EXTENSION AND GENERALIZATION INEQUALITIES INVOLVING THE KHATRI-RAO PRODUCT OF SEVERAL POSITIVE MATRICES

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Recently, there have been many authors, who established a number of inequalities involving Khatri-Rao and Hadamard products of two positive matrices. In this paper, the results are established in the following three ways. First, we find generalization of the inequalities involving Khatri-Rao product using results given by Liu (1999), Mond and Pečarić (1997), Cao et al. (2002), Chollet (1997), and Visick (2000). Second, we recover and develop some results of Visick. Third, the results are extended to the case of Khatri-Rao product of any finite number of matrices. These results lead to inequalities involving Hadamard product, as a special case.

1. Introduction

Consider matrices A and B of order $m\times n$ and $p\times q$, respectively. Let $A=[A_{ij}]$ be partitioned with A_{ij} of order $m_i\times n_j$ as the (i,j)th block submatrix and let $B=[B_{kl}]$ be partitioned with B_{kl} of order $p_k\times q_l$ as the (k,l)th block submatrix ($m=\sum_{i=1}^{t}m_i$, $n=\sum_{j=1}^{d}n_j$, $p=\sum_{k=1}^{u}p_k$, $q=\sum_{l=1}^{v}q_l$). For simplicity, we say that A and B are *compatible partitioned* if $A=[A_{ij}]_{i,j=1}^{t}$ and $B=[B_{ij}]_{i,j=1}^{t}$ are square matrices of order $m\times m$ and partitioned, respectively, with A_{ij} and B_{ij} of order $m_i\times m_j$ ($m=\sum_{i=1}^{t}m_i=\sum_{j=1}^{t}m_j$).

Let $A\otimes B$, $A\circ B$, $A\Theta B$, and $A*B$ be the Kronecker, Hadamard, Tracy-Singh, and Khatri-Rao products, respectively, of A and B. The definitions of the mentioned four matrix products are given by Liu in [5, 6] as follows:

(i) *Kronecker product*

$$A\otimes B=\left[a_{ij}B\right]_{ij},\tag{1.1}$$

where $A=[a_{ij}]$, $B=[b_{kl}]$ are scalar matrices of order $m\times n$ and $p\times q$, respectively, $a_{ij}B$ is of order $p\times q$, and $A\otimes B$ of order $mp\times nq$;

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(ii) *Hadamard product*

$$A \circ B = \left[a_{ij}b_{ij}\right]_{ij} = B \circ A, \tag{1.2}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$ are scalar matrices of order $m \times n$, $a_{ij}b_{ij}$ is a scalar, and $A \circ B$ is of order $m \times n$;

(iii) *Tracy-Singh product*

$$A \Theta B = \left[A_{ij} \Theta B\right]_{ij} = \left[\left[A_{ij} \otimes B_{kl}\right]_{kl}\right]_{ij}, \tag{1.3}$$

where $A = [A_{ij}]$, $B = [B_{kl}]$ are partitioned matrices of order $m \times n$ and $p \times q$, respectively, A_{ij} is of order $m_i \times n_j$, B_{kl} of order $p_k \times q_l$, $A_{ij} \otimes B_{kl}$ of order $m_i p_k \times n_j q_l$, $A_{ij} \Theta B$ of order $m_i p \times n_j q$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p = \sum_{k=1}^{u} p_k$, $q = \sum_{l=1}^{v} q_l$), and $A \Theta B$ of order $mp \times nq$;

(iv) *Khatri-Rao product*

$$A * B = \left[A_{ij} \otimes B_{ij}\right]_{ij}, \tag{1.4}$$

where $A = [A_{ij}]$, $B = [B_{ij}]$ are partitioned matrices of order $m \times n$ and $p \times q$, respectively, A_{ij} is of order $m_i \times n_j$, B_{kl} of order $p_i \times q_j$, $A_{ij} \otimes B_{ij}$ of order $m_i p_i \times n_j q_j$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p = \sum_{i=1}^{t} p_i$, $q = \sum_{j=1}^{d} q_j$), and $A * B$ of order $M \times N$ ($M = \sum_{i=1}^{t} m_i p_i$,$N = \sum_{j=1}^{d} n_j q_j$).

In general, $A \Theta B \neq B \Theta A$, $A \otimes B \neq B \otimes A$, $A * B \neq B * A$, but if $A = [a_{ij}]$ is a scalar matrix and $B = [B_{ij}]$ is a partitioned matrix, then $A * B = B * A$. Additionally, Liu [5] shows that the Khatri-Rao product can be viewed as a generalized Hadamard product and the Tracy-Singh product as a generalized Kronecker product, as follows:

(1) for a nonpartitioned matrix A, their $A \Theta B$ is $A \otimes B$, that is,

$$A \Theta B = \left[a_{ij} \Theta B\right]_{ij} = \left[\left[a_{ij} \otimes B_{kl}\right]_{kl}\right]_{ij} = \left[\left[a_{ij} B_{kl}\right]_{kl}\right]_{ij} = \left[a_{ij} B\right]_{ij} = A \otimes B; \tag{1.5}$$

(2) for nonpartitioned matrices A and B of order $m \times n$, their $A * B$ is $A \circ B$, that is,

$$A * B = \left[a_{ij} \otimes b_{ij}\right]_{ij} = \left[a_{ij} b_{ij}\right]_{ij} = A \circ B. \tag{1.6}$$

The Khatri-Rao and Tracy-Singh products are related by the following relation [5, 6]:

$$A * B = Z_1^T (A \Theta B) Z_2, \tag{1.7}$$

where $A = [A_{ij}]$ is partitioned with A_{ij} of order $m_i \times n_j$ and $B = [B_{kl}]$ is partitioned with B_{kl} of order $p_k \times q_l$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p = \sum_{k=1}^{u} p_k$, $q = \sum_{l=1}^{v} q_l$), Z_1 is an $mp \times r$ ($r = \sum_{i=1}^{t} m_i p_i$) matrix of zeros and ones, and Z_2 is an $nq \times s$ ($s = \sum_{j=1}^{d} n_j q_j$) matrix of zeros and ones such that $Z_1^T Z_1 = I_r$, $Z_2^T Z_2 = I_s$ (I_r and I_s are $r \times r$ and $s \times s$ identity matrices, resp.).

In particular, if $m = n$ and $p = q$, then there exists an $mp \times r$ ($r = \sum_{i=1}^{t} m_i p_i$) matrix Z such that $Z^T Z = I_r$ (I_r is an $r \times r$ identity matrix) and

$$A * B = Z^T (A \Theta B) Z. \tag{1.8}$$

Here

$$Z = \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_t \end{bmatrix}, \tag{1.9}$$

where each $Z_i = [\, 0_{i1} \ \cdots \ 0_{ii-1} \ I_{m_i p_i} \ 0_{ii+1} \ \cdots \ 0_{it} \,]^T$ is an real matrix of zeros and ones, and 0_{ik} is a $m_i p_i \times m_i p_k$ zero matrix for any $k \neq i$. Note also that $Z_i^T Z_i = I$ and

$$Z_i^T (A_{ij} \Theta B) Z_j = Z_i^T (A_{ij} \otimes B_{kl})_{kl} Z_j = A_{ij} \otimes B_{ij}, \quad i, j = 1, 2, \ldots, t. \tag{1.10}$$

In [5–8], the authors proved a number of equalities and inequalities involving Khatri-Rao and Hadamard products of two matrices. Here we extend these results in three ways. First, we establish new attractive equalities and inequalities involving Khatri-Rao product of matrices. Second, we recover and develop some results of Visick, for example, [8, Theorem 11, page 54]. This does not follow simply from the work of Visick. Third, the results are extended to the case of Khatri-Rao products of any finite number of matrices. This result leads to inequalities involving Hadamard product, as a special case.

We use the following notations:

(i) $M_{m,n}$—the set of all $m \times n$ matrices over the complex number field $\mathbb{C}$ and when $m = n$, we write M_m instead of $M_{m,n}$;

(ii) A^T, A^*, A^+, A^{-1}—the transpose, conjugate transpose, Moore-Penrose inverse, and inverse of matrix A, respectively.

For Hermitian matrices A and B, the relation $A > B$ means that $A - B > 0$ is a positive definite and the relation $A \geq B$ means $A - B \geq 0$ is a positive semidefinite. Given a positive definite matrix A, its positive definite square root is denoted by $A^{1/2}$. We use the known fact "for positive definite matrices A and B, the relation $A \geq B$ implies $A^{1/2} \geq B^{1/2}$" which is called the *Löwner-Heinz theorem*.

