WEIGHTED ESTIMATES FOR COMMUTATORS ON NONHOMOGENEOUS SPACES

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Let μ be a Borel measure on $\mathbb{R}^d$ which may be nondoubling. The only condition that μ must satisfy is $\mu(Q) \leq c_0 l(Q)^n$ for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes and for some fixed n with $0 < n \leq d$. This paper is to establish the weighted norm inequality for commutators of Calderón-Zygmund operators with RBMO(μ) functions by an estimate for a variant of the sharp maximal function in the context of the nonhomogeneous spaces.

1. Introduction

Let μ be some nonnegative Borel measure on $\mathbb{R}^d$ satisfying

$$\mu(Q) \leq c_0 l(Q)^n \tag{1.1}$$

for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \leq d$. Throughout this paper, all cubes we will consider will be those with sides parallel to the coordinate axes. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$. Moreover, $Q(x,r)$ will be the cube centered at x with side length r.

The classical theory of harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^d,\mu)$ has been developed under the assumption that the underlying measure μ satisfies the doubling property, that is, there exists a constant $c > 0$ such that $\mu(B(x,2r)) \leq c\mu(B(x,r))$ for every $x \in \mathbb{R}^d$ and $r > 0$. But recently, many classical results have been proved still valid without the doubling condition; see [1–18] and their references.

Orobitg and Pérez [11] have studied an analogue of the classical theory of $A_p(\mu)$ weights in $\mathbb{R}^d$ without assuming that the underlying measure μ is doubling. Then, they

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obtained weighted norm inequalities for the (centered) Hardy-Littlewood maximal function and corresponding weighted estimates for nonclassical Calderón-Zygmund operators. They also considered commutators of those Calderón-Zygmund operators with BMO(μ) functions. The purpose of this paper is to establish weighted estimates for commutators of those nonclassical Calderón-Zygmund operators with RBMO(μ) in this new setting.

Let us introduce some notations and definitions. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, we set

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n}, \tag{1.2}$$

where $N_{Q,R}$ is the first integer k such that $l(2^k Q) \geq l(R)$. $K_{Q,R}$ was introduced by Tolsa in [15].

Given β_d (depending on d) big enough (e.g., $\beta_d > 2^n$), we say that some cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \leq \beta_d \mu(Q)$.

Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer ≥ 0 such that $2^N Q$ is doubling. We denote this cube by $\widetilde{Q}$.

Let $\eta > 1$ be some fixed constant. We say that a function $b(x)$ is in RBMO(μ) if there exists some constant c_1 such that for any cube Q,

$$\begin{gathered} \frac{1}{\mu(\eta Q)} \int_Q |b - m_{\widetilde{Q}} b| \, d\mu \leq c_1, \\ |m_Q b - m_R b| \leq c_1 K_{Q,R} \quad \text{for any two doubling cubes } Q \subset R, \end{gathered} \tag{1.3}$$

where $m_Q b = 1/\mu(Q) \int_Q b \, d\mu$. The minimal constant c_1 is the RBMO(μ) norm of b, and it will be denoted by $\|b\|_*$. The RBMO(μ) function space was introduced by Tolsa in [15] and shares more properties with the classical BMO function space than BMO(μ) space.

We say a kernel $k(x,y) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\} \to \mathbb{C}$ is an n-dimensional Calderón-Zygmund kernel in the new setting if

(1) $|k(x,y)| \leq A/|x-y|^n$ if $x \neq y$,

(2) there exists $0 < \gamma \leq 1$ such that

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq \frac{A|x-x'|^\gamma}{|x-y|^{n+\gamma}} \tag{1.4}$$

if $|x-y| > 2|x-x'|$.

A bounded linear operator T from $L^2(\mu)$ to $L^2(\mu)$ is said to be a Calderón-Zygmund operator with n-dimensional kernel k if for every compacted supported function $f \in L^2(\mu)$,

$$Tf(x) = \int_{\mathbb{R}^d} k(x,y) f(y) d\mu(y) \quad \text{for } x \notin \operatorname{supp} f. \tag{1.5}$$

For $r > 0$, we define the truncated operators by

$$T_r f(x) = \int_{\mathbb{R}^d \setminus B(x,r)} k(x,y) f(y) d\mu(y) \tag{1.6}$$

and define the maximal operator associated with T as follows:

$$T_* f(x) = \sup_{r>0} |T_r f(x)|. \tag{1.7}$$

2. Sharp maximal function estimates for commutators

In [15], Tolsa defined a sharp maximal operator $M^{\#} f(x)$ such that

$$f \in \mathrm{RBMO}(\mu) \Longleftrightarrow M^{\#} f \in L^\infty(\mu), \tag{2.1}$$

where

$$M^{\#} f(x) = \sup_{x\in Q} \frac{1}{\mu((3/2)Q)} \int_Q |f - m_{\tilde{Q}} f|\, d\mu + \sup_{\substack{x\in Q\subset R \\ Q,R \text{ doubling}}} \frac{|m_Q f - m_R f|}{K_{Q,R}}. \tag{2.2}$$

