ON MULTIPLE HARDY-HILBERT INTEGRAL INEQUALITIES WITH SOME PARAMETERS

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By introducing some parameters and norm $\|x\|_\alpha$ $(x \in \mathbb{R}^n)$, we give multiple Hardy-Hilbert integral inequalities, and prove that their constant factors are the best possible when parameters satisfy appropriate conditions.

1. Introduction

If $p > 1$, $1/p + 1/q = 1$, $f \ge 0$, $g \ge 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then we have the well-known Hardy-Hilbert inequality (see [4]):

$$\iint_0^{+\infty} \frac{f(x)g(x)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{+\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Its equivalent form is

$$\int_0^{+\infty}\left(\int_0^{+\infty}\frac{f(x)}{x+y}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^{+\infty} f^p(x)dx, \tag{1.2}$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is also the best possible.

Hardy-Hilbert inequalities are important in analysis and in their applications (see [7]). In recent years, many results (see [1, 3, 8–10]) have been obtained in the research of Hardy-Hilbert inequality. At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert integral inequalities are researched (see [5, 6, 11]). Yang [11] obtains the following: if $\alpha \in \mathbb{R}$, $n \ge 2$, $p_i > 1$ $(i = 1,2,\dots,n)$, $\sum_{i=1}^n(1/p_i) = 1$, $\lambda > n - \min_{1\le i\le n}\{p_i\}$, $f_i \ge 0$, and

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$0 < \int_{\alpha}^{+\infty}(t-\alpha)^{n-1-\lambda} f_i^{p_i}(t)dt < +\infty$, $(i = 1,2,\dots,n)$, then

$$\int_{\alpha}^{+\infty}\cdots\int_{\alpha}^{+\infty}\frac{1}{\left(\sum_{i=1}^{n}x_i - n\alpha\right)^{\lambda}}\prod_{i=1}^{n}f_i(x_i)dx_1\dots dx_n < \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-\lambda}{p_i}\right)\left[\int_{\alpha}^{+\infty}(t-\alpha)^{n-1-\lambda}f_i^{p_i}(t)dt\right]^{1/p_i}, \tag{1.3}$$

where the constant factor $(1/\Gamma(\lambda))\prod_{i=1}^{n}\Gamma(1-(n-\lambda)/p_i)$ is the best possible.

In this paper, by introducing some parameters and norm $\|x\|_{\alpha}$ $(x \in \mathbb{R}^n)$, we give multiple Hardy-Hilbert integral inequalities, and discuss the problem of the best constant factor. For this reason, we introduce the notation

$$R_+^n = \{x = (x_1,\dots,x_n) : x_1,\dots,x_n > 0\},$$
$$\|x\|_{\alpha} = \left(x_1^{\alpha}+\cdots+x_n^{\alpha}\right)^{1/\alpha}, \quad (\alpha > 0), \tag{1.4}$$

and we agree on $\|x\|_{\alpha} < c$ representing $\{x \in \mathbb{R}_+^n : \|x\|_{\alpha} < c\}$.

2. Some lemmas

Lemma 2.1 (see [2]). *If $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, $(i = 1,2,\dots,n)$, $\Psi(u)$ is a measurable function, then*

$$\int\cdots\int_{x_1,\dots,x_n>0;\ (x_1/a_1)^{\alpha_1}+\cdots+(x_n/a_n)^{\alpha_n}\le 1}\Psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1}+\cdots+\left(\frac{x_n}{a_n}\right)^{\alpha_n}\right)\times x_1^{p_1-1}\dots x_n^{p_n-1}dx_1\dots dx_n = \frac{a_1^{p_1}\dots a_n^{p_n}\Gamma(p_1/\alpha_1)\dots\Gamma(p_n/\alpha_n)}{\alpha_1\dots\alpha_n\Gamma(p_1/\alpha_1+\cdots+p_n/\alpha_n)}\int_0^1\Psi(u)u^{p_1/\alpha_1+\cdots+p_n/\alpha_n-1}du, \tag{2.1}$$

where the $\Gamma(\cdot)$ is Γ-function.

