EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SOME THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS

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We study the existence and multiplicity of solutions for the three-point nonlinear boundary value problem $u''(t)+\lambda a(t)f(u)=0$, $0<t<1$; $u(0)=0=u(1)-\gamma u(\eta)$, where $\eta\in(0,1)$, $\gamma\in[0,1)$, $a(t)$ and $f(u)$ are assumed to be positive and have some singularities, and λ is a positive parameter. Under certain conditions, we prove that there exists $\lambda^*>0$ such that the three-point nonlinear boundary value problem has at least two positive solutions for $0<\lambda<\lambda^*$, at least one solution for $\lambda=\lambda^*$, and no solution for $\lambda>\lambda^*$.

1. Introduction

In this paper, we consider the following second-order three-point boundary value problem (BVP)

$$\begin{gathered} u''(t)+\lambda a(t)f(u)=0, \quad 0<t<1, \\ u(0)=0=u(1)-\gamma u(\eta), \end{gathered} \tag{1.1_λ}$$

where $\eta\in(0,1)$, $\gamma\in[0,1)$, $a\in C((0,1),(0,+\infty))$, and $f\in C(\mathbb{R}^+\setminus\{0\},\mathbb{R}^+)$, here λ is a positive parameter and $\mathbb{R}^+=[0,+\infty)$.

Now $a(t)$ may have a singularity at $t=0$ and $t=1$, $f(u)$ may have a singularity at $u=0$, so the BVP (1.1_λ) is a singular problem. The BVP (1.1_λ) in the case when $\gamma=0$ can be reduced to the Dirichlet BVP

$$\begin{gathered} u''(t)+\lambda a(t)f(u)=0, \quad 0<t<1, \\ u(0)=0=u(1). \end{gathered} \tag{1.2_λ}$$

The BVP (1.2_λ) has been studied extensively in the literature, see [1, 2, 5, 9, 12] and the references therein. Choi [1] studied the particular case where $f(u)=e^u$, $a\in C^1(0,1]$, $a>0$ in $(0,1)$, and a can be singular at $t=0$, but is at most $O(1/t^{2-\delta})$ as $t\to 0^+$ for some δ. Using the shooting method, he established the following result.

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Theorem 1.1 (see [1]). *There exists* $\lambda_0 > 0$ *such that the BVP* (1.2_λ) *has a solution in* $C^2(0,1] \cap C[0,1]$ *for* $0 < \lambda < \lambda_0$, *while there is no solution for* $\lambda > \lambda_0$.

Wong [9] studied the more general BVP (1.2_λ). Using also the shooting method, Wong proved some existence results for positive solutions of the BVP (1.2_λ). Recently, Dalmasso [2] improved Theorem 1.1 and the main results in [9]. Using the upper and lower solutions technique and the fixed point index method, Dalmasso [2] proved the following result.

Theorem 1.2 (see [2]). *Let a and f satisfy the following assumptions:*

(A_1) $a \in C((0,1),[0,\infty))$, $a \not\equiv 0$ *in* $(0,1)$, *and there exists* $\alpha, \beta \in [0,1)$ *such that*

$$\int_0^1 s^\alpha (1-s)^\beta a(s)\,ds < \infty; \tag{1.1}$$

(A_2) $f \in C([0,\infty),(0,\infty))$ *is nondecreasing.*

Then,

(i) *there exists* $\lambda_0 > 0$ *such that the BVP* (1.2_λ) *has at least one positive solution in* $C^2(0,1) \cap C[0,1]$ *for* $0 < \lambda < \lambda_0$,

(ii) *if in addition f satisfies the condition that*

(A_3) *there exists* $d > 0$ *such that* $f(u) \geq du$ *for* $u \geq 0$.

Then there exists $\lambda^* > 0$ *such that the BVP* (1.2_λ) *has at least one positive solution in* $C^2(0,1) \cap C[0,1]$ *for* $0 < \lambda < \lambda^*$ *while there is no such solution for* $\lambda > \lambda^*$.