We also consider the noncentered doubling maximal operator N:

$$N f(x) = \sup_{\substack{x\in Q \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f|\, d\mu. \tag{2.3}$$

By [15, Remark 2.3], for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} b(y)\, d\mu(y) = b(x). \tag{2.4}$$

So, $|f(x)| \leq N f(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Moreover, it is easy to show that N is of weak type (1,1) and bounded on $L^p(\mu)$, $p \in (1,\infty]$.

In order to obtain the estimate for a variant of the sharp maximal function for the commutators of Calderón-Zygmund operators defined as above with RBMO(μ) functions, we need the following definition.

A function $B : [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, increasing, and satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. We define the B-average of a function f over a cube Q by means of the following Luxemburg norm:

$$\|f\|_{B,Q,(\rho)} = \inf\left\{\lambda > 0 : \frac{1}{\mu(\rho Q)} \int_Q B\left(\frac{|f(y)|}{\lambda}\right) d\mu \leq 1\right\}. \tag{2.5}$$

The generalized Hölder's inequality

$$\frac{1}{\mu(\rho Q)} \int_Q |f(y) g(y)|\, d\mu(y) \leq \|f\|_{B,Q,(\rho)} \|g\|_{\overline{B},Q,(\rho)} \tag{2.6}$$

holds, where $\overline{B}$ is the complementary Young function associated to B. For every locally integrable function f, define its maximal operator $M_{B,(\rho)}$ by

$$M_{B,(\rho)} f(x) = \sup_{x\in Q} \|f\|_{B,Q,(\rho)}. \tag{2.7}$$

THEOREM 2.1. *Let $b \in \mathrm{RBMO}(\mu)$, let $0 < \delta < \epsilon < 1$, there exists $C = C_{\delta,\epsilon}$ such that*

$$M_\delta^\#([b,T]f)(x) \le C\|b\|_*\left(M_{\epsilon,(3/2)}(Tf)(x) + M_{(9/8)}^2 f(x) + T_* f(x)\right), \tag{2.8}$$

where $M_\delta^\# f(x) = M^\#(|f|^\delta)^{1/\delta}$, $M_{p,(\rho)} f(x) = \sup_{x\in Q}((1/\mu(\rho Q))\int_Q |f|^p d\mu)^{1/p}$, $0 < p < \infty$. Set $M_{(\rho)} f(x) = M_{1,(\rho)} f(x)$.

Before proving the theorem, another equivalent norm for $\mathrm{RBMO}(\mu)$ is needed. Suppose that for a given function $b \in L^1_{\mathrm{loc}}(\mu)$ there exist some c_2 and a collection of numbers $\{b_Q\}_Q$ (i.e., for each cube Q, there exists $b_Q \in \mathbb{R}$) such that

$$\begin{gathered} \sup_Q \frac{1}{\mu(\eta Q)} \int_Q |b - b_Q|\, d\mu \le c_2, \\ |b_Q - b_R| \le c_2 K_{Q,R} \quad \text{for any two cubes } Q \subset R. \end{gathered} \tag{2.9}$$

Then, set $\|b\|_{**} = \inf c_2$, where the infimum is taken over all the constants c_2 and all the numbers $\{b_Q\}$ satisfying (2.9). By [15, Lemma 2.8, page 99], for a fixed $\eta > 1$, the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are equivalent.

Proof of Theorem 2.1. We follow the argument from [15, proof of Theorem 9.1]. Let $Q = Q(x,r)$ be a cube with center x and side length r. For $0 < \delta < 1$ and $\alpha,\beta \in \mathbb{R}$, we have $||\alpha|^\delta - |\beta|^\delta| \le |\alpha - \beta|^\delta$. Let $\{b_Q\}_Q$ be a sequence of numbers satisfying

$$\int_Q |b - b_Q|\, d\mu \le 2\mu(2Q)\|b\|_{**}, \tag{2.10}$$

for all cubes Q and

$$|b_Q - b_R| \le 2K_{Q,R}\|b\|_{**} \tag{2.11}$$

for all cubes Q, R with $Q \subset R$. For any cube Q, we denote $h_Q := -m_Q(T((b - b_Q)f\chi_{\mathbb{R}^d \setminus (4/3)Q}))$. We will show that for all x, Q with $x \in Q$,

$$\frac{1}{\mu((3/2)Q)}\left(\int_Q |[b,T]f - h_Q|^\delta d\mu\right)^{1/\delta} \le C\|b\|_{**}\left(M_{\epsilon,(3/2)}(Tf)(x) + M_{(9/8)}^2 f(x)\right); \tag{2.12}$$

and for all cubes Q, R with $Q \subset R$, $x \in Q$,

$$|h_Q - h_R| \le C\|b\|_{**}\left(M_{(9/8)}^2 f(x) + T_* f(x)\right)K_{Q,R}^2. \tag{2.13}$$

