# TO A NONLOCAL GENERALIZATION OF THE DIRICHLET PROBLEM

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A mixed problem with a boundary Dirichlet condition and nonlocal integral condition is considered for a two-dimensional elliptic equation.The existence and uniqueness of a weak solution of this problem are proved in a weighted Sobolev space.



## 1. Introduction

The first paper devoted to a nonlocal boundary value problem with integral conditions goes back to Cannon [1]. More general nonlocal conditions for different types of partial differential equations were considered later (see, e.g., [2, 4–6, 9, 11, 13, 14]).

In the present paper, a mixed problem with a boundary Dirichlet condition and nonlocal integral condition is considered in a unit square for a second-order elliptic equation. The existence and uniqueness of a weak solution of this problem in the weighted Sobolev space $W_2^1(\Omega,\rho)$ are proved. The proof is based on the Lax-Milgram lemma. It is shown that a nonlocal problem can be regarded as a generalization of the Dirichlet boundary value problem.

## 2. Statement of the problem

Let $\Omega = \{(x_1,x_2) : 0 < x_k < 1,\ k = 1,2\}$ be a unit square with boundary $\Gamma$, $\Gamma_1 = \{(0,x_2) : 0 < x_2 < 1\}$, $\Gamma_* = \Gamma\backslash\Gamma_1$, $\Omega_\xi = (0,\xi)\times(0,1)$, let $\xi$ be a fixed point from $(0,1]$.

By $L_2(\Omega,\rho)$ we denote a weighted Lebesgue space of all real-valued functions $u(x)$ on $\Omega$ with the inner product and the norm

$$(u,v)_\rho = \int_\Omega \rho(x)u(x)v(x)\,dx, \qquad \|u\|_\rho = (u,u)_\rho^{1/2}. \tag{2.1}$$

The weighted Sobolev space $W_2^1(\Omega,\rho)$ is usually defined as a linear set of all functions $u(x) \in L_2(\Omega,\rho)$ whose derivatives $\partial u/\partial x_k$, $k = 1,2$ (in the generalized sense), belong

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to $L_2(\Omega,\rho)$. It is a normed linear space if equipped with the norm

$$\|u\|_{1,\rho} = \left(\|u\|_\rho^2 + |u|_{1,\rho}^2\right)^{1/2}, \qquad |u|_{1,\rho}^2 = \left\|\frac{\partial u}{\partial x_1}\right\|_\rho^2 + \left\|\frac{\partial u}{\partial x_2}\right\|_\rho^2. \tag{2.2}$$

Let us choose a weight function $\rho(x)$ as follows:

$$\rho(x) = \begin{cases} (x_1/\xi)^\varepsilon, & x_1 \le \xi, \\ 1, & x_1 > \xi, \end{cases} \qquad \varepsilon \in (0,1). \tag{2.3}$$

It is well known (see, e.g., [7, page 10], [10, Theorem 3.1]) that $W_2^1(\Omega,\rho)$ is a Banach space and $C^\infty(\bar{\Omega})$ is dense in $W_2^1(\Omega,\rho)$ and in $L_2(\Omega,\rho)$. As an immediate consequence, we can define the space $W_2^1(\Omega,\rho)$ as a closure of $C^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_{1,\rho}$ and these both definitions are equivalent.

Define the subspace of the space $W_2^1(\Omega,\rho)$ which can be obtained by closing the set $\overset{*}{C^\infty}(\bar{\Omega}) = \{u \in C^\infty(\bar{\Omega}) : \operatorname{supp} u \cap \Gamma_* = \varnothing,\ l(u) = 0,\ 0 < x_2 < 1\}$ with the norm $\|\cdot\|_{1,\rho}$. Denote it by $\overset{*}{W_2^1}(\Omega,\rho)$.

For $f \in L_\infty(\Omega)$, denote $\|f\|_\infty = \operatorname{vrai\,max}_{x\in\Omega} |f(x)|$.

We say that the function $b$ has the *property* (P) if $b \in L_\infty(\Omega)$ and $0 \le x_1^\varepsilon \partial(x_1^{1-\varepsilon} b)/\partial x_1 \in L_\infty(\Omega_\xi)$.

Consider the nonlocal boundary value problem

$$\mathcal{L}u = f(x), \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \Gamma_*, \qquad l(u) = 0, \quad 0 < x_2 < 1, \tag{2.4}$$

where

$$\mathcal{L}u = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - a_0 u, \qquad l(u) = \int_0^\xi \beta(x) u(x)\, dx_1, \tag{2.5}$$

$\beta(x) = \varepsilon x_1^{\varepsilon-1}/\xi^\varepsilon$ if $x_1 \le \xi$, $\rho(x) = 0$ if $x_1 > \xi$.