Ha and Lee [5] also considered the BVP (1.2_λ) in the case when $f(u) \geq e^u$. They proved Theorems 1.3 and 1.4.

Theorem 1.3 (see [5]). *Assume the following conditions hold*

(B_1) $a > 0$ *on* $(0,1)$;

(B_2) $a(t)$ *is singular at* $t = 0$ *satisfying* $\int_0^1 sa(s)\,ds < \infty$;

(B_3) $f(u) \geq e^u$ *for all* $u \in \mathbb{R}$.

Then there exists λ_0 *such that the BVP* (1.2_λ) *has no solution for* $\lambda > \lambda_0$ *and at least one solution for* $0 < \lambda < \lambda_0$.

Theorem 1.4 (see [5]). *Consider* (1.2_λ), *where a and f are continuous and satisfy* (B_1)–(B_3). *Also assume that*

(B_4) *f is nondecreasing.*

Then the number λ_0 *given by Theorem 1.3 is such that*

(i) (1.2_λ) *has no solution for* $\lambda > \lambda_0$;

(ii) (1.2_λ) *has at least one solution for* $\lambda = \lambda_0$;

(iii) (1.2_λ) *has at least two solutions for* $0 < \lambda < \lambda_0$.

Xu and Ma [12] generalized the main results of [1, 2, 5, 9] to an operator equation in a real Banach space E. In recent years, the multipoint BVP has been extensively studied (see [3, 4, 6–8, 10, 11, 13] and the references therein). For example, Ma and Castaneda [7] using the well-known fixed point theorem in cones established some results on the existence of at least one positive solution for some m-point boundary value problems if the nonlinearity f is either superlinear or sublinear. The purpose of this paper is to extend the main results of [1, 2, 5, 9] to the nonlinear three-point BVP (1.1_λ). We will consider the existence and multiplicity of positive solution for the nonlinear three-point BVP (1.1_λ). The results of this paper are improvements of the main results in [1, 2, 5, 9].

2. Several lemmas

Let us list some conditions to be used in this paper.

(H_1) $\gamma \in [0,1), a \in C((0,1),(0,\infty))$, and

$$\int_0^1 s(1-s)a(s)ds < \infty. \tag{2.1}$$

(H_2) $f(u) = g(u) + h(u)$, where $g : (0,\infty) \mapsto (0,\infty)$ is continuous and nonincreasing, $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is continuous, and

$$h(u) \geq b_0 u^w, \quad u \in \mathbb{R}^+, \tag{2.2}$$

for some $b_0 > 0$ and $w \geq 1$.

(H_3) There exists $M > 0$ such that

$$h(u_2) - h(u_1) \geq -M(u_2 - u_1) \tag{2.3}$$

for all $u_1, u_2 \in \mathbb{R}^+$ with $u_2 \geq u_1$

The main results of this paper are the following theorems.

Theorem 2.1. *Assume that (H_1) and (H_2) hold. Then there exists $\lambda^* > 0$ such that the BVP (1.1_λ) has at least one positive solution for $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.*

Moreover, the BVP (1.1_λ) has at least one positive solution if $\omega > 1$.

Theorem 2.2. *Assume that (H_1), (H_2), and (H_3) hold, $\omega > 1$, and there exists constant $c \geq 0$ such that $g(u) = c$ for all $u \in (0,+\infty)$. Then there exists $\lambda^* > 0$ such that the BVP (1.1_λ) has at least two positive solutions for $0 < \lambda < \lambda^*$, at least one solution for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$.*

Remark 2.3. Our theorems generalize Theorems 1.1–1.4 and the main results in [9]. In fact, Theorems 1.1–1.4 are corollaries of our theorems. Moreover, the nonlinear term $f(u)$ may have singularity at $u = 0$, therefore, even in the case when $\gamma = 0$, Theorem 2.1 cannot be obtained by Theorems 1.1–1.4 and the abstract results in [12].