2. Some notations and preliminary results

Let A be a positive definite $m \times m$ matrix. The *spectral decomposition* of matrix A assures that there exists a unitary matrix U such that

$$A = U^* D U = U^* \operatorname{diag}(\lambda_i) U, \quad U^* U = I_m, \tag{2.1}$$

where $D = \operatorname{diag}(\lambda_i) = \operatorname{diag}(\lambda_1,\ldots,\lambda_m)$ is the diagonal matrix with diagonal entries λ_i (λ_i are the positive eigenvalues of A). For any real number r, A^r is defined by

$$A^r = U^* D^r U = U^* \operatorname{diag}(\lambda_i^r) U. \tag{2.2}$$

If $A \in M_{m,n}$ is any matrix with rank (A) $= s$, the *singular value decomposition* of A assures that there are unitary matrices $U \in M_m$ and $V \in M_n$ such that

$$A = U \sum V^*. \tag{2.3}$$

Here $\sum = \left[\begin{smallmatrix} W & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_{m,n}$, where $W = \operatorname{diag}(\varsigma_1,\ldots,\varsigma_s) \in M_s$ is the diagonal matrix with diagonal entries ς_i ($i = 1,2,\ldots,s$) and $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_s > 0$ are the singular values of A, that is, $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_s > 0$ are positive square roots of positive eigenvalues of A^*A and AA^*. The *Moore-Penrose inverse* of A is defined by

$$A^+ = V \begin{bmatrix} W^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \in M_{n,m}, \tag{2.4}$$

where $W^{-1} = \operatorname{diag}(\varsigma_1^{-1},\varsigma_2^{-1},\ldots,\varsigma_s^{-1}) \in M_s$ is the diagonal matrix with diagonal entries ς_i^{-1} ($i = 1,2,\ldots,s$). A^+ is a unique matrix which satisfies the following conditions:

$$AA^+A = A, \qquad A^+AA^+ = A^+, \qquad (AA^+)^* = AA^+, \qquad (A^+A)^* = A^+A. \tag{2.5}$$

For any compatible partitioned matrices A, B, C, and D, we will make a frequent use of the following properties of the Tracy-Singh product (see e.g., [1, 3, 5, 10]):

(a) $(A\Theta B)(C\Theta D) = (AC)\Theta(BD)$ if AC and BD are well defined;
(b) $(A\Theta B)^r = A^r\Theta B^r$ if $A \in M_m$, $B \in M_n$ are positive semidefinite matrices and r is any real number;
(c) $(A\Theta B)^* = A^*\Theta B^*$;
(d) $(A\Theta B)^+ = A^+\Theta B^+$.

If $A \in M_m$ and $B \in M_n$ are positive semidefinite matrices, then (see, [3, 10])

(e) $A\Theta B \geq 0$;
(f) $\lambda_1(A\Theta B) = \lambda_1(A)\lambda_1(B)$, $\lambda_{mn}(A\Theta B) = \lambda_m(A)\lambda_n(B)$,

where $\lambda_1(A)$, $\lambda_m(A)$ are the largest and smallest eigenvalues, respectively, of a matrix A, and $\lambda_1(B)$, $\lambda_n(B)$ are the largest and smallest eigenvalues, respectively, of a matrix B.

The Khatri-Rao and Tracy-Singh products of k matrices A_i ($1 \leq i \leq k$, $k \geq 2$) will be denoted by $\prod_{i=1}^k *A_i = A_1 * A_2 * \cdots * A_k$ and $\prod_{i=1}^k \Theta A_i = A_1\Theta A_2\Theta\cdots\Theta A_k$, respectively.

For a finite number of matrices A_i ($i = 1,2,\ldots,k$), the properties (a)–(d) become as in Lemma 2.1 and the connection between the Khatri-Rao and Tracy-Singh products in (1.7) and (1.8) becomes as in Lemma 2.2.

Lemma 2.1. *Let A_i and B_i $(1 \le i \le k,\ k \ge 2)$ be compatible partitioned matrices. Then*

(i)

$$\left(\prod_{i=1}^{k} \Theta A_i\right)\left(\prod_{i=1}^{k} \Theta B_i\right) = \left(\prod_{i=1}^{k} \Theta (A_i B_i)\right) \tag{2.6}$$

if A_iB_i $(1 \le i \le k,\ k \ge 2)$ are well defined;

(ii)

$$\left(\prod_{i=1}^{k} \Theta A_i\right)^{+} = \prod_{i=1}^{k} \Theta A_i^{+}, \quad k = 2,3,\ldots; \tag{2.7}$$

(iii)

$$\left(\prod_{i=1}^{k} \Theta A_i\right)^{*} = \prod_{i=1}^{k} \Theta A_i^{*}, \qquad \left(\prod_{i=1}^{k} * A_i\right)^{*} = \prod_{i=1}^{k} * A_i^{*}, \quad k = 2,3,\ldots; \tag{2.8}$$

(iv)

$$\left(\prod_{i=1}^{k} \Theta A_i\right)^{r} = \prod_{i=1}^{k} \Theta A_i^{r} \quad \text{if } A_i \in M_{m(i)}\ (1 \le i \le k,\ k \ge 2) \tag{2.9}$$

are positive semidefinite matrices and r is any real number;

(v)

$$\left(\prod_{i=1}^{k} (A_i \Theta B_i)\right) = \left(\prod_{i=1}^{k} A_i\right) \Theta \left(\prod_{i=1}^{k} B_i\right), \quad k = 2,3,\ldots. \tag{2.10}$$

Proof. The proof is immediately derived by induction on k. □

Lemma 2.2. *Let $A_i = [A_{gh}^{(i)}] \in M_{m(i),n(i)}$ $(1 \le i \le k,\ k \ge 2)$ be partitioned matrices with $A_{gh}^{(i)}$ as the (g,h)th block submatrix ($m = \prod_{i=1}^{k} m(i)$, $n = \prod_{i=1}^{k} n(i)$, $r = \sum_{j=1}^{t} \prod_{i=1}^{k} m_j(i)$, $s = \sum_{j=1}^{t} \prod_{i=1}^{k} n_j(i)$, $m(i) = \sum_{j=1}^{t} m_j(i)$, $n(i) = \sum_{j=1}^{t} n_j(i)$). Then there exist two real matrices Z_1 of order $m \times r$ and Z_2 of order $n \times s$ such that $Z_1^T Z_1 = I_r$, $Z_2^T Z_2 = I_s$ (Z_1, Z_2 are real matrices of zeros and ones) and*

$$\prod_{i=1}^{k} * A_i = Z_1^T \left(\prod_{i=1}^{k} \Theta A_i\right) Z_2, \quad k = 2,3,\ldots, \tag{2.11}$$

where I_r and I_s are identity matrices of order $r \times r$ and $s \times s$, respectively. In particular, if

$m(i)=n(i)$ $(1\le i\le k, k\ge 2)$, then there exists an $m\times r$ matrix Z of zeros and ones such that $Z^TZ=I_r$,

$$\prod_{i=1}^{k}*A_i=Z^T\left(\prod_{i=1}^{k}\Theta A_i\right)Z,\quad k=2,3,\ldots,\tag{2.12}$$

and ZZ^T is an $m\times m$ diagonal matrix of zeros and ones, so

$$0\le ZZ^T\le I_m,\tag{2.13}$$

where $m=\prod_{i=1}^{k}m(i)$.

Proof. The special case in (2.12) of Lemma 2.2 is proved in [3, Corollary 2.2] and (2.13) follows immediately by the definition of matrix Z. We give proof of the general case in (2.11) of Lemma 2.2 for the sake of convenience. We proceed by induction on k. If $k=2$, then (2.11) is true by (1.7). Now suppose (2.11) holds for the Khatri-Rao product of k matrices, that is, there exist an $m\times r$ matrix P_{kr} of zeros and ones and an $n\times s$ matrix R_{ks} of zeros and ones such that $P_{kr}^TP_{kr}=I_r$, $R_{ks}^TR_{ks}=I_s$, and

$$\prod_{i=1}^{k}*A_i=P_{kr}^T\left(\prod_{i=1}^{k}\Theta A_i\right)R_{ks},\quad k=2,3,\ldots.\tag{2.14}$$

