A PYTHAGOREAN APPROACH IN BANACH SPACES

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Let X be a Banach space and let $S(X) = \{x \in X, \|x\| = 1\}$ be the unit sphere of X. Parameters $E(X) = \sup\{\alpha(x),\ x \in S(X)\}$, $e(X) = \inf\{\alpha(x),\ x \in S(X)\}$, $F(X) = \sup\{\beta(x),\ x \in S(X)\}$, and $f(X) = \inf\{\beta(x),\ x \in S(X)\}$, where $\alpha(x) = \sup\{\|x + y\|^2 + \|x - y\|^2,\ y \in S(X)\}$, and $\beta(x) = \inf\{\|x + y\|^2 + \|x - y\|^2,\ y \in S(X)\}$ are introduced and studied. The values of these parameters in the l_p spaces and function spaces $L_p[0,1]$ are estimated. Among the other results, we proved that a Banach space X with $E(X) < 8$, or $f(X) > 2$ is uniform nonsquare; and a Banach space X with $E(X) < 5$, or $f(X) > 32/9$ has uniform normal structure.

1. Preliminaries

Let X be a normed linear space, and let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \le 1\}$ be the unit sphere and unit ball of X respectively.

The following two concepts, the modulus of convexity: $\delta(\epsilon) = \inf\{1 - \|x + y\|/2 : \|x - y\| \le \epsilon,\ x, y \in S(X_2)\}$ where $0 \le \epsilon \le 2$, and the modulus of smoothness: $\rho_X(\epsilon) = \sup\{(\|x + y\| + \|x - y\| - 2)/2,\ x \in S(X),\ \|y\| = \epsilon\}$ where $\epsilon \ge 0$ have strong effect for studying and describing the shape of unit spheres and unit balls of Banach spaces.

Many other parameters were introduced and used to study the geometry of unit spheres and unit balls, more properties of Banach spaces were obtained and some results were improved, see [6–8, 11, 12, 14]. However most of the above-defined parameters are linear in the norm.

The Pythagorean theorem describes the shape of unit sphere of Euclidean spaces H by considering the inscribed triangle with two antipodal points x and $-x$ on $S(H)$, and characterizes the Euclidean spaces by the equation $\|x + y\|^2 + \|x - y\|^2 = 4$, for all $y \in S(H)$. In this paper we use this idea to introduce some quadratic parameters, then compute the values of these new parameters for some classical Banach spaces, and demonstrate the relationships between these parameters and uniformly nonsquare spaces, and the spaces with uniform normal structure.

Let X be a Banach space. A mapping $T : X \to X$ is called a nonexpansive mapping if $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in X$. Kirk [10] proved that if a weakly compact convex subset K of X has normal structure then any nonexpansive mapping on K has a fixed point.

Definition 1.1 [1]. A bounded convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$ such that $\sup\{\|x_0 - y\|, \, y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\|, \, x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded, convex subset of X has normal structure. A Banach space X is said to have weak normal structure if each weakly compact convex set K in X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any subset K as above, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\|, \, y \in K\} < c \cdot (d(K))$.

For a reflexive Banach space X, the normal structure and weak normal structure coincide.

Definition 1.2 [5]. Let X be a Banach space, a hexagon H in X is called a normal hexagon if the length of each side is 1 and each pair of two opposite sides are parallel.

LEMMA 1.3 [5]. *Let X be a Banach space without weak normal structure, then for any $0 < \epsilon < 1$, there exist infinitely many inscribed normal hexagons with vertices x_1, x_2, x_3 in $S(X)$ satisfying*
    (i) $x_2 - x_3 = x_1$;
    (ii) $\|(x_1 + x_2)/2\| > 1 - \epsilon$;
    (iii) $\|(x_3 + (-x_1))/2\| > 1 - \epsilon$.

A continuous mapping $x(t)$ from a closed interval $[a, b]$ to a Banach space X is called a curve in $X : C = x(t)$, $a \le t \le b$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces.

For a normed linear space X, we use X_2 to denote a two-dimensional subspace of X, then $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation.

THEOREM 1.4 [2, 12]. *Let X_2 be a two-dimensional Banach space, and K_1, K_2 be closed convex subsets of X_2, and have nonvoid interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \le l(\partial(K_2))$, where $l(\partial(K_i))$ denote the lengths of the circumferences of K_i, $i = 1, 2$.*

For $x \in S(X_2)$, let κ be one of the arcs of $S(X_2)$ from x to $-x$ counterclockwise, and let $g : [0, L] \to \kappa$ be the standard representation in terms of arc length, where L is the length of κ.