Lemma 2.2. *If $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $m \in \mathbb{R}$, $0 < n-m < \beta\lambda$, and setting weight function $\omega_{\alpha,\beta,\lambda}(m,n,y)$ as*

$$\omega_{\alpha,\beta,\lambda}(m,n,y) = \int_{R_+^n}\frac{1}{\left(\|x\|_{\alpha}^{\beta}+\|y\|_{\alpha}^{\beta}\right)^{\lambda}}\|x\|_{\alpha}^{-m}dx, \tag{2.2}$$

then

$$\omega_{\alpha,\beta,\lambda}(m,n,y) = \|y\|_\alpha^{n-\beta\lambda-m}\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{n-m}{\beta},\lambda-\frac{n-m}{\beta}\right), \tag{2.3}$$

where the $B(\cdot,\cdot)$ is β-function.

Proof. By Lemma 2.1, we have

$$\begin{aligned}
\omega_{\alpha,\beta,\lambda}(m,n,y) &= \int_{R_+^n}\frac{1}{\left(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta\right)^\lambda}\|x\|_\alpha^{-m}dy \\
&= \lim_{r\to+\infty}\int\cdots\int_{x_1,\ldots,x_n>0;\ x_1^\alpha+\cdots+x_n^\alpha<r^\alpha} \\
&\quad\times\frac{\left[r\left((x_1/r)^\alpha+\cdots+(x_n/r)^\alpha\right)^{1/\alpha}\right]^{-m}}{\left[r^\beta\left((x_1/r)^\alpha+\cdots+(x_n/r)^\alpha\right)^{\beta/\alpha}+\|y\|_\alpha^\beta\right]^\lambda}x_1^{1-1}\ldots x_n^{1-1}dx_1\ldots dx_n \\
&= \lim_{r\to+\infty}\frac{r^n\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^1\frac{\left(ru^{1/\alpha}\right)^{-m}}{\left(\|y\|_\alpha^\beta+r^\beta u^{\beta/\alpha}\right)^\lambda}u^{n/\alpha-1}du \\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\lim_{r\to+\infty}\int_0^r\frac{1}{\left(\|y\|_\alpha^\beta+t^\beta\right)^\lambda}t^{n-m-1}dt \\
&= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_0^{+\infty}\frac{1}{\left(\|y\|_\alpha^\beta+t^\beta\right)^\lambda}t^{n-m-1}dt \\
&= \|y\|_\alpha^{n-\beta\lambda-m}\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}\int_0^1\frac{1}{(1+u)^\lambda}u^{(n-m)/\beta-1}du \\
&= \|y\|_\alpha^{n-\beta\lambda-m}\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{n-m}{\beta},\lambda-\frac{n-m}{\beta}\right).
\end{aligned} \tag{2.4}$$

Hence (2.3) is valid. □

3. Main results

THEOREM 3.1. *If $p>1$, $1/p+1/q=1$, $n\in\mathbb{Z}_+$, $\alpha>0$, $\beta>0$, $\lambda>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$, $0<n-ap<\beta\lambda$, $0<n-bq<\beta\lambda$, $f\geq 0$, $g\geq 0$, and*

$$0<\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx<+\infty, \tag{3.1}$$

$$0<\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}g^q(y)dy<+\infty, \tag{3.2}$$

then

$$\iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\,dy$$
$$< C_{\alpha,\beta,\lambda}(a,b,p,q) \times \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x)dx \right)^{1/p} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y)dy \right)^{1/q}, \tag{3.3}$$

$$\int_{R_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left[\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy$$
$$< C^p_{\alpha,\beta,\lambda}(a,b,p,q) \times \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x)dx, \tag{3.4}$$

where $C_{\alpha,\beta,\lambda}(a,b,p,q) = (\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B^{1/p}((n-ap)/\beta,\lambda-(n-ap)/\beta)B^{1/q}((n-bq)/\beta,\lambda-(n-bq)/\beta)$.