We will prove that it is true for the Khatri-Rao product of $k+1$ matrices. Then by (1.7), there exist an $m(1)r\times r$ matrix Q_1 of zeros and ones and an $n(1)s\times s$ matrix Q_2 of zeros and ones such that $Q_1^TQ_1=I_r$, $Q_2^TQ_2=I_s$, and

$$\begin{aligned}\prod_{i=1}^{k+1}*A_i&=A_1*\left(\prod_{i=2}^{k+1}*A_i\right)=Q_1^T\left(A_1\Theta\prod_{i=2}^{k+1}*A_i\right)Q_2=Q_1^T\left\{A_1\Theta\left(P_{kr}^T\left(\prod_{i=2}^{k+1}\Theta A_i\right)R_{ks}\right)\right\}Q_2\\&=Q_1^T\left\{\left(I_{m(1)}A_1I_{n(1)}\right)\Theta\left(P_{kr}^T\left(\prod_{i=2}^{k+1}\Theta A_i\right)R_{ks}\right)\right\}Q_2\\&=Q_1^T\left(I_{m(1)}\Theta P_{kr}^T\right)\left\{A_1\Theta\left(\prod_{i=2}^{k+1}\Theta A_i\right)\right\}\left(I_{n(1)}\Theta R_{ks}\right)Q_2\\&=Q_1^T\left(I_{m(1)}\Theta P_{kr}^T\right)\left(\prod_{i=1}^{k+1}\Theta A_i\right)\left(I_{n(1)}\Theta R_{ks}\right)Q_2.\end{aligned}\tag{2.15}$$

Letting $Z_1=(I_{m(1)}\Theta P_{kr})Q_1$ and $Z_2=(I_{n(1)}\Theta R_{ks})Q_2$, the inductive step is complete. Here $Q_1=P_{2r}=P_r$, $Q_1=R_{2s}=R_s$, and it is a simple matter to verify that

$$\begin{aligned}Z_1&=(I_{m(1)}\Theta P_{kr})P_r=P_{(k+1)r}, & Z_1^T&=P_r^T(I_{m(1)}\Theta P_{kr}^T)=P_{(k+1)r}^T,\\Z_2&=(I_{n(1)}\Theta R_{ks})R_s=R_{(k+1)s}, & Z_2^T&=R_s^T(I_{n(1)}\Theta R_{ks}^T)=R_{(k+1)s}^T.\end{aligned}\tag{2.16}$$

Note that

$$\begin{aligned} Z_1^T Z_1 &= P_r^T (I_{m(1)} \Theta P_{kr}^T)(I_{m(1)} \Theta P_{kr}) P_r = Q_1^T (I_{m(1)} \Theta P_{kr}^T)(I_{m(1)} \Theta P_{kr}) Q_1 \\ &= Q_1^T (I_{m(1)} I_{m(1)} \Theta P_{kr}^T P_{kr}) Q_1 \\ &= Q_1^T (I_{m(1)} \Theta I_r) Q_1 \quad (I_{m(1)} \Theta I_r = I_{m(1)r}) \\ &= Q_1^T (I_{m(1)r}) Q_1 = Q_1^T Q_1 = I_r. \end{aligned} \tag{2.17}$$

Similarly, it is easy to verify that $Z_2^T Z_2 = I_s$. □

Lemma 2.3. *Let α be a nonempty subset of the set $\{1,2,\dots,m\}$ and let $A \in M_m$ be a positive semidefinite matrix. Then (see Chollet [4])*

(i) *if either $-1 \le r \le 0$ or $1 \le r \le 2$, then*

$$A^r(\alpha) \ge A(\alpha)^r, \quad \forall \alpha; \tag{2.18}$$

(ii) *if $0 \le r \le 1$, then*

$$A^r(\alpha) \le A(\alpha)^r, \quad \forall \alpha, \tag{2.19}$$

where $A(\alpha)$ is the principal submatrix of A whose entries are in the intersection of the rows and columns of A specified by α.

Lemma 2.4. *Let $X_j > 0$ $(j = 1,2,\dots,k)$ be $n \times n$ matrices with eigenvalues in the interval $[w,W]$ and U_j $(j = 1,2,\dots,k)$ are $r \times m$ matrices such that $\sum_{j=1}^k U_j U_j^* = I$. Then (see Mond and Pečarić [7])*

(i) *for every real $p > 1$ and $p < 0$,*

$$\sum_{j=1}^k U_j X_j^p U_j^* \le \mu \left\{ \sum_{j=1}^k U_j X_j U_j^* \right\}^p, \tag{2.20}$$

where

$$\mu = \frac{\delta^p - \delta}{(p-1)(\delta-1)} \left(\frac{p-1}{p} \frac{\delta^p - 1}{\delta^p - \delta} \right)^p, \quad \delta = \frac{W}{w}. \tag{2.21}$$

While for $0 < p < 1$, the reverse inequality holds in (2.20);

(ii) *for every real $p > 1$ and $p < 0$,*

$$\left\{ \sum_{j=1}^k U_j X_j^p U_j^* \right\} - \left\{ \sum_{j=1}^k U_j X_j U_j^* \right\}^p \le \gamma\{I\}, \tag{2.22}$$

where

$$\gamma = \frac{W w^p - w W^p}{W - w} + (p-1) \left(\frac{1}{p} \frac{W^p - w^p}{W - w} \right)^{p/(p-1)}. \tag{2.23}$$

While for $0 < p < 1$, the reverse inequality holds in (2.22).

3. New applications and results

Based on the basic results in Section 2 and the general connection between the Khatri-Rao and Tracy-Singh products in Lemma 2.2, we generalize and derive some equalities and inequalities in works of Visick [8, Corollary 3, Theorem 4], Chollet [4], and Mond and Pečarić [7] with respect to the Khatri-Rao product and extend these results to any finite number of matrices. These results lead to inequalities involving Hadamard products, as a special case.

Theorem 3.1. *Let* $A_i = [A_{gh}^{(i)}] \in M_{m(i),n(i)}$ $(1 \leq i \leq k,\ k \geq 2)$ *be partitioned matrices with* $A_{gh}^{(i)}$ *as the* (g,h)*th block submatrix* ($m = \prod_{i=1}^{k} m(i)$, $n = \prod_{i=1}^{k} n(i)$) *and let* Z_1 *and* Z_2 *be the real matrices of zeros and ones that satisfy (2.11). Then*

(i) *there exists an* $m \times (m-r)$ *matrix* $Q_{(m)}$ *of zeros and ones such that the block matrix* $\Omega = [\, Z_1 \; Q_{(m)} \,]$ *is an* $m \times m$ *permutation matrix.* $Q_{(m)}$ *is not unique but for any such choice of* $Q_{(m)}$,

$$Z_1^T Q_{(m)} = 0, \qquad Q_{(m)}^T Q_{(m)} = I_{m-r}, \qquad Q_{(m)} Q_{(m)}^T + Z_1 Z_1^T = I_m \tag{3.1}$$

(ii) *for any* $m \times n$ *matrix* L,

$$Z_1^T L L^* Z_1 \geq (Z_1^T L Z_2)(Z_1^T L Z_2)^* \geq 0. \tag{3.2}$$

Proof. Though the proof is quite similar to the proof of [8, Corollary 3(iii) and (vii)] for Hadamard product, we give proof for the sake of convenience.

(i) It is evident from the structure of Z_1 that it may be considered as part of an $m \times m$ permutation matrix $\Omega = [\, Z_1 \; Q_{(m)} \,]$, where $Q_{(m)}$ is an $m \times (m-r)$ matrix of zeros and ones. For example, when $k = 2$, then $Q_{(2)}$ is not unique (see, [8, page 49]). Using the properties of a permutation matrix together with the definition of $\Omega = [\, Z_1 \; Q_{(m)} \,]$, we have

$$\begin{gathered} I_m = \Omega\Omega^T = \begin{bmatrix} Z_1 & Q_{(m)} \end{bmatrix} \begin{bmatrix} Z_1^T \\ Q_{(m)}^T \end{bmatrix} = Q_{(m)} Q_{(m)}^T + Z_1 Z_1^T, \\ I_m = \begin{bmatrix} I_r & 0 \\ 0 & I_{m-r} \end{bmatrix} = \Omega^T \Omega = \begin{bmatrix} Z_1^T \\ Q_m^T \end{bmatrix} \begin{bmatrix} Z_1 & Q_{(m)} \end{bmatrix} = \begin{bmatrix} Z_1^T Z_1 & Z_1^T Q_{(m)} \\ Q_{(m)}^T Z_1 & Q_{(m)}^T Q_{(m)} \end{bmatrix}. \end{gathered} \tag{3.3}$$

From these come the required results in (i), that is,

$$Z_1^T Q_{(m)} = 0, \qquad Q_{(m)}^T Q_{(m)} = I_{m-r}, \qquad Q_{(m)} Q_{(m)}^T + Z_1 Z_1^T = I_m. \tag{3.4}$$

(ii) By (2.13) of Lemma 2.2, we have $I_n \geq Z_2 Z_2^T \geq 0$ and so

$$Z_1^T L L^* Z_1 \geq Z_1^T L Z_2 Z_2^T L^* Z_1 = (Z_1^T L Z_2)(Z_1^T L Z_2)^* \geq 0. \tag{3.5}$$

We now generalize [8, Theorem 4] to the case of Khatri-Rao product involving a finite number of matrices. □