LEMMA 1.5 [7]. *The functions $\Phi, \Psi : [0, L] \to [0, 2]$ defined by $\Phi(s) = \|g(s) - x\|$, $\Psi(s) = \|g(s) + x\|$ are continuously increasing, and decreasing respectively.*

2. Parameters $E(X)$, $e(X)$, $F(X)$, and $f(X)$

Definition 2.1. For $x \in S(X)$, let
  (i) $\alpha(x) = \sup\{\|x+y\|^2 + \|x-y\|^2, \, y \in S(X)\}$ and
  (ii) $\beta(x) = \inf\{\|x+y\|^2 + \|x-y\|^2, \, y \in S(X)\}$.
  We define

$$E(X) = \sup\{\alpha(x), \, x \in S(X)\}, \qquad e(X) = \inf\{\alpha(x), \, x \in S(X)\},$$
$$F(X) = \sup\{\beta(x), \, x \in S(X)\}, \qquad f(X) = \inf\{\beta(x), \, x \in S(X)\}. \tag{2.1}$$

LEMMA 2.2. *If $a \geq 0$, $b \geq 0$, and $a+b \geq 2$, then $a^2 + b^2 \geq 2$.*

Proof. $a^2 + b^2 \geq a^2 + (2-a)^2 = a^2 + 4 - 4a + a^2 = 2(a^2 - 2a + 2) \geq 2[(a-1)^2 + 1] \geq 2$. $\square$

PROPOSITION 2.3. *For any Banach space X, $2 \leq f(X) \leq e(X) \leq E(X) \leq 8$, and $2 \leq f(X) \leq F(X) \leq E(X) \leq 8$.*

Proof. Since $\|x \pm y\| \leq \|x\| + \|y\| \leq 2$, we have $E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2, \, x \in S(X), \, y \in S(X)\} \leq 4 + 4 = 8$; and since $\|x+y\| + \|x-y\| \geq \|2x\| = 2$, we have $f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2, \, x \in S(X), \, y \in S(X)\} \geq 2$, by the about lemma. $\square$

We now list the values of these parameters for some special two-dimensional normed linear spaces.

For any $x \in X$, $y \in X$ we use $[x,y]$ to denote the line segment connecting x and y in X.

LEMMA 2.4. *Let $x, y \in S(X)$ with $\|x+y\| = \|x-y\| = a$, and u, v be the normalization of $x+y$, $x-y$, respectively, then $\|u+v\| = \|u-v\| = 2/a$.*

Proof. Since $-x - y = -au$, $x - y = av$, then $2x = (x-y) - (-x-y) = a(u+v)$. Therefore $\|u+v\| = 2\|x\|/a = 2/a$. Similarly, we can prove $\|u-v\| = 2/a$. $\square$

PROPOSITION 2.5. *$S(X_2)$ is affinely homeomorphic to a parallelogram if and only if $E(X_2) = 8$, or $f(X_2) = 2$.*

PROPOSITION 2.6. *If $S(X_2)$ is affinely homeomorphic to a normal hexagon, then $E(X_2) \geq 5$, and $f(X_2) \leq 32/9$.*

LEMMA 2.7 [7]. *Let $x, y \in S(X)$, and $\|x+y\|/2 > 1 - \epsilon$, then for any $z \in [x,y]$, the line segment connecting x and y, $\|z\| > 1 - 2\epsilon$.*

THEOREM 2.8. *A Banach space with $E(X_2) < 8$, or $f(X) > 2$ is uniformly nonsquare.*

Proof. Suppose X is not uniformly nonsquare. For any $\epsilon > 0$, there exist x, $y \in S(X)$ such that both $\|x+y\|$ and $\|x-y\| > 2 - \epsilon/2$ [9]. Then, both $\|x-y\|^2$ and $\|x-y\|^2 \geq 4 - 2\epsilon$. Since ϵ can be arbitrarily small we have $E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2, \, x \in S(X), \, y \in S(X)\} = 8$.

If $\|x+y\| \neq \|x-y\|$, take z such that y and z are on the same arc κ from x to $-x$, and $\|x+z\| = \|x-z\|$. From Lemma 1.5 both $\|x+z\|$ and $\|x-z\| > 2 - \epsilon/2$. Let u and v be normalization of $x+z$ and $x-z$ respectively. From Lemma 2.4, both $\|u+v\|$ and

$\|u - v\| \leq 2/(2 - \epsilon/2) \leq 1 + \epsilon/2$. Since ϵ can be arbitrarily small we have $f(X) = \inf\{\|x + y\|^2 + \|x - y\|^2, \ x \in S(X), \ y \in S(X)\} = 2$. $\qquad\square$

3. Parameters $E(X)$ and $f(X)$, and normal structure

LEMMA 3.1. *Let $x \in S(X)$ and $\|y\| > 1$, then $\|x + t(y - x)\|$ is an increasing function of $t \in (1, \infty)$.*