Proof. By Hölder's inequality, we have

$$G := \iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\,dy$$
$$= \iint_{R_+^n} \left(\frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^{\lambda/p}} \frac{\|x\|_\alpha^b}{\|y\|_\alpha^a} \right) \left(\frac{g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^{\lambda/q}} \frac{\|y\|_\alpha^a}{\|x\|_\alpha^b} \right) dx\,dy$$
$$\leq \left(\iint_{R_+^n} \frac{f^p(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|x\|_\alpha^{bp}}{\|y\|_\alpha^{ap}} dx\,dy \right)^{1/p} \times \left(\iint_{R_+^n} \frac{g^q(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|y\|_\alpha^{aq}}{\|x\|_\alpha^{bq}} dx\,dy \right)^{1/q}, \tag{3.5}$$

according to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2, such that they are not all zero, and

$$\frac{C_1 f^p(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|x\|_\alpha^{bp}}{\|y\|_\alpha^{ap}} = \frac{C_2 g^q(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|y\|_\alpha^{aq}}{\|x\|_\alpha^{bq}}, \quad \text{a.e. } (x,y) \in R_+^n \times R_+^n. \tag{3.6}$$

Without losing generality, we suppose that $C_1 \neq 0$, we may get

$$\|x\|_\alpha^{b(p+q)} f^p(x) = \frac{C_2}{C_1} \|y\|_\alpha^{a(p+q)} g^q(y), \quad \text{a.e. } (x,y) \in R_+^n \times R_+^n, \tag{3.7}$$

hence, we obtain

$$\|x\|_\alpha^{b(p+q)} f^p(x) = C(\text{constant}), \quad \text{a.e. } x \in R_+^n, \tag{3.8}$$

hence, we have

$$\begin{aligned}\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx &= \int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)-bq-ap+b(p+q)}f^p(x)dx\\ &= C\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)-bq-ap}dx=\infty,\end{aligned}\tag{3.9}$$

which contradicts (3.1). Hence, and by Lemma 2.2, we obtain

$$\begin{aligned}G &< \left[\int_{R_+^n}\left(\int_{R_+^n}\frac{1}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}\frac{1}{\|y\|_\alpha^{ap}}dy\right)\|x\|_\alpha^{bp}f^p(x)dx\right]^{1/p}\\
&\quad\times\left[\int_{R_+^n}\left(\int_{R_+^n}\frac{1}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}\frac{1}{\|x\|_\alpha^{bq}}dx\right)\|y\|_\alpha^{aq}g^q(y)dy\right]^{1/q}\\
&= \left(\int_{R_+^n}\omega_{\alpha,\beta,\lambda,}(ap,n,x)\|x\|_\alpha^{bp}f^p(x)dx\right)^{1/p}\left(\int_{R_+^n}\omega_{\alpha,\beta,\lambda,}(bq,n,y)\|y\|_\alpha^{aq}g^q(y)dy\right)^{1/q}\\
&= \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{n-ap}{\beta},\lambda-\frac{n-ap}{\beta}\right)\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right]^{1/p}\\
&\quad\times\left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{n-bq}{\beta},\lambda-\frac{n-bq}{\beta}\right)\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}g^q(y)dy\right]^{1/q}\\
&= C_{\alpha,\beta,\lambda,}(a,b,p,q)\left[\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right]^{1/p}\times\left[\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}g^q(y)dy\right]^{1/q}.\end{aligned}\tag{3.10}$$

Hence, (3.3) is valid.

Let $k=((n-\beta\lambda)+q(a-b))/(1-q)$, for $0<h<l<+\infty$, setting

$$\begin{aligned}g_{h,l}(y) &= \begin{cases}\|y\|_\alpha^k\left(\int_{R_+^n}\dfrac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)^{p/q}, & h<\|y\|_\alpha<l,\\ 0, & 0<\|y\|_\alpha\le h \text{ or } \|y\|_\alpha\ge l,\end{cases}\\
\tilde{g}(y) &= \|y\|_\alpha^k\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)^{p/q},\quad y\in R_+^n,\end{aligned}\tag{3.11}$$

by (3.1), for sufficiently small $h>0$ and sufficiently large $l>0$, we have

$$0<\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}g_{h,l}^q(y)dy<+\infty.\tag{3.12}$$