Let the right-hand side $f(x)$ in (2.4) be a linear continuous functional on $\overset{*}{W_2^1}(\Omega,\rho)$ which can be represented as

$$f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \qquad f_k(x) \in L_2(\Omega,\rho),\ k = 0,1,2. \tag{2.6}$$

We assume that the coefficients $a_{ij}$ and $a_0$ satisfy the following conditions:

$$\begin{gathered} a_{1j} = a_{1j}(x) \in L_\infty(\Omega) \quad (j = 1,2) \qquad a_{21} = \text{const}; \\ a_{22} \text{ and } a_0 \text{ have property (P)}, \\ a_0 \ge 0, \quad \sum_{i,j=1}^{2} a_{ij} t_i t_j \ge \nu(t_1^2 + t_2^2) \quad \text{a.e. in } \Omega,\ \nu = \text{const} > 0. \end{gathered} \tag{2.7}$$

We say that the function $u \in \overset{*}{W}{}_2^1(\Omega,\rho)$ is a *weak solution* of problem (2.4)–(2.7) if the relation

$$a(u,v) = \langle f,v\rangle, \quad \forall v \in \overset{*}{W}{}_2^1(\Omega,\rho) \tag{2.8}$$

holds, where

$$a(u,v) = \left(a_{11}\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_\rho + \left((a_{12}+a_{21})\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_1}\right)_\rho + \left(a_{22}\frac{\partial u}{\partial x_2},K\frac{\partial v}{\partial x_2}\right)_\rho + (a_0u,Kv)_\rho, \tag{2.9}$$

$$\langle f,v\rangle = (f_0,Kv)_\rho - \left(f_1,\frac{\partial v}{\partial x_1}\right)_\rho - \left(f_2,K\frac{\partial v}{\partial x_2}\right)_\rho, \tag{2.10}$$

$$Kv(x) = v(x) - \frac{1}{\rho(x_1)}\int_0^{x_1}\beta(t)v(t,x_2)\,dt. \tag{2.11}$$

Equality (2.8) is formally obtained from $(\mathcal{L}u - f,Kv)_\rho = 0$ by integration by parts.

## 3. Solvability of a nonlocal problem

To prove the existence of a unique solution of problem (2.8) (a weak solution of the problem (2.4)–(2.7)), we will apply Lax-Milgram lemma [3]. First we will prove some auxiliary statements.

Lemma 3.1. *Let $u \in \overset{*}{W}{}_2^1(\Omega,\rho)$. Then*

$$|u|_{1,\rho} \le \|u\|_{1,\rho} \le c_2|u|_{1,\rho}, \quad c_2 = \sqrt{5}. \tag{3.1}$$

*Proof.* The first inequality of the lemma is obvious. Integrating by parts, we obtain

$$\begin{aligned}\int_\Omega \rho(x)u^2(x)\,dx = &-\frac{\varepsilon\xi}{\varepsilon+1}\int_0^1 u^2(\xi,x_2)\,dx_2 - 2\int_\xi^1\int_0^1 x_1u(x)\frac{\partial u}{\partial x_1}\,dx \\ &-\frac{2}{\varepsilon+1}\int_0^\xi\int_0^1 x_1\rho(x)u(x)\frac{\partial u}{\partial x_1}\,dx \le 2\int_\Omega \rho(x)\left|u\frac{\partial u}{\partial x_1}\right|dx.\end{aligned} \tag{3.2}$$

Therefore

$$\|u\|_\rho \le 2\left\|\frac{\partial u}{\partial x_1}\right\|_\rho, \tag{3.3}$$

which proves the lemma. □

Lemma 3.2. *Let $u,v \in L_2(\Omega,\rho)$ and let $v$ satisfy the condition $l(v) = 0$. Then*

$$\|v\|_\rho \le \|Kv\|_\rho \le c_1\|v\|_\rho; \tag{3.4}$$

*further, if $b$ belongs to $L_\infty(\Omega)$, then*

$$\left|(bu,Kv)_\rho\right| \le c_1\|b\|_\infty\|u\|_\rho\|v\|_\rho, \tag{3.5}$$

*where $c_1 = (1+\varepsilon)/(1-\varepsilon)$.*

*Proof.* Denote

$$J(v) = \int_{\Omega_\xi}\rho^{-1}(x)\left(\int_0^{x_1}\beta(t)v(t,x_2)\,dt\right)^2 dx. \tag{3.6}$$

It is not difficult to verify that

$$J(v) = -\frac{2\varepsilon}{1-\varepsilon}\int_{\Omega_\xi}v(x)\int_0^{x_1}\beta(t)v(t,x_2)\,dt\,dx \le \frac{2\varepsilon}{1-\varepsilon}\|v\|_\rho J(v). \tag{3.7}$$

Thus

$$\left(J(v)\right)^{1/2} \le 2\varepsilon(1-\varepsilon)^{-1}\|v\|_\rho. \tag{3.8}$$

Since

$$\int_0^{x_1}\beta(t)v(t,x_2)\,dt = 0, \quad x_1 \ge \xi, \tag{3.9}$$

from (2.11) we get

$$\|Kv\|_\rho^2 = \|v\|_\rho^2 + \left(\frac{1}{\varepsilon}\right)J(v), \tag{3.10}$$

which using (3.8) yields (3.4).