Proof. It is clear that $\|x + t(y - x)\| > 1$ for any $t > 1$. Otherwise

$$
\begin{aligned}
\|y\| &= \left\| \frac{x + t(y - x)}{t} + \frac{(t - 1)x}{t} \right\| \\
&\leq \frac{\|x + t(y - x)\|}{t} + \frac{(t - 1)\|x\|}{t} \\
&\leq \frac{1}{t} + \frac{t - 1}{t} = 1.
\end{aligned}
\tag{3.1}
$$

If there are $t_1 > t_2 > 1$ such that $1 \leq a = \|x + t_1(y - x)\| < \|x + t_2(y - x)\|$, then

$$
\begin{aligned}
\|x + t_2(y - x)\| &= \left\| \frac{t_2[x + t_1(y - x)]}{t_1} + \frac{(t_1 - t_2)x}{t_1} \right\| \\
&\leq \frac{t_2\|x + t_1(y - x)\|}{t_1} + \frac{(t_1 - t_2)\|x\|}{t_1} \\
&= \frac{t_2 a}{t_1} + \frac{t_1 - t_2}{t_1} \leq \frac{t_2 a}{t_1} + \frac{(t_1 - t_2)a}{t_1} = a.
\end{aligned}
\tag{3.2}
$$

This is a contradiction. $\qquad\square$

THEOREM 3.2. *A Banach space X with $E(X) < 5$, or $f(X) > 32/9$ has normal structure.*

Proof. Suppose X does not have normal structure, then there exist x_1, x_2 and $x_3 \in S(X)$ such that $x_1 = x_2 - x_3$, $\|(x_1 + x_2)/2\| > 1 - \epsilon$, and $\|(x_3 + (-x_1))/2\| > 1 - \epsilon$ by Lemma 1.3. Then, $\|x_3 - x_1\|^2 > 4 - 8\epsilon$ and $\|x_3 + x_1\|^2 = \|x_2\|^2 = 1$. Since ϵ can be arbitrarily small we have $E(X) = \sup\{\|x + y\|^2 + \|x - y\|^2, \ x \in S(X), \ y \in S(X)\} \geq 5$. The first assert is approved.

To prove the second assert, let $y = (x_2 + x_3)/2$, then $y - x_1 = (x_2 + x_3)/2 - x_1 = (x_2 + x_3 - 2x_1)/2 = (x_1 + x_3 + x_3 - 2x_1)/2 = (2x_3 - x_1)/2 = 3/2(2x_3/3 + (-x_1)/3)$. Since $\|(x_3 + (-x_1))/2\| > 1 - \epsilon$ and $2x_3/3 + (-x_1)/3 \in [-x_1, x_3]$, by Lemma 2.7 we have $\|y - x_1\| = 3/2\|2x_3/3 + (-x_1)/3\| > 3/2(1 - 2\epsilon)$. Let y' be normalization of $y = (x_2 + x_3)/2$. From Lemma 3.1, $\|y' - x_1\| \geq \|y - x_1\| > 3/2(1 - 2\epsilon)$.

Similarly we can prove $\|y' + x_1\| \geq \|y + x_1\| > 3/2(1 - 2\epsilon)$.

If $\|y' + x_1\| \neq \|y' - x_1\|$, let $z \in S(X)$ such that $\|z + x_1\| = \|z - x_1\|$. From Lemma 1.5 we have $\|z + x_1\| = \|z - x_1\| > 3/2(1 - 2\epsilon)$. Let u, v be the normalization of $z + x_1$, $z - x_1$ respectively, by Lemma 2.4 we have both $\|u + v\|$ and $\|u - v\| < 2/(3/2(1 - 2\epsilon)) < 4/3(1 + 4\epsilon)$. Since ϵ can be arbitrarily small, we have $f(X) = \inf\{\|x + y\|^2 + \|x - y\|^2, \ x \in S(X), \ y \in S(X)\} = 2(4/3)^2 = 32/9$. $\qquad\square$

4. Inequalities of parameters for some classical spaces

PROPOSITION 4.1. $E(H) = e(X) = F(X) = f(X) = 4$, where H is a Hilbert space.

This is a direct result of parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in H$, in the Hilbert spaces.

THEOREM 4.2. Let $p, q \geq 1$ and $1/p + 1/q = 1$, then
  (i) $f(l_p) = 2^{1+2/q}$, $F(l_p) \leq 2^{1+2/p} \leq e(l_p) \leq E(l_p)$, $1 < p \leq 2$;
  (ii) $f(l_p) \leq F(l_p) \leq 2^{1+2/p} \leq e(l_p)$, $E(l_p) = 2^{1+2/q}$, $p > 2$.