To obtain (2.12) for some fixed cube Q and x with $x \in Q$, we rewrite $[b,T]f$:

$$[b,T]f = (b - b_Q)Tf - T((b - b_Q)f_1) - T((b - b_Q)f_2), \tag{2.14}$$

where $f_1 = f\chi_{(4/3)Q}$, $f_2 = f - f_1$. Let us estimate the term $(b - b_Q)Tf$ first. Take $1 < r < \varepsilon/\delta$. By Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{\mu((3/2)Q)}\int_Q |(b(y)-b_Q)Tf(y)|^{\delta}d\mu(y)\right)^{1/\delta}\\
&\quad\le\left(\frac{1}{\mu((3/2)Q)}\int_Q |b(y)-b_Q|^{\delta r'}d\mu(y)\right)^{1/\delta r'}\left(\frac{1}{\mu((3/2)Q)}\int_Q |Tf(y)|^{\delta r}d\mu(y)\right)^{1/\delta r}\\
&\quad\le C\|b\|_{**}M_{\delta r,(3/2)}(Tf)(x)\le C\|b\|_{**}M_{\varepsilon,(3/2)}(Tf)(x).
\end{aligned}\tag{2.15}$$

Since $T : L^1(\mu) \to L^{1,\infty}(\mu)$ (see [9]) and $0 < \delta < 1$, Kolmogorov's inequality and generalized Hölder's inequality yield

$$\begin{aligned}
&\left(\frac{1}{\mu((3/2)Q)}\int_Q |T((b-b_Q)f_1(y))|^{\delta}d\mu(y)\right)^{1/\delta}\\
&\quad\le\frac{1}{\mu((3/2)Q)}\int_{(4/3)Q}|(b(y)-b_Q)f(y)|\,d\mu(y)\\
&\quad\le C||b-b_Q||_{\exp L,(4/3)Q,(9/8)}\|f\|_{L\operatorname{Log}L,(4/3)Q,(9/8)},
\end{aligned}\tag{2.16}$$

while John-Nirenberg inequality implies that

$$\frac{1}{\mu((3/2)Q)}\int_{(4/3)Q}\exp\left(\frac{|b(y)-b_Q|}{C\|b\|_*}\right)d\mu(y)\le C_0.\tag{2.17}$$

So there exists a positive constant C such that for all cubes Q,

$$||b-b_Q||_{\exp L,(4/3)Q,(\rho)}\le C\|b\|_*.\tag{2.18}$$

Therefore

$$\left(\frac{1}{\mu((3/2)Q)}\int_Q |T((b-b_Q)f_1(y))|^{\delta}d\mu(y)\right)^{1/\delta}\le C\|b\|_*M_{L\operatorname{Log}L,(9/8)}f(x).\tag{2.19}$$

In order to prove (2.12), we only need to estimate $|T((b-b_Q)f_2)-h_Q|^{\delta}$. Note that

$$K_{Q,2^k(4/3)Q}=1+\sum_{j=1}^{k+1}\frac{\mu(2^jQ)}{l(2^jQ)^n}\le 1+(k+1)C_0\le Ck.\tag{2.20}$$