Theorem 3.2. *Let $A_i = [A_{gh}^{(i)}] \in M_{m,n}$ $(1 \le i \le k,\ k \ge 2)$ be partitioned matrices with $A_{gh}^{(i)}$ as the (g,h)th block submatrix. Let Z_1 be an $m^k \times r$ matrix of zeros and ones that satisfies (2.12) and let $Q_{(n)}$ be an $n^k \times (n^k - s)$ matrix of zeros and ones that satisfies (3.1). Then*

$$\begin{aligned}\prod_{i=1}^{k} *(A_iA_i^*) &= \left(\prod_{i=1}^{k} *(A_i)\right)\left(\prod_{i=1}^{k} *A_i\right)^* + Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Q_{(n)}Q_{(n)}^T\left(\prod_{i=1}^{k} \Theta A_i\right)^* Z_1 \\ &= \left(\prod_{i=1}^{k} *(A_i)\right)\left(\prod_{i=1}^{k} *A_i\right)^* + \left(Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Q_{(n)}\right)\left(Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Q_{(n)}\right)^*,\end{aligned} \tag{3.6}$$

and hence

$$\prod_{i=1}^{k} *(A_iA_i^*) \ge \left(\prod_{i=1}^{k} *(A_i)\right)\left(\prod_{i=1}^{k} *A_i\right)^*, \quad k = 2,3,\dots. \tag{3.7}$$

Proof. From Lemma 2.1(i) and (iii), we have

$$\prod_{i=1}^{k} \Theta(A_iA_i^*) = \left(\prod_{i=1}^{k} \Theta A_i\right)\left(\prod_{i=1}^{k} \Theta A_i\right)^*. \tag{3.8}$$

But by Theorem 3.1(i), there exist an $n^k \times s$ matrix Z_2 of zeros and ones that satisfies (2.12) and an $n^k \times (n^k - s)$ matrix $Q_{(n)}$ of zeros and ones that satisfies (3.1) such that $Z_2Z_2^T + Q_{(n)}Q_{(n)}^T = I_{n^k}$ and

$$\begin{aligned}\prod_{i=1}^{k} \Theta(A_iA_i^*) &= \left(\prod_{i=1}^{k} \Theta A_i\right)\left(Z_2Z_2^T + Q_{(n)}Q_{(n)}^T\right)\left(\prod_{i=1}^{k} \Theta A_i\right)^* \\ &= \left(\prod_{i=1}^{k} \Theta A_i\right)(Z_2Z_2^T)\left(\prod_{i=1}^{k} \Theta A_i\right)^* + \left(\prod_{i=1}^{k} \Theta A_i\right)\left(Q_{(n)}Q_{(n)}^T\right)\left(\prod_{i=1}^{k} \Theta A_i\right)^*.\end{aligned} \tag{3.9}$$

Since A_i $(1 \le i \le k,\ k \ge 2)$ are rectangular partitioned matrices of order $m \times n$, then due to (2.11) of Lemma 2.2 there exist two real matrices Z_1 and Z_2 of zeros and ones of order $m^k \times r$ and $n^k \times s$, respectively, such that

$$\prod_{i=1}^{k} *A_i = Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Z_2, \quad k = 2,3,\dots. \tag{3.10}$$

But because $A_iA_i^*$ $(1 \le i \le k,\ k \ge 2)$ are square matrices of order $m \times m$, then due to (2.12) of Lemma 2.2 there exists a real matrix Z_1 of zeros and ones of order $m^k \times r$ such that

$$\prod_{i=1}^{k} *(A_iA_i^*) = Z_1^T\left(\prod_{i=1}^{k} \Theta(A_iA_i^*)\right)Z_1, \quad k = 2,3,\dots. \tag{3.11}$$

Due to (3.9), (3.10), and (3.11), we have

$$\begin{aligned}\prod_{i=1}^{k}*(A_iA_i^*) &= Z_1^T\left(\prod_{i=1}^{k}\Theta(A_iA_i^*)\right)Z_1 = Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Z_2Z_2^T\left(\prod_{i=1}^{k}\Theta A_i\right)^*Z_1\\ &\quad+Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)\left(Q_{(n)}Q_{(n)}^T\right)\left(\prod_{i=1}^{k}\Theta A_i\right)^*Z_1\\ &=\left(Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Z_2\right)\left(Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Z_2\right)^*\\ &\quad+Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)\left(Q_{(n)}Q_{(n)}^T\right)\left(\prod_{i=1}^{k}\Theta A_i\right)^*Z_1\\ &=\left(\prod_{i=1}^{k}*(A_i)\right)\left(\prod_{i=1}^{k}*A_i\right)^*+Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Q_{(n)}Q_{(n)}^T\left(\prod_{i=1}^{k}\Theta A_i\right)^*Z_1\\ &=\left(\prod_{i=1}^{k}*(A_i)\right)\left(\prod_{i=1}^{k}*A_i\right)^*+\left(Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Q_{(n)}\right)\left(Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Q_{(n)}\right)^*.\end{aligned}\tag{3.12}$$

□

If we put $k=2$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Let* $A_i=[A_{gh}^{(i)}]\in M_{m,n}$ $(1\le i\le 2)$ *be partitioned matrices with* $A_{gh}^{(i)}$ *as the* (g,h)*th block submatrix. Let* Z_1 *be an* $m^2\times r$ *matrix of zeros and ones that satisfies (1.8) and let* $Q_{(n)}$ *be an* $n^2\times(n^2-s)$ *matrix of zeros and ones that satisfies (3.1). Then*

$$A_1A_1^**A_2A_2^*=(A_1*A_2)(A_1*A_2)^*+Z_1^T(A_1\Theta A_2)Q_{(n)}Q_{(n)}^T(A_1\Theta A_2)^*Z_1,\tag{3.13}$$

and hence

$$A_1A_1^**A_2A_2^*\ge(A_1*A_2)(A_1*A_2)^*.\tag{3.14}$$

Corollary 3.4. *Let* $A_i=[A_{gh}^{(i)}]\in M_{m,n}$ $(1\le i\le k,\ k\ge 2)$ *be partitioned matrices with* $A_{gh}^{(i)}$ *as the* (g,h)*th block submatrix. Let* Z_1 *be an* $m^k\times r$ *matrix of zeros and ones that satisfies (2.12) and let* $Q_{(n)}$ *be an* $n^k\times(n^k-s)$ *matrix of zeros and ones that satisfies (3.1). Then the following statements are equivalent:*

(i)

$$\prod_{i=1}^{k}*(A_iA_i^*)=\left(\prod_{i=1}^{k}*(A_i)\right)\left(\prod_{i=1}^{k}*A_i\right)^*,\quad k=2,3,\ldots;\tag{3.15}$$

(ii)

$$Z_1^T\left(\prod_{i=1}^{k}\Theta A_i\right)Q_{(n)}=0,\quad k=2,3,\ldots;\tag{3.16}$$

(iii)

$$\prod_{i=1}^{k} *(A_iX_i) = \left(\prod_{i=1}^{k} *(A_i)\right)\left(\prod_{i=1}^{k} *(X_i)\right), \quad \textit{for } X_i \in M_{n,m}\ (1 \le i \le k,\ k \ge 2). \tag{3.17}$$

Proof. To arrive from (i) to (ii), notice that (i) holds if and only if the last term of (3.6) is zero, which is equivalent to $Z_1^T(\prod_{i=1}^k \Theta A_i)Q_{(n)} = 0$. To arrive from (ii) to (iii), notice that (ii) may be rewritten as $Z_1^T(\prod_{i=1}^k \Theta A_i)Q_{(n)}Q_{(n)}^T = 0$. By Theorem 3.1(i), there exist an $n^k \times s$ matrix Z_2 of zeros and ones that satisfies (2.12) and an $n^k \times (n^k - s)$ matrix $Q_{(n)}$ of zeros and ones that satisfies (3.1) such that $Q_{(n)}Q_{(n)}^T = I_{n^k} - Z_2Z_2^T$, this becomes

$$Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right) = Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Z_2Z_2^T. \tag{3.18}$$

By postmultiplying by $(\prod_{i=1}^k \Theta X_i)Z_1$ for any of the $n \times m$ matrices X_i $(1 \le i \le k)$, we have

$$Z_1^T\left(\prod_{i=1}^{k} \Theta (A_iX_i)\right)Z_1 = Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right)Z_2Z_2^T\left(\prod_{i=1}^{k} \Theta X_i\right)Z_1, \tag{3.19}$$

which is (iii) by (2.11) and (2.12) of Lemma 2.2. To arrive from (iii) to (i), assume (iii) holds for all $n \times m$ matrices X_i $(1 \le i \le k)$. It must therefore be true for $X_i = A_i^*$ $(1 \le i \le k)$, which is condition (i). Hence (iii) implies (3.6) which is (i). □

If we put $k = 2$ in Corollary 3.4, we obtain the following corollary.