Remark 2.4. The nonlinear term f was assumed to be nondecreasing in Theorems 1.2 and 1.4, but in Theorem 2.2 in this paper, we do not assume that the nonlinear term f is nondecreasing. Thus, even in the case when $\gamma = 0$, Theorem 2.2 cannot be obtained from Theorem 1.4.

Let $n \in \mathbb{N}$ and let $\mathbb{N}$ be the natural numbers set. First, let us consider the BVP of the form

$$\begin{gathered} u''(t) + \lambda a(t)\left(g\left(u + \frac{1}{n}\right) + h(u)\right) = 0, \quad 0 < t < 1, \\ u(0) = 0 = u(1) - \gamma u(\eta). \end{gathered} \tag{2.1_n^λ}$$

Definition 2.5. $\alpha \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$ is called a lower solution of (2.1_n^λ) if

$$\alpha''(t) + \lambda a(t)\left(g\left(\alpha(t) + \frac{1}{n}\right) + h(\alpha(t))\right) \geq 0, \quad t \in (0,1),$$
$$\alpha(0) \leq 0, \quad \alpha(1) - \gamma\alpha(\eta) \leq 0. \tag{2.4}$$

$\beta \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$ is called an upper solution of (2.1_n^λ) if

$$\beta''(t) + \lambda a(t)\left(g\left(\beta(t) + \frac{1}{n}\right) + h(\beta(t))\right) \leq 0, \quad t \in (0,1),$$
$$\beta(0) \geq 0, \quad \beta(1) - \gamma\beta(\eta) \geq 0. \tag{2.5}$$

According to [13, Lemma 4], we have the following lemma.

Lemma 2.6. *Assume that (H_1) holds and $\tau \geq 0$. Then the initial value problems*

$$\begin{gathered} u''(t) = \tau a(t)u(t), \quad 0 \leq \alpha < t < 1, \\ u(\alpha) = 0, \qquad u'(\alpha) = 1, \\ u''(t) = \tau a(t)u(t), \quad 0 < t < \beta \leq 1, \\ u(\beta) = 0, \qquad u'(\beta) = -1 \end{gathered} \tag{2.6}$$

have unique positive solutions $p_{\alpha,\tau}(t) \in AC[\alpha,1) \cap C^1[\alpha,1)$ and $q_{\beta,\tau}(t) \in AC(0,\beta] \cap C^1(0,\beta]$, respectively. Moreover, $p_{\alpha,\tau}$ and $q_{\beta,\tau}$ are strictly convex. As a result,

$$\begin{gathered} t - \alpha \leq p_{\alpha,\tau}(t) \leq p_{\alpha,\tau}(a)\frac{(t-\alpha)}{(a-\alpha)}, \quad \alpha \leq t \leq a \leq 1, \\ \beta - t \leq q_{\beta,\tau}(t) \leq q_{\beta,\tau}(b)\frac{(\beta-t)}{(\beta-b)}, \quad 0 \leq b \leq t \leq \beta \end{gathered} \tag{2.7}$$

for any $a \in [\alpha,1)$ and $b \in [0,\beta)$.

When $0 \leq \alpha < \beta \leq 1$, for $t \in [\alpha,\beta]$,

$$W_{[\alpha,\beta]}^{(\tau)}(t) = \begin{vmatrix} q_{\beta,\tau}(t), & p_{\alpha,\tau}(t) \\ q'_{\beta,\tau}(t), & p'_{\alpha,\tau}(t) \end{vmatrix} = q_{\beta,\tau}(\alpha) = p_{\alpha,\tau}(\beta). \tag{2.8}$$