Further, we can write

$$(bu,Kv)_\rho = (bu,v)_\rho - \int_{\Omega_\xi}bu\int_0^{x_1}\beta(t)v(t,x_2)\,dt\,dx, \tag{3.11}$$

and by virtue of the Cauchy inequality we have

$$\left|(bu,Kv)_\rho\right| \le \|b\|_\infty\|u\|_\rho\left(\|v\|_\rho + \left(J(v)\right)^{1/2}\right). \tag{3.12}$$

This together with (3.8) completes the proof of (3.5). □

Lemma 3.3. *Let $v \in L_2(\Omega,\rho)$ and $l(v) = 0$. If $b$ has property (P), then*

$$(bv,Kv)_\rho \ge (bv,v)_\rho. \tag{3.13}$$

The proof follows from the easily verifiable identity

$$(bv,Kv)_\rho = (bv,v)_\rho - \int_{\Omega_\xi}bv\int_0^{x_1}\beta(t)v(t,x_2)\,dt\,dx = (bv,v)_\rho + \frac{1}{2\varepsilon}\tilde{J}, \tag{3.14}$$

where

$$\tilde{J} = \int_{\Omega_\xi} x_1^{\varepsilon} \frac{\partial(x_1^{1-\varepsilon} b)}{\partial x_1} \rho^{-1} \left( \int_0^{x_1} \beta(t) v(t,x_2)\,dt \right)^2 dx < \infty. \tag{3.15}$$

By applying Lemmas 3.1, 3.2, 3.3, and conditions (2.7), from (2.9) we obtain the continuity

$$|a(u,v)| \leq c_3 \|u\|_{1,\rho} \|v\|_{1,\rho}, \quad c_3 > 0,\ \forall u,v \in \overset{*}{W}{}_2^1(\Omega,\rho) \tag{3.16}$$

and the $W_2^1$-ellipticity

$$a(u,u) \geq c_4 \|u\|_{1,\rho}^2, \quad c_4 > 0,\ \forall u \in \overset{*}{W}{}_2^1(\Omega,\rho) \tag{3.17}$$

of the bilinear form $a(u,v)$.

Analogously, from (2.10) follows the continuity of the linear form $\langle f,v\rangle$:

$$|\langle f,v\rangle| \leq c_5 \|v\|_{1,\rho}, \quad c_5 > 0,\ \forall v \in \overset{*}{W}{}_2^1(\Omega,\rho). \tag{3.18}$$

Thus, all conditions of the Lax-Milgram lemma are fulfilled. Therefore the following theorem is true.

Theorem 3.4. *The problem (2.4)–(2.7) has a unique weak solution from* $\overset{*}{W}{}_2^1(\Omega,\rho)$.

*Remark 3.5.* If we notice that

$$u(\xi,x_2) - l(u) = \int_0^{\xi} \rho(x_1) \frac{\partial u(x)}{\partial x_1}\,dx_1, \tag{3.19}$$

then, applying the Cauchy inequality, we get

$$\left| \int_0^1 |u(\xi,x_2) - l(u)|^2 dx_2 \right| \leq c_6 \|u\|_{1,\rho}^2, \quad c_6 = \frac{\xi}{\varepsilon+1}. \tag{3.20}$$

Consequently,

$$\lim_{\xi\to 0} l(u) = u(0,x_2). \tag{3.21}$$

Thus, passing to the limit as $\xi \to 0$, the nonlocal condition $l(u) = 0$ transforms to $u(0,x_2) = 0$, while Theorem 3.4 transforms to the well-known theorem on the existence and uniqueness of a solution of the Dirichlet problem. In this sense, the nonlocal problem (2.4)–(2.7) can be regarded as a generalization of the Dirichlet boundary value problem.

*Remark 3.6.* By definition (2.9), for all $u \in D(\mathcal{L})$ we have $a(u,u) = (\mathcal{L}u, Ku)_\rho$. Hence, using (3.4) it follows from (3.17) that

$$(\mathcal{L}u, Ku)_\rho \geq c\|u\|_\rho^2, \qquad (\mathcal{L}u, Ku)_\rho \geq c\|Ku\|_\rho^2. \tag{3.22}$$

Thus $\mathcal{L}$ is a $K$-positive definite operator [8, 12].

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