Corollary 3.5. *Let* $A_i = [A_{gh}^{(i)}] \in M_{m,n}$ $(1 \le i \le 2)$ *be partitioned matrices with* $A_{gh}^{(i)}$ *as the* (g,h)*th block submatrix. Let* Z_1 *be an* $m^2 \times r$ *matrix of zeros and ones that satisfies (1.8) and let* $Q_{(n)}$ *be an* $n^2 \times (n^2 - s)$ *matrix of zeros and ones that satisfies (3.1). Then the following statements are equivalent:*

(i)

$$A_1A_1^* * A_2A_2^* = (A_1 * A_2)(A_1 * A_2)^*; \tag{3.20}$$

(ii)

$$Z_1^T(A_1\Theta A_2)Q_{(n)} = 0; \tag{3.21}$$

(iii)

$$A_1X_1 * A_2X_2 = (A_1 * A_2)(X_1 * X_2), \quad \textit{for } X_1, X_2 \in M_{n,m}. \tag{3.22}$$

Theorem 3.6. *Let* $A_i \ge 0$ $(1 \le i \le k,\ k \ge 2)$ *be* $n \times n$ *compatible partitioned matrices. Then*

(i) *if either* $-1 \le r \le 0$ *or* $1 \le r \le 2$, *then*

$$\prod_{i=1}^{k} *A_i^r \ge \left(\prod_{i=1}^{k} *A_i\right)^r; \tag{3.23}$$

(ii) *if* $0 \le r \le 1$, *then*

$$\prod_{i=1}^{k} * A_i^r \le \left(\prod_{i=1}^{k} * A_i\right)^r. \tag{3.24}$$

Proof. If we put $s = 1$, replace r by $1/r$ and A_i by A_i^r in [3, Theorem 3.1(i)], we obtain (i). But, if we put $s = -1$, replace r by $1/-r$ and A_i by A_i^{-r} in [3, Theorem 3.1(i)], we obtain (ii). □

Remark 3.7. It is easy to give another proof of Theorem 3.6 by replacing A by $\prod_{i=1}^{k} \Theta A_i$ in Lemma 2.3 and applying (2.12) of Lemma 2.2.

THEOREM 3.8. *Let* $A_i > 0$ *be compatible partitioned matrices such that* $\prod_{i=1}^{k} \Theta A_i > 0$ $(1 \le i \le k,\ k \ge 2)$. *Let* W *and* w *be the largest and smallest eigenvalues of* $\prod_{i=1}^{k} \Theta A_i$, *respectively. Then*

(i) *for every real* $p > 1$ *and* $p < 0$,

$$\prod_{i=1}^{k} * A_i^p \le \mu \left(\prod_{i=1}^{k} * A_i\right)^p, \quad k = 2,3,\ldots, \tag{3.25}$$

where

$$\mu = \frac{\delta^p - \delta}{(p-1)(\delta-1)} \left(\frac{p-1}{p} \frac{\delta^p - 1}{\delta^p - \delta}\right)^p, \quad \delta = \frac{W}{w}. \tag{3.26}$$

While for every $0 < p < 1$, *the reverse inequality holds in (3.25);*

(ii) *for every real* $p > 1$ *and* $p < 0$,

$$\prod_{i=1}^{k} * A_i^p - \left(\prod_{i=1}^{k} * A_i\right)^p \le \gamma I, \quad k = 2,3,\ldots, \tag{3.27}$$

where

$$\gamma = \frac{W w^p - w W^p}{W - w} + (p-1)\left(\frac{1}{p} \frac{W^p - w^p}{W - w}\right)^{p/(p-1)}. \tag{3.28}$$

While for every $0 < p < 1$, *the reverse inequality holds in (3.27).*

Proof. This theorem follows from [3, Theorem 3.1(ii) and (iii)]. We give proof for the sake of convenience. In (2.20) and (2.22) of Lemma 2.4, set $k = 1$ and replace U by Z^T, U^* by Z, and X by $\prod_{i=1}^{k} \Theta A_i$, where Z, is the selection matrix of zeros and ones that satisfies (2.12). By using Lemma 2.1(iv), we establish Theorem 3.8. □

From (3.25), we have the following special cases:

(i) for $p = 2$, we have

$$\prod_{i=1}^{k} * A_i^2 \le \left\{\frac{(W+w)^2}{4wW}\right\} \left(\prod_{i=1}^{k} * A_i\right)^2, \quad k = 2,3,\ldots; \tag{3.29}$$

(ii) for $p = -1$, we have

$$\prod_{i=1}^{k} *A_i^{-1} \leq \left\{ \frac{(W+w)^2}{4wW} \right\} \left(\prod_{i=1}^{k} *A_i \right)^{-1}, \quad k = 2,3,\ldots. \tag{3.30}$$

From (3.27), we have the following special cases:

(i) for $p = 2$, we have

$$\prod_{i=1}^{k} *A_i^2 - \left(\prod_{i=1}^{k} *A_i \right)^2 \leq \frac{1}{4}(W-w)^2\{I\}, \quad k = 2,3,\ldots; \tag{3.31}$$

(ii) for $p = -1$, we have

$$\prod_{i=1}^{k} *A_i^{-1} - \left(\prod_{i=1}^{k} *A_i \right)^{-1} \leq \left\{ \frac{\sqrt{W}-\sqrt{w}}{wW} \right\} I, \quad k = 2,3,\ldots. \tag{3.32}$$

4. Further developments and applications

Due to *Albert's theorem* in [2] and [9, Theorem 6.13], for a partitioned matrix $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right]$ with a positive (semi) definite matrix $A \in M_m$,

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \geq 0 \quad \text{iff } D \geq B^* A^+ B, \tag{4.1}$$

for any positive semidefinite matrix $D \in M_n$. It is also known that if matrix A is square and nonsingular, then $A^+ = A^{-1}$ and $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \geq 0$ if and only if $D \geq B^* A^{-1} B$.

Let Z_1 and Z_2 be the real matrices of zeros and ones of order $m \times r$ and $n \times s$, respectively, that satisfy (2.11) in Lemma 2.2. Now another way to use Lemma 2.2 to generate inequalities involving the Khatri-Rao product is by using the following obvious inequality:

$$TT^* = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} T_1^* & T_2^* \end{bmatrix} = \begin{bmatrix} T_1 T_1^* & T_1 T_2^* \\ T_2 T_1^* & T_2 T_2^* \end{bmatrix} \geq 0, \tag{4.2}$$

where T_1 and T_2 are $n \times l$ and $m \times l$ matrices, respectively. Note that $T_1 T_1^*$ and $T_2 T_2^*$ are positive semidefinite (positive definite) matrices for every (nonsingular) complex matrices T_1 and T_2. This leads to

$$\begin{bmatrix} Z_2^T & 0 \\ 0 & Z_1^T \end{bmatrix} \begin{bmatrix} T_1 T_1^* & T_1 T_2^* \\ T_2 T_1^* & T_2 T_2^* \end{bmatrix} \begin{bmatrix} Z_2 & 0 \\ 0 & Z_1 \end{bmatrix} = \begin{bmatrix} Z_2^T T_1 T_1^* Z_2 & Z_2^T T_1 T_2^* Z_1 \\ Z_1^T T_2 T_1^* Z_2 & Z_1^T T_2 T_2^* Z_1 \end{bmatrix} \geq 0, \tag{4.3}$$

if and only if

$$Z_1^T T_2 T_2^* Z_1 \geq (Z_1^T T_2 T_1^* Z_2)(Z_2^T T_1 T_1^* Z_2)^+ (Z_2^T T_1 T_2^* Z_1). \tag{4.4}$$

Therefore (4.4) can be considered to be more general than (3.2). In order to prove this we set $T_1 = I$ and $T_2 = L$ in (4.4), we have

$$\begin{aligned} Z_1^T LL^* Z_1 &\geq (Z_1^T LI^* Z_2)(Z_2^T II^* Z_2)^+ (Z_2^T IL^* Z_1) \\ &= (Z_1^T LZ_2)(Z_2^T Z_2)^+ (Z_2^T L^* Z_1) \quad (Z_2^T Z_2 = I) \\ &= (Z_1^T LZ_2)(Z_1^T LZ_2)^*. \end{aligned} \tag{4.5}$$

Returning to (4.4) and (3.2), it can be easily seen that various other choices of the matrices T_1, T_2, and L are possible which lead to quite different inequalities involving Khatri-Rao products. However, there exist some inequalities that do not seem to follow directly from (1.7) or (2.11), but follow easily from (4.4) and (3.2). Based on (4.4) and (3.2) we generalize some inequalities in works of Visick [8, Corollary 13, Remark in page 56, Theorems 11, 17, and 20] and establish some new inequalities involving Khatri-Rao products of several positive matrices.