Proof. (i) To prove $e(l_p) \geq 2^{1+2/p}$, for any $0 < \delta < 1$ and $y = (y_1, y_2, y_3, \ldots, y_n, \ldots) \in S(l_p)$ take N such that $|y_N| < \delta$. Let $e_N = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the Nth entry is 1 and others are 0. Since $(1 - x)^p \geq 1 - px$ for $0 \leq x \leq 1$ and $p > 1$, we have

$$
\begin{aligned}
\|y \pm e_N\|^p &= |y_1|^p + |y_2|^p + \cdots + |y_N \pm 1|^p + |y_{N+1}|^p + \cdots \\
&\geq |y_1|^p + |y_2|^p + \cdots + |1 - |y_N||^p + |y_{N+1}|^p + \cdots \\
&\geq (1 + |y_1|^p + |y_2|^p + \cdots + |y_N|^p + |y_{N+1}|^p + \cdots) \\
&\quad - p|y_N| - |y_N|^p \geq 2 - 2p\delta.
\end{aligned}
\tag{4.1}
$$

Since δ can be arbitrarily small, by definition of $\alpha(y)$, $\alpha(y) \geq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $y \in S(l_p)$ is arbitrary, by definition of $e(X)$, we have $e(l_p) \geq 2^{1+2/p}$.

To prove $F(l_p) \leq 2^{1+2/p}$, we need the inequality: $(1 + x)^p \leq 1 + 2px$ if x is small enough and $p > 1$. For any $y = (y_1, y_2, y_3, \ldots, y_n, \ldots) \in S(l_p)$.

Take N such that $|y_N| < \delta < 1$. Let $e_N = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the Nth entry is 1 and others are 0, we have

$$
\begin{aligned}
\|y \pm e_N\|^p &= |y_1|^p + |y_2|^p + \cdots + |y_N \pm 1|^p + |y_{N+1}|^p + \cdots \\
&\leq |y_1|^p + |y_2|^p + \cdots + |1 + |y_N||^p + |y_{N+1}|^p + \cdots \\
&\leq (|y_1|^p + |y_2|^p + \cdots + |y_N|^p + |y_{N+1}|^p + \cdots) \\
&\quad + 1 + 2p|y_N| \leq 2 + 2p\delta.
\end{aligned}
\tag{4.2}
$$

Since δ can be arbitrarily small, by definition of $\beta(y)$, $\beta(y) \leq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $y \in S(l_p)$ is arbitrary, by definition of $F(X)$, we have $F(l_p) \leq 2^{1+2/p}$.

To prove $f(l_p) = 2^{1+2/q}$, let $x = 2^{-1/p}(1, 1, 0, \ldots, 0, \ldots)$, $y = 2^{-1/p}(1, -1, 0, \ldots, 0, \ldots) \in S(l_p)$. Then $\|x + y\|^2 + \|x - y\|^2 = 2^{-2/p}(\|(2, 0, 0, \ldots, 0, \ldots)\|_{l_p}^2 + \|(0, 2, 0, \ldots, 0, \ldots)\|_{l_p}^2) = 2 \cdot 2^{-2/p} \cdot 4 = 2^{3-2/p} = 2^{1+2/q}$. From definition of $f(X)$, we have $f(l_p) \leq 2^{1+2/q}$.

On the other hand, by using Lagrange multipliers in basic calculus, the function $u^2 + v^2$, under the constraint $u^p + v^p = a$ for $1 < p \leq 2$, assumes its minimum $2^{1-2/p} \cdot a^{2/p}$ at $u = v = (a/2)^{1/p}$.

The Clarkson inequality [3, 4]: $\|x + y\|^p + \|x - y\|^p \geq (\|x\| + \|y\|)^p + |\|x\| - \|y\||^p$, for all $x, y \in X$, implies that $\|x + y\|^p + \|x - y\|^p \geq 2^p$, for all $x, y \in S(l_p)$. Therefore $\|x + y\|^2 + \|x - y\|^2 \geq 2^{1-2/p} \cdot (2^p)^{2/p} = 2^{3-2/p} = 2^{1+2/q}$, for all $x, y \in S(l_p)$. By definition of $f(X)$ we have $f(l_p) \geq 2^{1+2/q}$.

Finally, $f(l_p) = 2^{1+2/q}$.

(ii) To prove $F(l_p) \leq 2^{1+2/p}$, we need the inequality: $(1+x)^p \leq 1 + 2px$ if x is small enough and $p > 1$. For any $y = (y_1, y_2, y_3, \ldots, y_n, \ldots) \in S(l_p)$.

Take N such that $|y_N| < \delta < 1$. Let $e_N = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the Nth entry is 1 and others are 0, we have

$$
\begin{aligned}
\|y \pm e_N\|^p &= |y_1|^p + |y_2|^p + \cdots + |y_N \pm 1|^p + |y_{N+1}|^p + \cdots \\
&\leq |y_1|^p + |y_2|^p + \cdots + |1 + |y_N||^p + |y_{N+1}|^p + \cdots \\
&\leq (|y_1|^p + |y_2|^p + \cdots + |y_N|^p + |y_{N+1}|^p + \cdots) \\
&\quad + 1 + 2p|y_N| \leq 2 + 2p\delta.
\end{aligned}
\tag{4.3}
$$

Since δ can be arbitrarily small, by definition of $\beta(y)$, $\beta(y) \leq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $y \in S(l_p)$ is arbitrary, by definition of $F(X)$, we have $F(l_p) \leq 2^{1+2/p}$.