It is well known that $C[0,1]$ is a Banach space with maximum norm $\|\cdot\|$. For $\tau \geq 0$, denote θ_τ by

$$\theta_\tau = \frac{\gamma(1-\eta)}{p_{0,\tau}(\eta) + q_{1,\tau}(\eta)} \min\left\{\frac{p_{0,\tau}(\eta)}{p_{0,\tau}(1) + p_{0,\tau}(\eta)}, \frac{q_{1,\tau}(\eta)}{q_{1,\tau}(0) + q_{1,\tau}(\eta)}\right\}. \tag{2.9}$$

Let $P = \{x \in C[0,1] \mid x(t) \geq 0$ *for* $t \in [0,1]\}$ *and* $Q_\tau = \{x \in P \mid x(t) \geq \theta_\tau \|x\| t$ *for* $t \in [0,1]\}$. *It is easy to see that* P *and* Q_τ *are cones in* $C[0,1]$. *For* $\tau \geq 0$ *and each* $n \in \mathbb{N}$, *define operators* L_τ *and* $F_n : C[0,1] \mapsto C[0,1]$ *by*

$$(L_\tau x)(t) = \begin{cases} \dfrac{p_{0,\tau}(1)}{p_{0,\tau}(1) - \gamma p_{0,\tau}(\eta)} \displaystyle\int_0^1 G^{(\tau)}_{[0,1]}(\eta,s)a(s)x(s)ds, & t = \eta, \\ \displaystyle\int_0^\eta G^{(\tau)}_{[0,\eta]}(t,s)a(s)x(s)ds + (L_\tau x)(\eta)\dfrac{p_{0,\tau}(t)}{p_{0,\tau}(\eta)}, & t \in [0,\eta], \\ \displaystyle\int_\eta^1 G^{(\tau)}_{[\eta,1]}(t,s)a(s)x(s)ds + (L_\tau x)(\eta)\dfrac{q_{1,\tau}(t) + \gamma p_{\eta,\tau}(t)}{q_{1,\tau}(\eta)}, & t \in [\eta,1], \end{cases} \tag{2.10}$$

and $(F_n x)(t) = g(x(t) + 1/n) + h(x(t))$ *for* $t \in [0,1]$, *where*

$$G^{(\tau)}_{[\alpha,\beta]}(t,s) := \begin{cases} q_{\beta,\tau}(t)\dfrac{p_{\alpha,\tau}(s)}{p_{\alpha,\tau}(\beta)}, & \alpha \leq s \leq t \leq \beta, \\ p_{\alpha,\tau}(t)\dfrac{q_{\beta,\tau}(s)}{q_{\beta,\tau}(\alpha)}, & \alpha \leq t \leq s \leq \beta. \end{cases} \tag{2.11}$$

From [13, Theorem 5], we have Lemmas 2.7 and 2.9.

Lemma 2.7. *Assume that* (H_1) *holds,* $\tau \geq 0$, *and* $h \in C([0,1],R)$. *Then* $w(t)$ *is the solution of the three-point BVP*

$$\begin{gathered} -w''(t) + \tau a(t)w(t) = a(t)h(t), \quad 0 \leq \alpha < t \leq 1, \\ w(\alpha) = 0 = w(1) - \gamma w(\eta) \end{gathered} \tag{2.12}$$

if and only if $w \in C[0,1]$ *is the solution of the integral equation*

$$w(t) = (L_\tau h)(t), \quad t \in [0,1]. \tag{2.13}$$

Remark 2.8. To ensure that $p_{\alpha,\tau}(1) - \gamma p_{\alpha,\tau}(\eta) > 0$, the following condition is assumed in [13, Theorem 5]:

$$\tau a(t) > \frac{3\gamma}{(1-\eta)^2}. \tag{2.14}$$

If $0 \leq \gamma < 1$, we have

$$p_{\alpha,\tau}(1) - \gamma p_{\alpha,\tau}(\eta) > p_{\alpha,\tau}(\eta)\left(1 + \int_\eta^1 \tau a(s) q_{1,\tau}(s)ds - \gamma\right) > 0. \tag{2.15}$$

Thus, if $0 \leq \gamma < 1$, condition (2.14) can be removed.