Theorem 4.1. *Let A_1 and A_2 be compatible partitioned matrices. Then*

$$\begin{aligned} &A_1A_1^* * A_2A_2^* + A_2A_2^* * A_1A_1^* + A_1A_2^* * A_2A_1^* + A_2A_1^* * A_1A_2^* \\ &\quad \geq (A_1 * A_2 + A_2 * A_1)((A_1 * A_2)^* + (A_2 * A_1)^*). \end{aligned} \tag{4.6}$$

Proof. Set $T_1 = I\Theta I$ and $T_2 = A_1\Theta A_2 + A_2\Theta A_1$. Then calculations show that

$$\begin{gathered} T_2T_2^* = A_1A_1^*\Theta A_2A_2^* + A_2A_2^*\Theta A_1A_1^* + A_1A_2^*\Theta A_2A_1^* + A_2A_1^*\Theta A_1A_2^*, \\ T_2T_1^* = A_1\Theta A_2 + A_2\Theta A_1, \qquad T_1T_2^* = (A_1\Theta A_2)^* + (A_2\Theta A_1)^*, \qquad T_1T_1^* = I\Theta I. \end{gathered} \tag{4.7}$$

Substituting these into (4.4) and using (1.7), we get (4.6). □

Corollary 4.2. *Let A_i $(1 \leq i \leq 2)$ be Hermitian compatible partitioned matrices. Then*

(i)

$$A_1^2 * A_2^2 \geq (A_1 * A_2)^2; \tag{4.8}$$

(ii)

$$A^2 * A^{-2} \geq (A * A^{-1})^2 \quad \text{if } A \text{ is nonsingular}; \tag{4.9}$$

(iii)

$$I * A^2 \geq (I * A)^2. \tag{4.10}$$

Proof. (i) Set $A_1^* = A_1$ and $A_2^* = A_2$ in (3.14) of Corollary 3.3, we get (4.8).

(ii) Set $A_1 = A$ and $A_2 = A^{-1}$ in (4.8), we get (4.9).

(iii) Set $A_1 = I$ and $A_2 = A$ in (4.8), we get (4.10). □

Corollary 4.3. *Let $A_i > 0$ $(1 \leq i \leq 2)$ be compatible partitioned matrices. Then*

$$(A_1^2 * A_2^2)^{1/2} \geq A_1 * A_2. \tag{4.11}$$

Proof. It follows immediately by (4.8) and *Löwner-Heinz theorem.* □

Theorem 4.4. *Let* $A_i \geq 0$ $(1 \leq i \leq k,\ k \geq 2)$ *be compatible partitioned matrices and let* $A_i^0 = A_i^{1/2}A_i^{+1/2} = A_i^{+1/2}A_i^{1/2}$ $(1 \leq i \leq k)$. *Then*

$$2\left(\prod_{i=1}^{k} * A_i^0\right) + \left(A_1 * \prod_{i=2}^{k} * A_i^+\right) + \left(A_1^+ * \prod_{i=2}^{k} * A_i\right)$$
$$\geq \left(A_1 * \prod_{i=2}^{k} * A_i^0 + A_1^0 * \prod_{i=2}^{k} * A_i\right)\left(\prod_{i=1}^{k} * A_i\right)^+ \left(A_1 * \prod_{i=2}^{k} * A_i^0 + A_1^0 * \prod_{i=2}^{k} * A_i\right). \tag{4.12}$$

Proof. Since $A_i \geq 0$ $(1 \leq i \leq k,\ k \geq 2)$, then $A_i^* = A_i$. Set $T_1 = \prod_{i=1}^{k} \Theta A_i^{1/2}$ and $T_2 = A_1^{1/2}\Theta\prod_{i=2}^{k}\Theta A_i^{+1/2} + A_1^{+1/2}\Theta\prod_{i=2}^{k}A_i^{1/2}$. Since $A_i^{1/2}A_i^{1/2} = A_i$, $A_i^{+1/2}A_i^{+1/2} = A_i^+$, and $A_i^0 = A_i^{1/2}A_i^{+1/2} = A_i^{+1/2}A_i^{1/2}$ $(1 \leq i \leq k)$, then calculations show that

$$T_2T_2^* = 2\left(\prod_{i=1}^{k}\Theta A_i^0\right) + \left(A_1\Theta\prod_{i=2}^{k}\Theta A_i^+\right) + \left(A_1^+\Theta\prod_{i=2}^{k}\Theta A_i\right), \qquad T_1T_1^* = \prod_{i=1}^{k}\Theta A_i,$$
$$T_2T_1^* = \left(A_1\Theta\prod_{i=2}^{k}\Theta A_i^0 + A_1^0\Theta\prod_{i=2}^{k}\Theta A_i\right), \qquad T_1T_2^* = \left(A_1\Theta\prod_{i=2}^{k}\Theta A_i^0 + A_1^0\Theta\prod_{i=2}^{k}\Theta A_i\right). \tag{4.13}$$

Substituting these into (4.4) and using Lemma 2.2, we get (4.12). □

If we put $k = 2$ and replace A_i by A_i^r $(1 \leq i \leq 2)$ in Theorem 4.4, we obtain the following theorem.

Theorem 4.5. *Let* $A_1 \geq 0$, $A_2 \geq 0$ *be compatible partitioned and let* r *be any nonzero real number such that* $A_1^0 = A_1^{r/2}A_1^{+r/2} = A_1^{+r/2}A_1^{r/2}$ *and* $A_2^0 = A_2^{r/2}A_2^{+r/2} = A_2^{+r/2}A_2^{r/2}$. *Then*

$$\begin{aligned} &2A_1^0 * A_2^0 + A_1^r * A_2^{+r} + A_1^{+r} * A_2^r \\ &\quad \geq \left(A_1^r * A_2^0 + A_1^0 * A_2^r\right)\left(A_1^r * A_2^r\right)^+\left(A_1^r * A_2^0 + A_1^0 * A_2^r\right). \end{aligned} \tag{4.14}$$

If $A_1 > 0$, $A_2 > 0$ in Theorem 4.5, we obtain the following theorem.

Theorem 4.6. *Let* $A_1 > 0$, $A_2 > 0$ *be compatible partitioned and let* I *be a compatible partitioned identity matrix. Then for any nonzero real number* r,

$$2I + A_1^r * A_2^{-r} + A_1^{-r} * A_2^r \geq \left(A_1^r * I + I * A_2^r\right)\left(A_1^r * A_2^r\right)^{-1}\left(A_1^r * I + I * A_2^r\right). \tag{4.15}$$

If we put $r = 1$ and $A_1 = A_2$ in Theorem 4.6, we obtain the following theorem.

Theorem 4.7. *Let* $A > 0$ *be compatible partitioned and let* I *be a compatible partitioned identity matrix. Then*

$$2I + A * A^{-1} + A^{-1} * A \geq (A * I + I * A)(A * A)^{-1}(A * I + I * A). \tag{4.16}$$

In particular, if I is a nonpartitioned identity matrix, then

$$2I + A * A^{-1} + A^{-1} * A \geq 4(I * A)(A * A)^{-1}(I * A). \tag{4.17}$$

Theorem 4.8. *Let $A_1 > 0$ and $A_2 > 0$ be compatible partitioned matrices. Then for any nonzero real number r*

$$A_1^r * A_2^{-r} + A_1^{-r} * A_2^r + 2I \geq \left(A_1^{r/2} * A_2^{-r/2} + A_1^{-r/2} * A_2^{r/2}\right)^2. \tag{4.18}$$

In particular, if $A_1 = A_2 = A$, Then

$$A^r * A^{-r} + A^{-r} * A^r + 2I \geq \left(A^{r/2} * A^{-r/2} + A^{-r/2} * A^{r/2}\right)^2. \tag{4.19}$$

Proof. Since $A_1 > 0$ and $A_2 > 0$, then $A_1^* = A_1$ and $A_2^* = A_2$. Set $L = A_1^{r/2} \Theta A_2^{-r/2} + A_1^{-r/2} \Theta A_2^{r/2}$. Compute

$$\begin{aligned} Z_1^T L L^* Z_1 = Z_1^T L L Z_1 &= Z_1^T \left(A_1^{r/2} \Theta A_2^{-r/2} + A_1^{-r/2} \Theta A_2^{r/2}\right)\left(A_1^{r/2} \Theta A_2^{-r/2} + A_1^{-r/2} \Theta A_2^{r/2}\right) Z_1 \\ &= Z_1^T \left(A_1^r \Theta A_2^{-r}\right) Z_1 + Z_1^T (I \Theta I) Z_1 + Z_1^T (I \Theta I) Z_1 + Z_1^T \left(A_1^{-r} \Theta A_2^r\right) Z_1 \\ &= A_1^r * A_2^{-r} + 2I + A_1^{-r} * A_2^r. \end{aligned} \tag{4.20}$$