To prove $e(l_p) \geq 2^{1+2/p}$, for any $0 < \delta < 1$ and $y = (y_1, y_2, y_3, \ldots, y_n, \ldots) \in S(l_p)$ take N such that $|y_N| < \delta$. Let $e_N = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the Nth entry is 1 and others are 0. Since $(1-x)^p \geq 1 - px$ for $0 \leq x \leq 1$ and $p > 1$, we have

$$
\begin{aligned}
\|y \pm e_N\|^p &= |y_1|^p + |y_2|^p + \cdots + |y_N \pm 1|^p + |y_{N+1}|^p + \cdots \\
&\geq |y_1|^p + |y_2|^p + \cdots + |1 - |y_N||^p + |y_{N+1}|^p + \cdots \\
&\geq (1 + |y_1|^p + |y_2|^p + \cdots + |y_N|^p + |y_{N+1}|^p + \cdots) \\
&\quad - p|y_N| - |y_N|^p \geq 2 - 2p\delta.
\end{aligned}
\tag{4.4}
$$

Since δ can be arbitrarily small, by definition of $\alpha(y)$, $\alpha(y) \geq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $y \in S(l_p)$ is arbitrary, by definition of $e(X)$, we have $e(l_p) \geq 2^{1+2/p}$.

To prove $E(l_p) = 2^{1+2/q}$, let $x = 2^{-1/p}(1, 1, 0, \ldots, 0, \ldots)$, $y = 2^{-1/p}(1, -1, 0, \ldots, 0, \ldots)$ $\in S(l_p)$. Then $\|x + y\|^2 + \|x - y\|^2 = 2^{-2/p}(\|(2, 0, 0, \ldots, 0, \ldots)\|_{l_p}^2 + \|(0, 2, 0, \ldots, 0, \ldots\|_{l_p})^2) = 2 \cdot 2^{-2/p} \cdot 4 = 2^{3-2/p} = 2^{1+2/q}$. From definition of $E(X)$, we have $E(l_p) \geq 2^{1+2/q}$.

On the other hand, by using Lagrange multipliers in basic calculus, the function $u^2 + v^2$, under the constraint $u^p + v^p = a$ for $p > 2$, assumes its maximum $2^{1-2/p} \cdot a^{2/p}$ at $u = v = (a/2)^{1/p}$.

The Clarkson inequality: $\|x + y\|^p + \|x - y\|^p \leq (\|x\| + \|y\|)^p + |\|x\| - \|y\||^p$, for all $x, y \in X$, implies that $\|x + y\|^p + \|x - y\|^p \leq 2^p$, for all $x, y \in S(l_p)$. Therefore $\|x + y\|^2 + \|x - y\|^2 \leq 2^{1-2/p} \cdot (2^p)^{2/p} = 2^{3-2/p} = 2^{1+2/q}$, for all $x, y \in S(l_p)$. By the definition of $E(X)$ we have $E(l_p) \leq 2^{1+2/q}$.

Finally, we have $E(l_p) = 2^{1+2/q}$. $\qquad \square$

THEOREM 4.3. *Let $p, q \geq 1$ and $1/p + 1/q = 1$, then*
(i) $f(L_p[0,1]) = F(L_p[0,1]) = 2^{1+2/q}$,
$\quad 2^{1+2/p} \leq e(L_p[0,1]) \leq E(L_p[0,1])$, $1 < p \leq 2$;
(ii) $f(L_p[0,1]) \leq F(L_p[0,1]) \leq 2^{1+2/p}$,
$\quad e(L_p[0,1]) = E(L_p[0,1]) = 2^{1+2/q}$, $p > 2$.

Proof. (i) To prove $e(L_p[0,1]) \geq 2^{1+2/p}$, let $x(t) \in S(L_p[0,1])$.

For any $0 < \delta < 1$, we can find a $\gamma > 0$, a subset $E \subseteq [0,1]$ with $mE = \gamma$ and $\int_E |x(t)|^p \, dt < \delta$, such that $|x(t)| < \gamma^{-1/p} (t \in E)$. To prove the existence of such a set E, for the given

$\gamma > 0$ it is easy to take $E \subseteq [0,1]$ such that $mE = \gamma$ and $\int_E |x(t)|^p \, dt < \delta$. Consider $E_1 = \{t \in E, \; |x(t)| < \gamma^{-1/p}\}$. We claim then $\gamma \geq \gamma_1 = mE_1 > 0$. Otherwise, if $mE_1 = 0$ then $\int_E |x(t)|^p \, dt \geq 1$, and hence $\delta \geq 1$, this is a contradiction with $0 < \delta < 1$. We now can take E_1 as E, γ_1 as γ, and for $t \in E_1$, $|x(t)| < \gamma^{-1/p}$ implies $|x(t)| < \gamma_1^{-1/p}$. Let

$$z(t) = \begin{cases} \gamma^{-1/p}, & t \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