Hence, by (3.3), we have

$$\begin{aligned}
&\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}\widetilde{g}^q(y)dy\\
&\quad=\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{k(1-q)}\widetilde{g}^q(y)dy=\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{k}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|x\|_\alpha^\beta)^\lambda}dx\right)^p dy\\
&\quad=\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{k}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)^{p/q}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)dy\\
&\quad=\iint_{R_+^n}\frac{f(x)g_{h,l}(y)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\,dy<C_{\alpha,\beta,\lambda,}(a,b,p,q)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right)^{1/p}\\
&\qquad\times\left(\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}g_{h,l}^q(y)dy\right)^{1/q}=C_{\alpha,\beta,\lambda,}(a,b,p,q)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right)^{1/p}\\
&\qquad\times\left(\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}\widetilde{g}^q(y)dy\right)^{1/q},
\end{aligned}\tag{3.13}$$

it follows that

$$\int_{h<\|y\|_\alpha<l}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}\widetilde{g}^q(y)dy<C_{\alpha,\beta,\lambda,}^p(a,b,p,q)\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx.\tag{3.14}$$

For $h\to 0^+$, $l\to+\infty$, we obtain

$$\begin{aligned}
0&<\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}\widetilde{g}^q(y)dy\\
&\le C_{\alpha,\beta,\lambda,}^p(a,b,p,q)\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx<+\infty,
\end{aligned}\tag{3.15}$$

hence, by (3.3), we obtain

$$\begin{aligned}
&\int_{R_+^n}\|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)^p dy\\
&\quad=\iint_{R_+^n}\frac{f(x)\widetilde{g}(y)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\,dy<C_{\alpha,\beta,\lambda,}(a,b,p,q)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right)^{1/p}\\
&\qquad\times\left(\int_{R_+^n}\|y\|_\alpha^{(n-\beta\lambda)+q(a-b)}\widetilde{g}^q(y)dy\right)^{1/q}=C_{\alpha,\beta,\lambda,}(a,b,p,q)\left(\int_{R_+^n}\|x\|_\alpha^{(n-\beta\lambda)+p(b-a)}f^p(x)dx\right)^{1/p}\\
&\qquad\times\left[\int_{R_+^n}\|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)}\left(\int_{R_+^n}\frac{f(x)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\right)^p dy\right]^{1/q}.
\end{aligned}\tag{3.16}$$

Hence, we can obtain (3.4). □

Remark 3.2. If f and g do not satisfy (3.1) and (3.2), by the proof of Theorem 3.1, we can obtain

$$\iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\,dy \leq C_{\alpha,\beta,\lambda}(a,b,p,q) \times \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x)dx \right)^{1/p} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y)dy \right)^{1/q}, \tag{3.17}$$

$$\int_{R_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left[\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy \leq C_{\alpha,\beta,\lambda}^p(a,b,p,q) \times \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x)dx. \tag{3.18}$$

Remark 3.3. By (3.4), we can also obtain (3.3), hence (3.4) and (3.3) are equivalent.

THEOREM 3.4. *If $p > 1$, $1/p + 1/q = 1$, $n \in Z_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $a \in R$, $b \in R$, $0 < n - ap < \beta\lambda$, $ap + bq = 2n - \beta\lambda$, $f \geq 0$, $g \geq 0$, and*

$$0 < \int_{R_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x)dx < +\infty, \quad 0 < \int_{R_+^n} \|y\|_\alpha^{a(p+q)-n} g^q(y)dy < +\infty, \tag{3.19}$$

then

$$\iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\,dy < \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-ap}{\beta}, \lambda - \frac{n-ap}{\beta} \right) \times \left(\int_{R_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x)dx \right)^{1/p} \left(\int_{R_+^n} \|y\|_\alpha^{a(p+q)-n} g^q(y)dy \right)^{1/q}, \tag{3.20}$$

$$\int_{R_+^n} \|y\|_\alpha^{(a(p+q)-n)/(1-q)} \left[\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy < \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-ap}{\beta}, \lambda - \frac{n-ap}{\beta} \right) \right]^p \int_{R_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x)dx, \tag{3.21}$$

where the constant factors $(\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n-ap)/\beta, \lambda-(n-ap)/\beta)$ and $[(\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n-ap)/\beta, \lambda-(n-ap)/\beta)]^p$ are all the best possible.