Similarly,

$$\begin{aligned} \left(Z_1^T L Z_2\right)\left(Z_1^T L Z_2\right)^* &= \left(Z_1^T L Z_2\right)^2 = \left\{Z_1^T \left(A_1^{r/2} \Theta A_2^{-r/2} + A_1^{-r/2} \Theta A_2^{r/2}\right) Z_2\right\}^2 \\ &= \left(A_1^{r/2} * A_2^{-r/2} + A_1^{-r/2} * A_2^{r/2}\right)^2. \end{aligned} \tag{4.21}$$

Substituting (4.20) and (4.21) into (3.2), we get (4.18). □

From (4.18), we have the following special cases:

(i) for $r = 1$, we have

$$A_1^* A_2^{-1} + A_1^{-1} * A_2 + 2I \geq \left(A_1^{1/2} * A_2^{-1/2} + A_1^{-1/2} * A_2^{1/2}\right)^2; \tag{4.22}$$

(ii) for $r = 2$, we have

$$A_1^2 * A_2^{-2} + A_1^{-2} * A_2^2 + 2I \geq \left(A_1 * A_2^{-1} + A_1^{-1} * A_2\right)^2. \tag{4.23}$$

From (4.19), we have the following special cases:

(i) for $r = 1$, we have

$$A * A^{-1} + A^{-1} * A + 2I \geq \left(A^{1/2} * A^{-1/2} + A^{-1/2} * A^{1/2}\right)^2; \tag{4.24}$$

(ii) for $r = 2$, we have

$$A^2 * A^{-2} + A^{-2} * A^2 + 2I \geq \left(A * A^{-1} + A^{-1} * A\right)^2. \tag{4.25}$$

Theorem 4.9. *Let $A_1 \geq 0$, $A_2 \geq 0$ be compatible partitioned and let I be a compatible partitioned identity matrix. Then*

$$A_1^2 \infty A_2^2 + 2(A_1 * A_2) \geq \left(A_1 \infty A_2\right)^2, \tag{4.26}$$

where $A_1 \infty A_2 = A_1 * I + I * A_2$ is called the Khatri-Rao sum.

Proof. Set $L = A_1 \nabla A_2 = A_1 \Theta I + I \Theta A_2$ (*Tracy-Singh sum*). Since $A_1 \geq 0$ and $A_2 \geq 0$, then $A_1^* = A_1$ and $A_2^* = A_2$. Calculations show that

$$\begin{aligned} Z_1^T L L^* Z_1 = Z_1^T L L Z_1 &= Z_1^T \left(A_1 \Theta I + I \Theta A_2\right)\left(A_1 \Theta I + I_2 \Theta A_2\right) Z_1 \\ &= A_1^2 * I + I * A_2^2 + 2\left(A_1 * A_2\right) = A_1^2 \infty A_2^2 + 2\left(A_1 * A_2\right). \end{aligned} \tag{4.27}$$

Similarly,

$$\begin{aligned} \left(Z_1^T L Z_2\right)\left(Z_1^T L Z_2\right)^* &= \left\{Z_1^T \left(A_1 \Theta I + I \Theta A_2\right) Z_2\right\}\left\{Z_1^T \left(A_1 \Theta I + I \Theta A_2\right) Z_2\right\}^* \\ &= \left(A_1 * I + I * A_2\right)^2 = \left(A_1 \infty A_2\right)^2. \end{aligned} \tag{4.28}$$

Substituting (4.27) and (4.28) into (3.2), we get (4.26). □

Theorem 4.10. *Let $A_1 > 0$ and $A_2 > 0$ be compatible partitioned matrices. Then for any positive real number r,*

$$\begin{aligned} & r\left(A_1^2 * A_2^2\right) + \left(A_1 A_2 * A_2 A_1\right) + \left(A_2 A_1 * A_1 A_2\right) + \frac{1}{r}\left(A_2^2 * A_1^2\right) \\ & \quad \geq r\left(A_1 * A_2\right)^2 + \left(A_1 * A_2\right)\left(A_2 * A_1\right) + \left(A_2 * A_1\right)\left(A_1 * A_2\right) + \frac{1}{r}\left(A_2 * A_1\right)^2. \end{aligned} \tag{4.29}$$

Proof. Set $L = \varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1$, where ε_1 and ε_2 are both positive. Since $A_1 > 0$ and $A_2 > 0$, then $A_1^* = A_1$ and $A_2^* = A_2$. Compute

$$\begin{aligned} Z_1^T L L^* Z_1 = Z_1^T L L Z_1 &= Z_1^T \left(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1\right)\left(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1\right) Z_1 \\ &= Z_1^T \left\{\varepsilon_1^2 \left(A_1^2 \Theta A_2^2\right) + \varepsilon_1 \varepsilon_2 \left(A_1 A_2 \Theta A_2 A_1\right) + \varepsilon_1 \varepsilon_2 \left(A_2 A_1 \Theta A_1 A_2\right) + \varepsilon_2^2 \left(A_2^2 \Theta A_1^2\right)\right\} Z_1 \\ &= \left\{\varepsilon_1^2 \left(A_1^2 * A_2^2\right) + \varepsilon_1 \varepsilon_2 \left(A_1 A_2 * A_2 A_1\right) + \varepsilon_1 \varepsilon_2 \left(A_2 A_1 * A_1 A_2\right) + \varepsilon_2^2 \left(A_2^2 * A_1^2\right)\right\}. \end{aligned} \tag{4.30}$$

Similarly,

$$\begin{aligned}(Z_1^T L Z_2)(Z_1^T L Z_2)^* &= \{Z_1^T(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1) Z_2\}\{Z_1^T(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1) Z_2\}^* \\ &= \{Z_1^T(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1) Z_2\}\{Z_1^T(\varepsilon_1 A_1 \Theta A_2 + \varepsilon_2 A_2 \Theta A_1) Z_2\} \\ &= \{(\varepsilon_1 A_1 * A_2 + \varepsilon_2 A_2 * A_1)\}^2 \\ &= \varepsilon_1^2 (A_1 * A_2)^2 + \varepsilon_1 \varepsilon_2 (A_1 * A_2)(A_2 * A_1) \\ &\quad + \varepsilon_1 \varepsilon_2 (A_2 * A_1)(A_1 * A_2) + \varepsilon_2^2 (A_2 * A_1)^2.\end{aligned} \tag{4.31}$$

Substituting (4.30) and (4.31) into (3.2), we have

$$\begin{aligned}&\{\varepsilon_1^2(A_1^2 * A_2^2) + \varepsilon_1\varepsilon_2(A_1A_2 * A_2A_1) + \varepsilon_1\varepsilon_2(A_2A_1 * A_1A_2) + \varepsilon_2^2(A_2^2 * A_1^2)\} \\ &\quad \geq \{\varepsilon_1^2(A_1^* A_2)^2 + \varepsilon_1\varepsilon_2(A_1 * A_2)(A_2 * A_1) + \varepsilon_1\varepsilon_2(A_2 * A_1)(A_1 * A_2) + \varepsilon_2^2(A_2^* A_1)^2\}.\end{aligned} \tag{4.32}$$

Set $r = \varepsilon_1/\varepsilon_2$, we get (4.29). ☐

Remark 4.11. Let A_i $(1 \leq i \leq k,\ k \geq 2)$ be compatible partitioned matrices. Then (3.7) can be proved by setting $T_1 = \prod_{i=1}^k \Theta I$ and $T_2 = \prod_{i=1}^k \Theta A_i$. Calculations show that

$$T_2 T_2^* = \prod_{i=1}^k \Theta A_i A_i^*, \qquad T_2 T_1^* = \prod_{i=1}^k \Theta A_i, \qquad T_1 T_2^* = \left(\prod_{i=1}^k \Theta A_i\right)^*, \qquad T_1 T_1^* = \prod_{i=1}^k \Theta I. \tag{4.33}$$

Substituting these into (4.4) and using (2.11), we get (3.7).

Remark 4.12. Let A_i $(1 \leq i \leq 2)$ be compatible partitioned matrices. Then (3.14) can be proved by putting $k = 2$ in Remark 4.11.

Remark 4.13. All results obtained in Sections 3 and 4 are quite general. These results lead to inequalities involving Hadamard product, as a special case, for nonpartitioned matrices A_i $(i = 1,2,\dots,k,\ k \geq 2)$ with the Hadamard product and Kronecker product replacing the Khatri-Rao product and Tracy-Singh product, respectively.