For $x, y \in Q$, we have

$$
\begin{aligned}
&|(T((b-b_Q)f_2))(x)-(T((b-b_Q)f_2))(y)| \\
&\quad \le C\int_{\mathbb{R}^d\setminus(4/3)Q}\frac{|y-x|^\gamma}{|z-x|^{n+\gamma}}|b(z)-b_Q||f(z)|d\mu(z) \\
&\quad \le C\sum_{k=1}^{\infty}\int_{2^k(4/3)Q\setminus 2^{k-1}(4/3)Q}\frac{l(Q)^\gamma}{|z-x|^{n+\gamma}}(|b(z)-b_{2^k(4/3)Q}|+|b_Q-b_{2^k(4/3)Q}|)|f(z)|d\mu(z) \\
&\quad \le C\sum_{k=1}^{\infty}2^{-k\gamma}\frac{1}{l(2^kQ)^n}\int_{2^k(4/3)Q}|b(z)-b_{2^k(4/3)Q}||f(z)|d\mu(z) \\
&\qquad + C\sum_{k=1}^{\infty}k2^{-k\gamma}\|b\|_*\frac{1}{l(2^kQ)^n}\int_{2^k(4/3)Q}|f(z)|d\mu(z) \\
&\quad \le C\sum_{k=1}^{\infty}2^{-k\gamma}\frac{1}{\mu((9/8)2^k(4/3)Q)}\int_{2^k(4/3)Q}|b(z)-b_{2^k(4/3)Q}||f(z)|d\mu(z) \\
&\qquad + C\sum_{k=1}^{\infty}k2^{-k\gamma}\|b\|_*M_{(9/8)}f(x) \\
&\quad \le C\sum_{k=1}^{\infty}2^{-k\gamma}\|b-b_{2^k(4/3)Q}\|_{\exp L,2^k(4/3)Q,(9/8)}\|f\|_{L\operatorname{Log}L,2^k(4/3)Q,(9/8)}+C\|b\|_*M_{(9/8)}f(x) \\
&\quad \le C\|b\|_*M_{L\operatorname{Log}L,(9/8)}f(x)+C\|b\|_*M_{(9/8)}f(x).
\end{aligned} \tag{2.21}
$$

For $\rho > 1$, it is easy to see $M_{(\rho)}f(x) \le M_{L\operatorname{Log}L,(\rho)}f(x)$. Thus

$$
|(T((b-b_Q)f_2))(x)-(T((b-b_Q)f_2))(y)| \le C\|b\|_*M_{L\operatorname{Log}L,(9/8)}f(x). \tag{2.22}
$$

According to Jensen's inequality, we obtain

$$
\begin{aligned}
&\left(\frac{1}{\mu((3/2)Q)}\int_Q|T((b-b_Q)f_2)(y)-m_Q(T(b-b_Q)f_2)|^\delta d\mu(y)\right)^{1/\delta} \\
&\qquad \le \frac{1}{\mu((3/2)Q)}\int_Q|T((b-b_Q)f_2)(y)-m_Q(T(b-b_Q)f_2)|d\mu(y) \\
&\qquad \le C\|b\|_*M_{L\operatorname{Log}L,(9/8)}f(x).
\end{aligned} \tag{2.23}
$$

Note that for $\rho > 1$, $M^2_{(\rho)}f(x) \approx M_{L\operatorname{Log}L,(\rho)}f(x)$. By (2.15), (2.16), and (2.23) we obtain (2.12).

For $\{h_Q\}_Q$, we want to prove (2.13). Consider two cubes $Q \subset R$ and $x \in Q$. We denote $N = N_{Q,R} + 1$. We write $h_Q - h_R$ in the following way:

$$\begin{aligned}
&|m_Q(T((b-b_Q)f\chi_{\mathbb{R}^d\setminus(4/3)Q})) - m_R(T((b-b_R)f\chi_{\mathbb{R}^d\setminus(4/3)R}))| \\
&\quad \le |m_Q(T((b-b_Q)f\chi_{2Q\setminus(4/3)Q}))| + |m_Q(T((b_Q-b_R)f\chi_{\mathbb{R}^d\setminus 2Q}))| \\
&\qquad + |m_Q(T((b-b_R)f\chi_{2^NQ\setminus 2Q}))| \\
&\qquad + |m_Q(T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ})) - m_R(T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2^NQ}))| \\
&\qquad + |m_R(T((b-b_R)f\chi_{2^NQ\setminus(4/3)R}))| \\
&\quad = M_1 + M_2 + M_3 + M_4 + M_5.
\end{aligned} \tag{2.24}$$

Let us estimate M_1. For $y \in Q$ we have

$$\begin{aligned}
|T((b-b_Q)f\chi_{2Q\setminus(4/3)Q})(y)| &\le \frac{C}{l(2Q)^n}\int_{2Q}|b-b_Q||f|d\mu \\
&\le C\|b-b_Q\|_{\exp L,2Q,(9/8)}\|f\|_{L\operatorname{Log}L,2Q,(9/8)} \\
&\le C\|b\|_* M_{L\operatorname{Log}L,(9/8)}f(x) \le C\|b\|_* M^2_{(9/8)}f(x).
\end{aligned} \tag{2.25}$$