Lemma 2.9. *Assume that (H_1) holds, $\tau,\alpha,\xi^*,\eta^* \geq 0$, $h \in C([0,1],\mathbb{R}^+)$. Also suppose that $w \in C[\alpha,1]$ satisfies*

$$\begin{gathered} -w''(t)+\tau a(t)w(t)=a(t)h(t), \quad \alpha<t<1, \\ w(\alpha)=\xi^*, \qquad w(1)-\gamma w(\eta)=\eta^*. \end{gathered} \tag{2.16}$$

Then $w(t) \geq 0$ for $t \in [\alpha,1]$.

Lemma 2.10. *Assume that (H_1) holds and $\tau \geq 0$. Then $L_\tau : P \mapsto Q_\tau$ is a completely continuous and increasing operator.*

Proof. From Lemma 2.6, we have for any $x \in P$ and $t \in [0,1]$,

$$\begin{aligned} (L_\tau x)(t) &\geq \begin{cases} (L_\tau x)(\eta)\dfrac{p_{0,\tau}(t)}{p_{0,\tau}(\eta)}, & t\in[0,\eta], \\[2ex] (L_\tau x)(\eta)\dfrac{q_{1,\tau}(t)+\gamma p_{\eta,\tau}(t)}{q_{1,\tau}(\eta)}, & t\in[\eta,1], \end{cases} \\ &\geq \begin{cases} (L_\tau x)(\eta)\dfrac{t}{p_{0,\tau}(\eta)}, & t\in[0,\eta], \\[2ex] (L_\tau x)(\eta)\dfrac{1-t+\gamma(t-\eta)}{q_{1,\tau}(\eta)}, & t\in[\eta,1], \end{cases} \\ &\geq (L_\tau x)(\eta)\frac{\gamma(1-\eta)t}{p_{0,\tau}(\eta)+q_{1,\tau}(\eta)}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} (L_\tau x)(\eta) &= \frac{p_{0,\tau}(1)}{p_{0,\tau}(1)-\gamma p_{0,\tau}(\eta)}\left(\int_0^\eta q_{1,\tau}(\eta)\frac{p_{0,\tau}(s)}{p_{0,\tau}(1)}a(s)x(s)ds\right. \\ &\qquad\left.+\int_\eta^1 p_{0,\tau}(\eta)\frac{q_{1,\tau}(s)}{q_{1,\tau}(0)}a(s)x(s)ds\right) \\ &\geq \frac{q_{1,\tau}(\eta)}{p_{0,\tau}(1)-\gamma p_{0,\tau}(\eta)}\int_0^\eta p_{0,\tau}(s)a(s)x(s)ds, \end{aligned} \tag{2.18}$$

$$\begin{aligned} (L_\tau x)(\eta) &= \frac{p_{0,\tau}(1)}{p_{0,\tau}(1)-\gamma p_{0,\tau}(\eta)}\left(\int_0^\eta q_{1,\tau}(\eta)\frac{p_{0,\tau}(s)}{p_{0,\tau}(1)}a(s)x(s)ds\right. \\ &\qquad\left.+\int_\eta^1 p_{0,\tau}(\eta)\frac{q_{1,\tau}(s)}{q_{1,\tau}(0)}a(s)x(s)ds\right) \\ &\geq \frac{p_{0,\tau}(\eta)}{p_{0,\tau}(1)-\gamma p_{0,\tau}(\eta)}\int_\eta^1 q_{1,\tau}(s)a(s)x(s)ds. \end{aligned} \tag{2.19}$$