Proof. Since $ap + bq = 2n - \beta\lambda$, we have

$$n - bq = n - (2n - \beta\lambda - ap) = \beta\lambda - (n - ap), \tag{3.22}$$

hence, by $0 < n - ap < \beta\lambda$, we obtain $0 < n - bq < \beta\lambda$, and

$$(n-\beta\lambda)+p(b-a)=b(p+q)-n, \qquad (n-\beta\lambda)+q(a-b)=a(p+q)-n,$$
$$\frac{n-ap}{\beta}=\lambda-\frac{n-bq}{\beta}, \qquad \lambda-\frac{n-ap}{\beta}=\frac{n-bq}{\beta}. \tag{3.23}$$

By Theorem 3.1, (3.20) and (3.21) are valid.

If the constant factor $K_1 := (\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n-ap)/\beta,\lambda-(n-ap)/\beta)$ in (3.20) is not the best possible, then there exists a positive constant $K < K_1$, such that (3.20) is still valid when we replace K_1 by K.

In particular, for $0 < \varepsilon < q(n - ap)$, we take

$$f_\varepsilon(x)=\|x\|_\alpha^{-bq-\varepsilon/p}, \qquad g_\varepsilon(y)=\|y\|_\alpha^{-ap-\varepsilon/q}, \tag{3.24}$$

by (3.17) and the properties of limit, when $\delta > 0$ is sufficiently small, we have

$$\begin{aligned}
&\int_{\|x\|_\alpha>\delta}\int_{R_+^n}\frac{f_\varepsilon(x)g_\varepsilon(y)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\,dy \\
&\quad\le K\left(\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{b(p+q)-n}f_\varepsilon^p(x)dx\right)^{1/p}\left(\int_{\|y\|_\alpha>\delta}\|y\|_\alpha^{a(p+q)-n}g_\varepsilon^q(y)dy\right)^{1/q} \\
&\quad=K\left(\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{-n-\varepsilon}\right)^{1/p}\left(\int_{\|y\|_\alpha>\delta}\|y\|_\alpha^{-n-\varepsilon}dy\right)^{1/q}=K\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{-n-\varepsilon}dx.
\end{aligned} \tag{3.25}$$

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}
&\int_{\|x\|_\alpha>\delta}\int_{R_+^n}\frac{f_\varepsilon(x)g_\varepsilon(y)}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}dx\,dy \\
&\quad=\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{-bq-\varepsilon/p}\int_{R_+^n}\frac{1}{(\|x\|_\alpha^\beta+\|y\|_\alpha^\beta)^\lambda}\|y\|_\alpha^{-ap-\varepsilon/q}dy\,dx \\
&\quad=\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{-bq-\varepsilon/p}\widetilde{\omega}_{\alpha,\beta,\lambda}\left(ap+\frac{\varepsilon}{q},n,x\right)dx \\
&\quad=\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{1}{\beta}\left(n-ap-\frac{\varepsilon}{q}\right),\lambda-\frac{1}{\beta}\left(n-ap-\frac{\varepsilon}{q}\right)\right)\int_{\|x\|_\alpha>\delta}\|x\|_\alpha^{-n-\varepsilon}dx.
\end{aligned} \tag{3.26}$$

Hence, we obtain

$$\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{1}{\beta}\left(n-ap-\frac{\varepsilon}{q}\right),\lambda-\frac{1}{\beta}\left(n-ap-\frac{\varepsilon}{q}\right)\right)\le K, \tag{3.27}$$

for $\varepsilon \to 0^+$, we have

$$K_1=\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)}B\left(\frac{n-ap}{\beta},\lambda-\frac{n-ap}{\beta}\right)\le K, \tag{3.28}$$

which contradicts the fact that $K < K_1$. Hence the constant factor in (3.20) is the best possible.

Since (3.21) and (3.20) are equivalent, the constant factor in (3.21) is also the best possible. □

4. Some corollaries

Corollary 4.1. *If* $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $f \geq 0$, $g \geq 0$, *and*

$$0 < \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x)dx < +\infty,$$
$$0 < \int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)(q-1)} g^q(y)dy < +\infty, \tag{4.1}$$

then

$$\iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\,dy < \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x)dx\right)^{1/p} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)(q-1)} g^q(y)dy\right)^{1/q},$$

$$\int_{R_+^n} \|y\|_\alpha^{\beta\lambda - n} \left[\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\right]^p dy < \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x)dx, \tag{4.2}$$

where the constant factors in (4.2) are all the best possible.