By using the inequality: $(1 - x)^p \geq 1 - px$ for $0 \leq x \leq 1$, $p > 1$, and Hölder's inequality:

$$\int_E |x(t)| \, dt \leq \left(\int_E 1^q \, dt \right)^{1/q} \left(\int_E |x(t)|^p \, dt \right)^{1/p},$$

$$\begin{aligned} \|x(t) \pm z(t)\|^p &= \int_E |x(t) \pm \gamma^{-1/p}|^p \, dt + \int_{[0,1]-E} |x(t)|^p \, dt \\ &\geq \gamma^{-1} \int_E |1 \pm \gamma^{1/p} x(t)|^p \, dt + 1 - \delta \geq \gamma^{-1} \int_E (1 - \gamma^{1/p} |x(t)|)^p \, dt \\ &\quad + 1 - \delta \geq \gamma^{-1} \int_E (1 - p\gamma^{1/p} |x(t)|) \, dt + 1 - \delta \\ &= \gamma^{-1}\gamma - p\gamma^{-1+1/p} \int_E |x(t)| \, dt + 1 - \delta \\ &\geq 1 + 1 - p\gamma^{-1+1/p+1/q} \left(\int_E |x(t)|^p \, dt \right)^{1/p} - \delta \geq 2 - p\delta^{1/p} - \delta. \end{aligned} \tag{4.6}$$

Since δ is arbitrary, $\alpha(x(t)) \geq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $x(t)$ is also arbitrary, we have $e(L_p[0,1]) \geq 2^{1+2/p}$.

To prove $F(L_p[0,1]) \leq 2^{1+2/q}$, let $x(t) \in S(L_p[0,1])$, then $\int_0^1 |x(t)|^p \, dt = 1$. Take $\gamma \in [0,1]$, such that $\int_0^\gamma |x(t)|^p \, dt = 1/2$, then $\int_\gamma^1 |x(t)|^p \, dt = 1/2$ also. Let

$$y(t) = \begin{cases} x(t), & 0 \leq t < \gamma, \\ -x(t), & \gamma \leq t \leq 1, \end{cases} \tag{4.7}$$

then $y(t) \in S(L_p[0,1])$, and $\|x(t) + y(t)\|^p = \int_0^\gamma 2^p |x(t)|^p \, dt = 2^{p-1}$.

Similarly $\|x(t) - y(t)\|^p = 2^{p-1}$ too.

By definition of $\beta(x(t))$, $\beta(x(t)) \leq 2 \cdot 2^{(2p-2)/p} = 2^{3-2/p} = 2^{1+2/q}$. Since $x(t)$ is arbitrary, we have $F(L_p[0,1]) \leq 2^{1+2/q}$.

On the other hand, recall the function $u^2 + v^2$, under the constraint $u^p + v^p = a$ for $1 < p \leq 2$, assumes its minimum $2^{1-2/p} \cdot a^{2/p}$ at $u = v = (a/2)^{1/p}$.

The Clarkson inequality [3, 4]: $\|x + y\|^p + \|x - y\|^p \geq (\|x\| + \|y\|)^p + |\|x\| - \|y\||^p$, for all $x, y \in X$, implies that $\|x + y\|^p + \|x - y\|^p \geq 2^p$, for all $x, y \in S(l_p)$. Therefore $\|x + y\|^2 + \|x - y\|^2 \geq 2^{1-2/p} \cdot (2^p)^{2/p} = 2^{3-2/p} = 2^{1+2/q}$, for all $x, y \in S(L_p[0,1])$. By definition of $f(X)$, we have $f(L_p[0,1]) \geq 2^{1+2/q}$.

From $2^{1+2/q} \leq f(L_p[0,1]) \leq F(L_p[0,1]) \leq 2^{1+2/q}$, we have $f(L_p[0,1]) = F(L_p[0,1]) = 2^{1+2/q}$.

(ii) To prove $F(L_p[0,1]) \leq 2^{1+2/p}$, for all $y(t) \in S(L_p[0,1])$, take $\delta > 0$ such that $(1 + \delta)^p < 1 + 2p\delta$. We can find $\gamma > 0$ with $\int_0^\gamma |x(t)|^p \, dt < \delta$.

