')\sum_{i=1}^{n}\alpha_i's_it_i\right)^{\alpha'}}ds_1dt_1\cdots ds_ndt_n\\
&\quad\geq\prod_{i=1}^{n}\int_0^{x_i}\int_0^{y_i}\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\\
&\qquad\times\left(\int_0^{t_i}\int_0^{s_i}\left(p_i(\sigma_i)q_i(\tau_i)\cdot\phi_i\left(\frac{D_2D_1f_i(\sigma_i,\tau_i)}{p_i(\sigma_i)q_i(\tau_i)}\right)\right)^{1/\alpha_i}d\sigma_i\,d\tau_i\right)^{\alpha_i}ds_i\,dt_i\\
&\quad\geq\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right)^{1/\alpha_i'}ds_i\,dt_i\right)^{\alpha_i'}\\
&\qquad\times\left(\int_0^{x_i}\int_0^{y_i}\left(\int_0^{t_i}\int_0^{s_i}\left(p_i(\sigma_i)q_i(\tau_i)\cdot\phi_i\left(\frac{D_2D_1f_i(\sigma_i,\tau_i)}{p_i(\sigma_i)q_i(\tau_i)}\right)\right)^{1/\alpha_i}d\sigma_id\tau_i\right)ds_i\,dt_i\right)^{\alpha_i}
\end{aligned}$$

$$= G(x_1y_1,\dots,x_ny_n)$$
$$\times\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left(p_i(s_i)q_i(t_i)\cdot\phi_i\left(\frac{D_2D_1f_i(s_i,t_i)}{p_i(s_i)q_i(t_i)}\right)\right)^{1/\alpha_i}ds_i\,dt_i\right)^{\alpha_i}. \tag{2.15}$$

The proof is complete. □

Remark 2.2. Let $f_i(s_i,t_i)$, x_i, y_i, $f_i(0,t_i)$, $f_i(s_i,0)$, $D_1f_i(s_i,t_i)$, $D_2f_i(s_i,t_i)$, $D_2D_1f_i(s_i,t_i)$, $p_i(\sigma_i)$, $q_i(\tau_i)$, and $P_i(s_i,t_i)$ be as in Theorem 1.6. Let ϕ_i, α_i, α_i', α, and α' be as in Theorem 1.3. Similar to the proof of Theorem 1.6, we have

$$\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\prod_{i=1}^{n}\frac{\phi_i(f_i(s_i,t_i))}{\left(\sum_{i=1}^{n}\alpha_i's_it_i\right)^{\alpha'}}ds_1dt_1\cdots ds_ndt_n$$
$$\le \overline{G}(x_1y_1,\dots,x_ny_n)$$
$$\times\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left(p_i(s_i)q_i(t_i)\cdot\phi_i\left(\frac{D_2D_1f_i(s_i,t_i)}{p_i(s_i)q_i(t_i)}\right)\right)^{1/\alpha_i}ds_i\,dt_i\right)^{\alpha_i}, \tag{2.16}$$

where

$$\overline{G}(x_1y_1,\dots,x_ny_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right)^{1/\alpha_i'}ds_i\,dt_i\right)^{\alpha_i'}. \tag{2.17}$$

This is just a *general form* of inequality (1.7) and an *inverse form* of the inequality (1.14).

On the other hand, let f_i, $f_i(0)$, $f_i'(s_i)$, $p_i(\sigma_i)$, σ_i, k_i, s_i, x_i, $P_i(s_i)$ be as in Theorem 1.4, *and* let ϕ_i, α_i, α_i', α, α' be as in Theorem 1.5. Then

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}\phi_i(f_i(s_i))}{(1/\alpha')\left(\sum_{i=1}^{n}\alpha_i's_i\right)^{\alpha'}}ds_1\cdots ds_n$$
$$\ge \overline{L}(x_1,\dots,x_n)\prod_{i=1}^{n}\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\phi_i\left(\frac{f_i'(s_i)}{p_i(s_i)}\right)\right)^{1/\alpha_i}ds_i\right)^{\alpha_i}, \tag{2.18}$$

where

$$\overline{L}(x_1,\dots,x_n)=\prod_{i=1}^{n}\left(\int_0^{x_i}\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{1/\alpha_i'}ds_i\right)^{\alpha_i'}. \tag{2.19}$$

This is just an *inverse form* of inequality (1.7) which was given by Handley et al. [4].

Let $n = 2$, $\alpha_1 = \alpha_2 = 2$, then $\alpha_1' = \alpha_2' = -1$, taking them in inequality (2.18), we have

$$\begin{aligned}\int_0^{x_1}\int_0^{x_2} \frac{\phi_1(f_1(s_1))\phi_2(f_2(s_2))}{(s_1+s_2)^{-2}}\,ds_1\,ds_2 &\geq \overline{L}(x_1,x_2)\left(\int_0^{x_1}(x_1-s_1)\left(p_1(s_1)\phi_1\left(\frac{f_1'(s_1)}{p_1(s_1)}\right)\right)^{1/2}ds_1\right)^2 \\ &\quad\times\left(\int_0^{x_2}(x_2-s_2)\left(p_2(s_2)\phi_2\left(\frac{f_2'(s_2)}{p_2(s_2)}\right)\right)^{1/2}ds_2\right)^2,\end{aligned} \tag{2.20}$$

where

$$\overline{L}(x_1,x_2) = 4\left(\int_0^{x_1}\left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_0^{x_2}\left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}. \tag{2.21}$$

This is just an *inverse form* of inequality (1.3) which was given by Pachpatte [9].

For interrelated research on similar inequalities, one is referred to [2, 8, 11, 14, 15], and the references cited therein.

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Chang-Jian Zhao: Department of Information and Mathematics Sciences, College of Science, China Jiliang University, Hangzhou 310018, China
E-mail address: chjzhao@cjlu.edu.cn

Wing-Sum Cheung: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: wscheung@hku.hk