So we derive $M_1 \le C\|b\|_* M^2_{9/8)}f(x)$. Let us consider M_2. For $x, y \in Q$,

$$\begin{aligned}
|Tf(\chi_{\mathbb{R}^d\setminus 2Q})(y)| &= \left|\int_{\mathbb{R}^d\setminus 2Q} f(z)k(y,z)d\mu(z)\right| \\
&\le \left|\int_{\mathbb{R}^d\setminus 2Q} f(z)(k(y,z)-k(x,z))d\mu(z)\right| + \left|\int_{\mathbb{R}^d\setminus 2Q} k(x,z)f(z)d\mu(z)\right| \\
&\le \left|\int_{\mathbb{R}^d\setminus 2Q} \frac{|y-z|^\gamma}{|y-z|^{n+\gamma}}|f(z)|d\mu(z)\right| + T_*f(x) \\
&\le C\sup_{Q_0\ni x}\frac{1}{l(Q_0)^n}\int_{Q_0}|f|d\mu + T_*f(x) \le CM_{(9/8)}f(x) + T_*f(x).
\end{aligned} \tag{2.26}$$

Thus

$$M_2 = |(b_R - b_Q)Tf(\chi_{\mathbb{R}^d\setminus 2Q})| \le CK_{Q,R}\|b\|_*(T_*f(x) + CM^2_{(9/8)}f(x)). \tag{2.27}$$

For the term M_4, we execute the process as in (2.21). For any $y, z \in \mathbb{R}^d$, we get

$$\begin{aligned}
&|T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2Q})(y) - T((b-b_R)f\chi_{\mathbb{R}^d\setminus 2Q})(z)| \\
&\quad \le C\|b\|_* M_{L\operatorname{Log}L,(9/8)}f(x) \le C\|b\|_* M^2_{(9/8)}f(x).
\end{aligned} \tag{2.28}$$

The term M_5 can be estimated as M_1. We can obtain

$$M_5 \leq C\|b\|_* M^2_{(9/8)} f(x). \tag{2.29}$$

Finally we have to deal with M_3. For $y \in Q$, we have

$$|b_{2^{k+1}Q} - b_R| \leq CK_{2^{k+1}Q,R}\|b\|_* \leq CK_{Q,R}\|b\|_*. \tag{2.30}$$

Then,

$$\begin{aligned}
&|T((b-b_R)f\chi_{2^N\setminus 2Q})(y)| \\
&\quad \leq C\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\int_{2^{k+1}Q\setminus 2^kQ}|b-b_R||f|\,d\mu \\
&\quad \leq C\sum_{k=1}^{N-1}\frac{1}{l(2^kQ)^n}\int_{2^{k+1}Q}|b-b_{2^{k+1}Q}||f|\,d\mu + C\sum_{k=1}^{N-1} \\
&\qquad \times \frac{1}{l(2^kQ)^n}\int_{2^{k+1}Q}|b_{2^{k+1}Q}-b_R||f|\,d\mu \\
&\quad \leq C\sum_{k=1}^{N-1}\|b-b_{2^{k+1}Q}\|_{\exp L,2^{k+1}Q,(9/8)}\|f\|_{L\operatorname{Log}L,2^{k+1}Q,(9/8)} \\
&\qquad + C\sum_{k=1}^{N-1}K_{Q,R}\|b\|_*\frac{\mu(2^{k+1}Q)}{l(2^kQ)^n}\frac{1}{\mu(2^{k+1}Q)}\int_{2^{k+1}Q}|f|\,d\mu \\
&\quad \leq C\|b\|_* M_{L\operatorname{Log}L,(9/8)}f(x) + CK_{Q,R}\|b\|_*\sum_{k=1}^{N-1}\frac{\mu(2^{k+1}Q)}{l(2^kQ)^n}M_{(9/8)}f(x) \\
&\quad \leq C\|b\|_* M_{L\operatorname{Log}L,(9/8)}f(x) + CK^2_{Q,R}\|b\|_* M_{(9/8)}f(x) \\
&\quad \leq C\|b\|_* M^2_{(9/8)}f(x)K^2_{Q,R}.
\end{aligned} \tag{2.31}$$

Taking the mean over Q, we get

$$M_3 \leq C\|b\|_* M^2_{(9/8)} f(x) K^2_{Q,R}. \tag{2.32}$$

By the estimates on M_1, M_2, M_3, M_4, M_5, we can get (2.13).