Let

$$y(t) = \begin{cases} \gamma^{-1/p}, & 0 \leq t < \gamma, \\ 0, & \gamma \leq t \leq 1, \end{cases} \tag{4.8}$$

then by Minkowski inequality:

$$\left(\int_a^b |x(t) + y(t)|^p \, dt \right)^{1/p} \leq \left(\int_a^b |x(t)|^p \, dt \right)^{1/p} + \left(\int_a^b |y(t)|^p \, dt \right)^{1/p},$$

$$||x(t) \pm y(t)||^p = \int_0^\gamma |x(t) \pm \gamma^{-1/p}|^p \, dt + \int_\gamma^1 |x(t)|^p \, dt$$

$$\leq \left[\left(\int_0^\gamma |x(t)|^p \, dt \right)^{1/p} + \left(\int_0^\gamma |\gamma^{-1/p}|^p \, dt \right)^{1/p} \right]^p + 1 \tag{4.9}$$

$$\leq (\delta^{1/p} + 1)^p + 1.$$

Since δ is arbitrary, from definition of $\beta(x(t))$, $\beta(x(t)) \leq 2 \cdot 2^{2/p} = 2^{1+2/p}$. Since $x(t)$ is also arbitrary, we have $F(L_p[0,1]) \leq 2^{1+2/p}$.

To prove $e(L_p[0,1]) \geq 2^{1+2/q}$, let $x(t) \in S(L_p[0,1])$, then $\int_0^1 |x(t)|^p \, dt = 1$. Take $\gamma \in [0,1]$, such that $\int_0^\gamma |x(t)|^p \, dt = 1/2$, then $\int_\gamma^1 |x(t)|^p \, dt = 1/2$ also. Let

$$y(t) = \begin{cases} x(t), & 0 \leq t < \gamma, \\ -x(t), & \gamma \leq t \leq 1, \end{cases} \tag{4.10}$$

then $y(t) \in S(L_p[0,1])$, and $||x(t) + y(t)||^p = \int_0^\gamma 2^p |x(t)|^p \, dt = 2^{p-1}$.

Similarly $||x(t) - y(t)||^p = 2^{p-1}$ too.

By definition of $\alpha(x(t))$, $\alpha(x(t)) \geq 2 \cdot 2^{(2p-2)/p} = 2^{3-2/p} = 2^{1+2/q}$. Since $x(t)$ is arbitrary we have $e(L_p[0,1]) \geq 2^{1+2/q}$.

On the other hand, recall the function $u^2 + v^2$, under the constraint $u^p + v^p = a$ for $p > 2$, assumes its maximum $2^{1-2/p} \cdot a^{2/p}$ at $u = v = (a/2)^{1/p}$.

The Clarkson inequality for $p \geq 2$: $||x + y||^p + ||x - y||^p \leq (||x|| + ||y||)^p + |\,||x|| - ||y||\,|^p$, for all $x, y \in X$, implies that $||x + y||^p + ||x - y||^p \leq 2^p$, for all $x, y \in S(l_p)$. Therefore $||x + y||^2 + ||x - y||^2 \leq 2^{1-2/p} \cdot (2^p)^{2/p} = 2^{3-2/p} = 2^{1+2/q}$ for all $x, y \in S(L_p[0,1])$. By the definition of $E(X)$ we have $E(L_p[0,1]) \leq 2^{1+2/q}$.

From $2^{1+2/q} \leq e(L_p[0,1]) \leq E(L_p[0,1]) \leq 2^{1+2/q}$, we have $e(L_p[0,1]) = E(L_p[0,1]) = 2^{1+2/q}$. $\qquad\square$

5. The parameters and uniform normal structure

Let U be an ultrafilter on an index set I. Let $\{X_i\}_{i \in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $||(x_i)|| = \sup_{i \in I} ||x_i|| < \infty$.

Definition 5.1 [13]. Let U be an ultrafilter on I and let $N_U = \{(x_i) \in l_\infty(I,X_i) : \lim_U \|x_i\|$ $= 0\}$. The ultraproduct of $\{X_i\}_{i \in I}$ is the quotient space $l_\infty(I,X_i)/N_U$ equipped with the quotient norm.

We will use $(x_i)_U$ to denote the element of the ultraproduct. It follows from (ii) of the fact, and the definition of quotient norm that

$$\|(x_i)_U\| = \lim_U \|x_i\|. \tag{5.1}$$

In the following we will restrict our index set I to be N, the set of natural numbers, and let $X_i = X$, $i \in N$ for some Banach space X. For an ultrafilter U on N, we use X_U to denote the ultraproduct.

THEOREM 5.2. *For any Banach space X, and for any nontrivial ultrafilter U on N, $K(X_U) = K(X)$, where K stands for either f, F, e, or E.*

Proof. We first prove this theorem for $K = f$, the proof for $K = E$ is similar.

For any $\delta > 0$, from definition of $f(X)$, there exists a two-dimensional subspace $X_2 \subseteq X$, and $x, y \in S(X_2)$ such that $\|x + y\|^2 + \|x - y\|^2 < f(X) + \delta$.