Proof. If we take $a = n/p - \beta\lambda/p^2$, $b = n/q - \beta\lambda/q^2$ in Theorem 3.4, (4.2) can be obtained. □

Remark 4.2. If we take $n = \lambda = 1$ in (4.2), we can obtain the results of [10]:

$$\iint_0^{+\infty} \frac{f(x)g(y)}{x^\beta + y^\beta} dx\,dy < \frac{\pi}{\beta\sin(\pi/p)} \left(\int_0^{+\infty} x^{(p-1)(1-\beta)} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\beta)} g^q(y)dy\right)^{1/q},$$
$$\int_0^{+\infty} y^{\beta-1} \left(\int_0^{+\infty} \frac{f(x)}{x^\beta + y^\beta} dx\right)^p dy < \left(\frac{\pi}{\beta\sin(\pi/p)}\right)^p \int_0^{+\infty} x^{(p-1)(1-\beta)} f^p(x)dx, \tag{4.3}$$

where the constant factors in (4.3) are all the best possible.

If we take $n=\beta=1$ in (4.2), we can obtain

$$\iint_0^{+\infty}\frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy$$
$$<B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^{+\infty}x^{(1-\lambda)(p-1)}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1-\lambda)(q-1)}g^q(y)dy\right)^{1/q}, \tag{4.4}$$
$$\int_0^{+\infty}y^{\lambda-1}\left(\int_0^{+\infty}\frac{f(x)}{(x+y)^\lambda}dx\right)^p dy<B^p\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\int_0^{+\infty}x^{(1-\lambda)(p-1)}f^p(x)dx,$$

where the constant factors in (4.4) are all the best possible.

Corollary 4.3. *If $p>1$, $1/p+1/q=1$, $n\in\mathbb{Z}_+$, $\lambda>0$, $np+\lambda-2n>0$, $nq+\lambda-2n>0$ $f\geq 0$, $g\geq 0$, and*

$$0<\int_{R_+^n}\|x\|_\alpha^{n-\lambda}f^p(x)dx<+\infty,$$
$$0<\int_{R_+^n}\|y\|_\alpha^{n-\lambda}g^q(y)dy<+\infty, \tag{4.5}$$

then

$$\iint_{R_+^n}\frac{f(x)g(y)}{\left(\|x\|_\alpha+\|y\|_\alpha\right)^\lambda}dx\,dy$$
$$<B\left(\frac{np+\lambda-2n}{p},\frac{nq+\lambda-2n}{q}\right)\left(\int_{R_+^n}\|x\|_\alpha^{n-\lambda}f^p(x)dx\right)^{1/p}\left(\int_{R_+^n}\|y\|_\alpha^{n-\lambda}g^q(y)dy\right)^{1/q},$$
$$\int_{R_+^n}\|y\|_\alpha^{(n-\lambda)/(1-q)}\left[\int_{R_+^n}\frac{f(x)}{\left(\|x\|_\alpha+\|y\|_\alpha\right)^\lambda}dx\right]^p dy<B^p\left(\frac{np+\lambda-2n}{p},\frac{nq+\lambda-2n}{q}\right)\int_{R_+^n}\|x\|_\alpha^{n-\lambda}f^p(x)dx, \tag{4.6}$$

where the constant factors in (4.6) are all the best possible.

Proof. If we take $\beta=1$, $a=b=(2n-\lambda)/pq$ in Theorem 3.4, (4.6) can be obtained. □

Remark 4.4. If we take $n=1$ in (4.6), we can obtain the results of [1]:

$$\iint_0^{+\infty}\frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy$$
$$<B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left(\int_0^{+\infty}x^{1-\lambda}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{1-\lambda}g^q(y)dy\right)^{1/q},$$
$$\int_0^{+\infty}y^{(1-\lambda)/(1-q)}\left(\int_0^{+\infty}\frac{f(x)}{(x+y)^\lambda}dx\right)^p dy<B^p\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\int_0^{+\infty}x^{1-\lambda}f^p(x)dx, \tag{4.7}$$

where the constant factors in (4.7) are all the best possible.

If we take other appropriate parameters, we can obtain many new inequalities.

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