Let us see how from (2.12) and (2.13) one obtains (2.8). If Q is a doubling cube and $x \in Q$, then we have by (2.12)

$$\begin{aligned}
\left| m_Q(|[b,T]f|^\delta) - |h_Q^\delta| \right|^{1/\delta} &\leq \left(\frac{1}{\mu(Q)}\int_Q \left||[b,T]f|^\delta - h_Q^\delta\right| d\mu\right)^{1/\delta} \\
&\leq C\|b\|_*\left(M_{\epsilon,(3/2)}(Tf)(x) + M^2_{(9/8)}f(x) + T_*f(x)\right).
\end{aligned} \tag{2.33}$$

Also, for any cube $Q \ni x$, $K_{Q,\tilde{Q}} \le C$, and then by (2.12) and (2.13) we get

$$\begin{aligned}
&\left(\frac{1}{\mu((3/2)Q)}\int_Q \left| |[b,T]f|^\delta - m_{\tilde{Q}}(|[b,T]f|^\delta)\right| d\mu\right)^{1/\delta} \\
&\quad\le \left(\frac{1}{\mu((3/2)Q)}\int_Q \left| |[b,T]f|^\delta - |h_Q|^\delta\right| d\mu\right)^{1/\delta} + \left| |h_Q|^\delta - |h_{\tilde{Q}}|^\delta\right|^{1/\delta} \\
&\qquad + \left| |h_{\tilde{Q}}|^\delta - m_{\tilde{Q}}(|[b,T]f|^\delta)\right|^{1/\delta} \\
&\quad\le \left(\frac{1}{\mu((3/2)Q)}\int_Q |[b,T]f - h_Q|^\delta d\mu\right)^{1/\delta} + |h_Q - h_{\tilde{Q}}| + \left|h_{\tilde{Q}}^\delta - m_{\tilde{Q}}(|[b,T]f|^\delta)\right|^{1/\delta} \\
&\quad\le C\|b\|_*(M_{\epsilon,(3/2)}(Tf)(x) + M^2_{(9/8)}f(x) + T_*f(x)).
\end{aligned} \tag{2.34}$$

On the other hand, for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \le P_0$, where P_0 is the constant in [15, Lemma 9.3, page 143]. By (2.13) we have

$$|h_Q - h_R| \le C\|b\|_*(M^2_{(9/8)}f(x) + T_*f(x))K_{Q,R}P_0. \tag{2.35}$$

So by [15, Lemma 9.3, page 143], we get

$$|h_Q - h_R| \le C\|b\|_*(M^2_{(9/8)}f(x) + T_*f(x))K_{Q,R} \tag{2.36}$$

for all doubling cubes $Q \subset R$ with $x \in Q$, using (2.13) again, we get

$$\begin{aligned}
&\left|m_Q\left(|[b,T]f|^\delta\right) - m_R\left(|[b,T]f|^\delta\right)\right| \\
&\quad\le \left|m_Q\left(|[b,T]f|^\delta\right) - h_Q^\delta\right| + |h_Q^\delta - h_R^\delta| + \left|h_R^\delta - m_R\left(|[b,T]f|^\delta\right)\right| \\
&\quad\le C(\|b\|_*(M_{\epsilon,(3/2)}(Tf)(x) + M^2_{(9/8)}f(x) + T_*f(x))K_{Q,R})^\delta.
\end{aligned} \tag{2.37}$$

From the above estimates, we can obtain

$$M^\#_\delta([b,T]f)(x) \le C\|b\|_*(M_{\epsilon,(3/2)}(Tf)(x) + M^2_{(9/8)}f(x) + T_*f(x)). \tag{2.38}$$

□

Now we are in the position to give the definition of weights we will consider. Here we will consider the $A_p(\mu)$ weights introduced by Orobitg and Pérez in [11]. So we need the assumption that $\mu(\partial Q) = 0$ for any cube Q with sides parallel to the coordinates axes.

Let $1 < p < \infty$ and let $p' = p/(p-1)$. We say that a weight w satisfies the $A_p(\mu)$ condition if there exists a constant K such that for all cubes Q

$$\left(\frac{1}{\mu(Q)}\int_Q w\,d\mu\right)\left(\frac{1}{\mu(Q)}\int_Q w^{1-p'}d\mu\right)^{p-1} \le K. \tag{2.39}$$

And we define the $A_\infty(\mu)$ class as $A_\infty(\mu) = \bigcup_{p>1} A_p(\mu)$.

Theorem 2.2. *Let $0 < p < \infty$, let $\rho > 1$, $w(x) \in A_\infty(\mu)$ defined above, then*

$$\int_{\mathbb{R}^d} |Tf(x)|^p w(x)d\mu(x) \le C\int_{\mathbb{R}^d} (M_{(\rho)}f(x))^p w(x)d\mu(x) \tag{2.40}$$

holds for every function f for which the left-hand side is finite.