Let $x_i = x$, and $y_i = y$, for all $i \in N$. Then $(x_i)_U, (y_i)_U \in S((X_U)_2)$, where $(X_U)_2$ is a two-dimensional subspace, spanned by $(x_i)_U$ and $(y_i)_U$, of X_U. Then $\|(x_i)_U + (y_i)_U\|^2 + \|(x_i)_U - (y_i)_U\|^2 = \|x + y\|^2 + \|x - y\|^2 < f(X) + \delta$. Therefore $f(X_U) < f(X) + \delta$.

Since δ can be arbitrarily small, we proved $f(X_U) \le f(X)$.

To prove the reverse inequality, we choose a $(X_U)_2 \subseteq X_U$, $(x_i)_U, (y_i)_U \in S((X_U)_2)$ such that $\|(x_i)_U + (y_i)_U\|^2 + \|(x_i)_U - (y_i)_U\|^2 < f(X_U) + \delta$.

From (5.1) above, the sets

$I = \{i \in N : |\|x_i\| - 1| < \delta\}$,

$J = \{i \in N : |\|y_i\| - 1| < \delta\}$,

$L = \{i \in N : \|x_i + y_i\|^2 + \|x_i - y_i\|^2 < f(X_U) + \delta\}$ are all in U. So the intersection $I \cap J \cap L$ is in U too, and is hence not empty.

Let $i \in I \cap J \cap L$, we have $|\|x_i\| - 1| < \delta$, $|\|y_i\| - 1| < \delta$, and $\|x_i + y_i\|^2 + \|x_i - y_i\|^2 < f(X_U) + \delta$. Let $x_i', y_i' \in S(X_2)$ be the normalization of x_i, y_i respectively. Since $\|x_i' - x_i\|$ and $\|y_i' - y_i\| \le \delta$, then $\|x_i' + y_i'\| \le \|x_i + y_i\| + 2\delta$, $\|x_i' - y_i'\| \le \|x_i - y_i\| + 2\delta$, and therefore $\|x_i' + y_i'\|^2 + \|x_i' - y_i'\|^2 \le \|x_i + y_i\|^2 + \|x_i - y_i\|^2 + 18\delta < f(X_U) + 19\delta$. Hence $f(X) < f(X_U) + 19\delta$. Since δ can be arbitrarily small, $f(X) \le f(X_U)$.

We then prove this theorem for $K = F$, the proof for $K = e$ is similar.

For any $\delta > 0$, from definition of $F(X)$, there exists a $x \in S(X)$ such that $\|x + y\|^2 + \|x - y\|^2 > F(X) - \delta$ for all $y \in S(X)$.

Let $x_i = x$, for all $i \in N$, then $(x_i)_U \in S((X_U))$. For any $(y_i)_U \in X_U$, from definition of the norm in X_U, we may assume that $y_i \in S(X)$, for all $i \in N$. Therefore

$$\begin{aligned}
\|(x_i)_U &+ (y_i)_U\|^2 + \|(x_i)_U - (y_i)_U\|^2 \\
&= \|(x + y_i)_U\|^2 + \|(x - y_i)_U\|^2 \\
&= (\|x + y_i\|^2 + \|x - y_i\|^2)_U > F(X) - \delta.
\end{aligned} \tag{5.2}$$

Therefore $F(X_U) > F(X) - \delta$.

Since δ can be arbitrarily small, we proved $F(X_U) \geq F(X)$.

To prove the reverse inequality, for any $\delta > 0$, from definition of $F(X_U)$ there exists a $(x_i)_U \in S((X_U))$ such that $\|(x_i)_U + (y_m)_U\|^2 + \|(x_i)_U - (y_m)_U\|^2 > f(X_U) - \delta$ for all $(y_m)_U \in S((X_U))$. From definition of the norm in X_U, we may assume $x_i, y_m \in S(X)$, for all $i, m \in N$.

Let $L = \{i \in N : \|x_i + y_m\|^2 + \|x_i - y_m\|^2 > f(X_U) - \delta\}$, then L is in U, and is hence not empty. Let $j \in L$, from (5.1) above, we have $\|x_j + y_m\|^2 + \|x_j - y_m\|^2 > f(X_U) - \delta$ for $y_m \in S(X)$. But $y_m \in S(X)$ may be any vector on $S(X)$. We have $\|x_j + y\|^2 + \|x_j - y\|^2 > f(X_U) - \delta$ for all $y \in S(X)$. Therefore $f(X) > f(X_U) - \delta$. Since δ can be arbitrarily small, $f(X) \geq f(X_U)$.

Finally $f(X) = f(X_U)$. □

Theorem 5.3. *A Banach space X with either $f(X) > 32/9$ or $E(X) < 5$ has uniform normal structure.*

Proof. The idea of the proof is same as the proof of [7, Theorem 4.4]. □

Acknowledgment

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