Now we utilize the commutativity of the Hadamard product to develop, for instance, (3.7) of Theorem 3.2. This result leads to the following inequality involving Hadamard product, as a special case:

$$\prod_{i=1}^k \circ (A_i A_i^*) \geq \left(\prod_{i=1}^k \circ (A_i)\right)\left(\prod_{i=1}^k \circ A_i\right)^*. \tag{4.34}$$

It is possible to develop (4.34) in a different direction from (3.6). For example, Visick [8, Theorem 11, page 54] proved that if $A_1, A_2 \in M_{m,n}$ and $s \in [-1,1]$, then

$$A_1 A_1^* \circ A_2 A_2^* + s(A_1 A_2^* \circ A_2 A_1^*) \geq (1+s)(A_1 \circ A_2)(A_1 \circ A_2)^*. \tag{4.35}$$

We will extend this inequality to the case of products involving any finite number of matrices.

If the Tracy-Singh and Khatri-Rao products are replaced by the Kronecker and Hadamard products in Lemma 2.2, respectively, we obtain the following corollary.

Corollary 4.14. *Let* $A_i \in M_{m,n}$ $(1 \leq i \leq k,\ k \geq 2)$. *Then*

$$\prod_{i=1}^{k} \circ A_i = P_{km}^{T} \left(\prod_{i=1}^{k} \otimes A_i \right) P_{kn}, \tag{4.36}$$

where $P_{km} = (\, E_{11}^{(m)}\ 0^{(m)} \cdots\ 0^{(m)}\ E_{22}^{(m)}\ 0^{(m)} \cdots\ 0^{(m)}\ \cdots\ 0^{(m)}\ \cdots\ 0^{(m)}\ E_{mm}^{(m)}\,)^{T}$ *is of order* $m^k \times m$, $0^{(m)}$ *is an* $m \times m$ *matrix with all entries equal to zero, and* $E_{ij}^{(m)}$ *is an* $m \times m$ *matrix of zeros except for a one in the* (i, j)*th position.*

Theorem 4.15. *Let* $A_i \in M_{m,n}$ $(1 \leq i \leq k,\ k \geq 2)$. *Then for any real scalars* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *which are not all zero,*

$$\begin{aligned} &(\alpha_1^2 + \cdots + \alpha_k^2) \left(\prod_{i=1}^{k} \circ (A_i A_i^*) \right) + \left(\sum_{r=1}^{k-1} \mu_r \prod_{w=1}^{k} \circ \left(A_w A_{(w+r)'}^* \right) \right) \\ &\quad \geq (\alpha_1 + \cdots + \alpha_k)^2 \left(\prod_{i=1}^{k} \circ A_i \right) \left(\prod_{i=1}^{k} \circ A_i \right)^{*}, \end{aligned} \tag{4.37}$$

where $\mu_r = \sum_{w=1}^{k} \alpha_w \alpha_{(w+r)'}$ *and* $w + r \equiv (w + r)' \bmod k$ *with* $1 \leq (w + r)' \leq k$.

Proof. Let

$$L = \alpha_1 (A_1 \otimes A_2 \otimes \cdots \otimes A_k) + \alpha_2 (A_2 \otimes \cdots \otimes A_k \otimes A_1) + \cdots + \alpha_k (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1}), \tag{4.38}$$

where $A_i \in M_{m,n}$ $(1 \leq i \leq k,\ k \geq 2)$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are real scalars which are not all zero. Taking indices "mod k," Lemma 2.1(i), (iii) (by setting $\otimes$ instead of Θ) give

$$\begin{aligned} LL^* &= \sum_{i=1}^{k} \alpha_i (A_i \otimes A_{i+1} \otimes \cdots \otimes A_{i-1}) \sum_{i=1}^{k} \alpha_i (A_i^* \otimes A_{i+1}^* \otimes \cdots \otimes A_{i-1}^*) \\ &= \alpha_1^2 (A_1 A_1^* \otimes \cdots \otimes A_k A_k^*) + \cdots + \alpha_k^2 (A_k A_k^* \otimes A A_1^* \otimes \cdots \otimes A_k A_{k-1}^*) \\ &\quad + \sum_{i \neq j} \alpha_i \alpha_j (A_i A_j^* \otimes A_{j+1} A_{j+1}^* \otimes \cdots \otimes A_{j-1} A_{j-1}^*). \end{aligned} \tag{4.39}$$

Now the application of (4.36) and the commutativity of the Hadamard product yield

$$P_{km}^{T} L L^* P_{km} = (\alpha_1^2 + \cdots + \alpha_k^2) \left(\prod_{i=1}^{k} \circ (A_i A_i^*) \right) + \left(\sum_{r=1}^{k-1} \mu_r \prod_{w=1}^{k} \circ \left(A_w A_{(w+r)'}^* \right) \right), \tag{4.40}$$

where $\mu_r = \sum_{w}^{k} \alpha_w \alpha_{(w+r)'}$ and $w + r \equiv (w + r)' \bmod k$ with $1 \leq (w + r)' \leq k$.

Also by (4.36) and the commutativity of the Hadamard product, we obtain

$$\begin{aligned}(P_{km}^{T}LP_{kn}) &= P_{km}^{T}\{\alpha_1(A_1\otimes A_2\otimes\cdots\otimes A_k)+\alpha_2(A_2\otimes\cdots\otimes A_k\otimes A_1)\\ &\qquad +\cdots+\alpha_k(A_k\otimes A_1\otimes\cdots\otimes A_{k-1})\}P_{kn}\\ &= \alpha_1P_{km}^{T}(A_1\otimes A_2\otimes\cdots\otimes A_k)P_{kn}+\alpha_2P_{km}^{T}(A_2\otimes\cdots\otimes A_k\otimes A_1)P_{kn}\\ &\quad +\cdots+\alpha_kP_{km}^{T}(A_k\otimes A_1\otimes\cdots\otimes A_{k-1})P_{kn}\\ &= \alpha_1(A_1\circ A_2\circ\cdots\circ A_k)+\alpha_2(A_2\circ\cdots\circ A_k\circ A_1)\\ &\quad +\cdots+\alpha_k(A_k\circ A_1\circ\cdots\circ A_{k-1})\\ &= (\alpha_1+\cdots+\alpha_k)\left(\prod_{i=1}^{k}\circ A_i\right),\\ (P_{km}^{T}LP_{kn})^{*} &= (\alpha_1+\cdots+\alpha_k)\left(\prod_{i=1}^{k}\circ A_i\right)^{*}.\end{aligned} \tag{4.41}$$

Now

$$(P_{km}^{T}LP_{kn})(P_{km}^{T}LP_{kn})^{*}=(\alpha_1+\cdots+\alpha_k)^{2}\left(\prod_{i=1}^{k}\circ A_i\right)\left(\prod_{i=1}^{k}\circ A_i\right)^{*}. \tag{4.42}$$

Since $P_{km}^{T}LL^{*}P_{km}\geq(P_{km}^{T}LP_{kn})(P_{km}^{T}LP_{kn})^{*}$ by (3.2) and from (4.40) and (4.42), we get (4.37). □

Now, we examine some special cases briefly.

In order to see that (4.37) really is an extension in (4.34), it is sufficient to set $\alpha_1=1$ and $\alpha_2=\cdots=\alpha_k=0$. Thus we recover the result of Visick in (4.35) which we mentioned before the statement of Corollary 4.14. Let $k=2$, then $\mu_1=\sum_{w=1}^{2}\alpha_w\alpha_{(w+1)'}$ with $w+1\equiv(w+1)'$ mod 2, that is, $\mu_1=2\alpha_1\alpha_2$. Then Theorem 4.15 asserts that

$$(\alpha_1^2+\alpha_2^2)(A_1A_1^{*}\circ A_2A_2^{*})+2\alpha_1\alpha_2(A_1A_2^{*}\circ A_2A_1^{*})\geq(\alpha_1+\alpha_2)^{2}(A_1\circ A_2)(A_1\circ A_2)^{*}. \tag{4.43}$$

Simplification gives

$$A_1A_1^{*}\circ A_2A_2^{*}+s(A_1A_2^{*}\circ A_2A_1^{*})\geq(1+s)(A_1\circ A_2)(A_1\circ A_2)^{*} \tag{4.44}$$

for any $s\in[-1,1]$, just as we wanted. Finally, we present an attractive inequality using three matrices. Let $k=3$, $\alpha_1=1$, $\alpha_2=\alpha_3=-1/2$. Theorem 4.15 asserts that

$$A_1A_1^{*}\circ A_2A_2^{*}\circ A_3A_3^{*}\geq\frac{1}{2}\{A_1A_2^{*}\circ A_2A_3^{*}\circ A_3A_1^{*}+A_2A_1^{*}\circ A_3A_2^{*}\circ A_1A_3^{*}\}. \tag{4.45}$$

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