Proof. For each $\epsilon > 0$ we define the maximal operator

$$T_\epsilon^* f(x) = \sup_{\delta>\epsilon} |T_\delta f(x)|. \tag{2.41}$$

We only need to prove that for $w \in A_\infty(\mu)$, there exist suitable constants α, β, ε such that

$$w(\{x : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le \alpha w(\{x : T_\epsilon^* f(x) > t\}), \quad t > 0, \tag{2.42}$$

for all $\alpha^p < (1+\beta)^{-1}$. We may assume f is nonnegative and locally integrable. Follow the idea of [11], we first consider the special case when $w = 1$, then (2.42) turns to

$$\mu(\{x : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le \alpha\mu(\{x : T_\epsilon^* f(x) > t\}). \tag{2.43}$$

Since $\Omega = \{x \in \mathbb{R}^d : T_\epsilon^* f(x) > t\}$ is open, we decompose it into disjoint Whitney cubes $\Omega = \bigcup_j Q_j$, where Q_j are disjoint and $2\rho\,\mathrm{diam}(Q_j) \le \mathrm{dist}(Q_j,\Omega^c) \le 8\rho\,\mathrm{diam}(Q_j)$, and every point of $\mathbb{R}^d$ at most lies in $4\rho Q_j$ cubes. Obviously $4\rho Q_j \subset \Omega$. We will show that for given $\beta > 0$, $0 < \alpha < 1$, there exists $c = c(\beta,\alpha,n)$ such that for all j,

$$\mu(\{x \in Q_j : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le \alpha\mu(4Q_j). \tag{2.44}$$

Summing over all j, we have

$$\mu(\{x \in \mathbb{R}^d : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le \alpha 4^n \mu(\Omega). \tag{2.45}$$

Choose α such that $\alpha 4^n < 1$, then we can obtain (2.42) in the special case. For the general case w, recall that if $w \in A_\infty(\mu)$, then by [11, Lemma 2.3, page 2017], there exist positive constants c, δ such that for all cubes Q and all $E \subset Q$,

$$\frac{w(E)}{w(Q)} \le c\left(\frac{\mu(E)}{\mu(Q)}\right)^\delta. \tag{2.46}$$

Looking back at (2.44), we get

$$w(\{x \in Q_j : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le c\alpha^\delta w(4Q_j). \tag{2.47}$$

Summing again over j, we obtain

$$w(\{x : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \le \varepsilon t\}) \le c\alpha^\delta 4^n w(\Omega). \tag{2.48}$$

Choosing α such that $c\alpha^\delta 4^n < (1+\beta)^{-1}$, we can get (2.42).

It remains to prove (2.44). Fix j and let $Q = Q_j$ and let $r = l(Q)$. Assume that there exists $b \in Q$ such that $M_{(\rho)} f(x) \leq \varepsilon t$ (otherwise the left-hand set of (2.44) would be empty). Set $z \in \Omega^c$, that is, $T_\epsilon^* f(z) \leq t$ such that $\operatorname{dist}(z, Q) = \operatorname{dist}(Q, \Omega^c)$. By a simple computation, we get

$$Q \subset P \equiv Q\left(b, \frac{5}{2}r\right) \subset 4Q \subset B \equiv Q(z, 18r). \tag{2.49}$$

Set $f_1 = f\chi_B$, $f_2 = f - f_1$. Then for $x \in Q$, $\gamma > \epsilon$, by the growth condition (1.1),

$$\begin{aligned} |T_\gamma f_1(x)| &\leq |T_\gamma (f\chi_P)(x)| + \int_{\mathbb{R}^d} \frac{f\chi_{B\setminus P}}{|x-y|^n} d\mu(y) \leq T_\epsilon^*(f\chi_P)(x) + \frac{c}{r^n}\int_B f(y)d\mu(y) \\ &\leq T_\epsilon^*(f\chi_P)(x) + cM_{(\rho)} f(x)(b) \leq T_\epsilon^*(f\chi_P)(x) + c\varepsilon t, \end{aligned} \tag{2.50}$$

and so

$$|T_\gamma f(x)| \leq |T_\gamma f_2(x)| + T_\epsilon^*(f\chi_P)(x) + c\varepsilon t. \tag{2.51}$$

To compare $T_\gamma f_2(x)$ with $T_\gamma f_2(z)$, we use the standard arguments. We get

$$\begin{gathered} |T_\gamma f_2(x) - T_\gamma f_2(z)| \leq cM_{(\rho)} f(x)(b), \\ |T_\gamma f_2(z)| \leq T_\epsilon^* f(z) \leq t. \end{gathered} \tag{2.52}$$

Therefore

$$T_\epsilon^* f(x) \leq T_\epsilon^*(f\chi_P)(x) + (1 + c\varepsilon)t. \tag{2.53}$$

Now choose ε such that $2c\varepsilon < \beta$ and consequently

$$\{x \in Q : T_\epsilon^* f(x) > (1+\beta)t,\ M_{(\rho)} f(x) \leq \varepsilon t\} \subset \left\{x \in Q : T_\epsilon^*(f\chi_P)(x) > \frac{\beta}{2}t\right\}. \tag{2.54}$$

Finally, since T_ϵ^* is of weak type $(1,1)$ (see [9]), we get

$$\begin{aligned} \mu\left(\left\{x \in Q : T_\epsilon^*(f\chi_P)(x) > \frac{\beta}{2}t\right\}\right) &\leq \frac{c}{\beta t}\int_P |f(y)|d\mu(y) \\ &= \frac{c\mu(\rho P)}{\beta t \mu(\rho P)}\int_P |f(y)|d\mu(y) \\ &\leq \frac{c\mu(\rho P)}{\beta t} M_{(\rho)} f(x)(b) \\ &\leq \frac{c}{\beta}\varepsilon\mu(4\rho Q) \leq \alpha\mu(4\rho Q) \end{aligned} \tag{2.55}$$

always provided that ε is chosen small enough so that $c\varepsilon/\beta \leq \alpha$. $\square$

Lemma 2.3. *Let* $1 < p < \infty$, *let* $\rho > 1$, $w \in A_p(\mu)$, *then*

$$\int_{\mathbb{R}^d} (M_{(\rho)} f(x))^p w(x) d\mu(x) \le C \int_{\mathbb{R}^d} |f(x)|^p w(x) d\mu(x). \tag{2.56}$$

Proof. Lemma 2.3 is a part of [5, Lemma 1]. Here we can give a more direct proof. By [6, Theorem 3], $M_{(\rho)}$ is weighted weak type (q,q) if $w \in A_q(\mu)$, $1 < q < \infty$. Since $w \in A_p(\mu)$, then by [11, Corollary 2.5], there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}(\mu)$. Finally by the Marcinkiewicz interpolation theorem, we can get the desired result. □

Theorem 2.4. *Let* $0 < p < \infty$, *let* $\rho > 1$, $w \in A_\infty(\mu)$, $b \in \mathrm{RBMO}(\mu)$. *Then there exists constant C such that*

$$\int_{\mathbb{R}^d} |[b,T]f|^p w(x) d\mu(x) \le C \int_{\mathbb{R}^d} (M_{(\rho)} f(x))^p w(x) d\mu(x) \tag{2.57}$$

holds for every function f for which the left-hand side is finite.

Proof. For $w \in A_\infty(\mu)$ and $b \in \mathrm{RBMO}(\mu)$, by the estimate for the variant of the sharp maximal function, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |[b,T]f|^p w(x) d\mu(x) &\le C \int_{\mathbb{R}^d} (N_\delta([b,T]f)(x))^p w(x) d\mu(x) \\
&\le C \int_{\mathbb{R}^d} (M_\delta^\#([b,T]f(x)))^p w \, d\mu(x) \\
&\le C \int_{\mathbb{R}^d} |M_{\epsilon,(3/2)}(Tf)(x)|^p w(x) d\mu(x) \\
&\quad + C \int_{\mathbb{R}^d} (M_{(9/8)}^2 f(x))^p w(x) d\mu(x) \\
&\quad + C \int_{\mathbb{R}^d} |T_* f(x)|^p w(x) d\mu(x).
\end{aligned} \tag{2.58}$$

Here we have to justify the second inequality, precisely

$$\int_{\mathbb{R}^d} (N_\delta([b,T]f)(x))^p w(x) d\mu(x) \le C \int_{\mathbb{R}^d} (M_\delta^\#([b,T]f(x)))^p w \, d\mu(x). \tag{2.59}$$

This inequality can be obtained by using a good-λ argument similar to [15, Theorem 6.2]. For brevity, we omit the details. Since $w \in A_\infty(\mu)$, there exists $1 < r < \infty$ such that $w \in A_r(\mu)$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < p/r$, then by Lemma 2.3, we have

$$\int_{\mathbb{R}^d} (M_{\epsilon,(3/2)}(Tf)(x))^p w \, d\mu \le C \int_{\mathbb{R}^d} |Tf|^p w \, d\mu. \tag{2.60}$$

From Theorem 2.2 and Lemma 2.3, we can get the proof of Theorem 2.4. □

Corollary 2.5. *Let $w \in A_p(\mu)$, let $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^d} |[b,T]f|^p w(x)d\mu(x) \leq C\int_{\mathbb{R}^d} |f(x)|^p w(x)d\mu(x). \tag{2.61}$$

Remark 2.6. Han in [5] obtained a similar result with Corollary 2.5 for higher-order commutators. But Theorems 2.1, 2.2, and 2.4 in our paper are new and are of independent interest in themselves.

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