By (2.18) and Lemma 2.6, we have for any $t \in [0,\eta]$,

$$\begin{aligned}(L_\tau x)(t) &= \int_0^t q_{\eta,\tau}(t)\frac{p_{0,\tau}(s)}{p_{0,\tau}(\eta)}a(s)x(s)ds \\ &\quad + \int_t^\eta p_{0,\tau}(t)\frac{q_{\eta,\tau}(s)}{q_{\eta,\tau}(0)}a(s)x(s)ds + (L_\tau x)(\eta)\frac{p_{0,\tau}(t)}{p_{0,\tau}(\eta)} \\ &\le \int_0^t q_{\eta,\tau}(0)\frac{p_{0,\tau}(s)}{p_{0,\tau}(\eta)}a(s)x(s)ds + \int_t^\eta p_{0,\tau}(s)\frac{q_{\eta,\tau}(0)}{q_{\eta,\tau}(0)}a(s)x(s)ds + (L_\tau x)(\eta) \\ &= \int_0^\eta p_{0,\tau}(s)a(s)x(s)ds + (L_\tau x)(\eta) \\ &\le \frac{q_{1,\tau}(0)+q_{1,\tau}(\eta)}{q_{1,\tau}(\eta)}(L_\tau x)(\eta);\end{aligned} \tag{2.20}$$

here we have used the facts that $q_{\eta,\tau}(0) = p_{0,\tau}(\eta)$ and $p_{0,\tau}(1) = q_{1,\tau}(0)$. From (2.19) and Lemma 2.6, we have for any $t \in [\eta, 1]$,

$$\begin{aligned}&(L_\tau x)(t) \\ &\quad \le \int_\eta^t q_{1,\tau}(s)\frac{p_{\eta,\tau}(1)}{p_{\eta,\tau}(1)}a(s)x(s)ds \\ &\qquad + \int_t^1 p_{\eta,\tau}(1)\frac{q_{1,\tau}(s)}{q_{1,\tau}(\eta)}a(s)x(s)ds + (L_\tau x)(\eta)\frac{q_{1,\tau}(t)+\gamma p_{\eta,\tau}(t)}{q_{1,\tau}(\eta)} \\ &\quad \le \int_\eta^1 q_{1,\tau}(s)a(s)x(s)ds + (L_\tau x)(\eta)\frac{q_{1,\tau}(\eta)\big((1-t)/(1-\eta)\big)+\gamma p_{\eta,\tau}(1)\big((t-\eta)/(1-\eta)\big)}{q_{1,\tau}(\eta)} \\ &\quad \le \int_\eta^1 q_{1,\tau}(s)a(s)x(s)ds + (L_\tau x)(\eta) \\ &\quad \le \frac{p_{0,\tau}(1)+p_{0,\tau}(\eta)}{p_{0,\tau}(\eta)}(L_\tau x)(\eta);\end{aligned} \tag{2.21}$$

here we have used the fact $p_{\eta,\tau}(1) = q_{1,\tau}(\eta)$. By (2.20) and (2.21), we have

$$(L_\tau)(\eta) \ge \min\left\{\frac{q_{1,\tau}(\eta)}{q_{1,\tau}(0)+q_{1,\tau}(\eta)}, \frac{p_{0,\tau}(\eta)}{p_{0,\tau}(1)+p_{0,\tau}(\eta)}\right\}\|L_\tau x\|. \tag{2.22}$$

By (2.17) and (2.22), we have

$$(L_\tau x)(t) \ge \theta_\tau \|L_\tau x\| t. \tag{2.23}$$

This implies that $L_\tau : P \mapsto Q_\tau$.

Now we will show that $L_\tau : P \mapsto Q_\tau$ is completely continuous. It is easy to show that $L_\tau : P \mapsto Q_\tau$ is continuous and bounded. Let $B \subset P$ be a bounded set such that $\|x\| \le R_0$

and $\|L_\tau x\| \le R_0$ for some $R_0 > 0$. For any $\varepsilon > 0$, by (H_1) there exists $\delta_1 > 0$ such that

$$\begin{aligned}
&2R_0\int_0^{\delta_1} G^{(\tau)}_{[0,\eta]}(s,s)a(s)ds + 2R_0\int_{\eta-\delta_1}^{\eta} G^{(\tau)}_{[0,\eta]}(s,s)a(s)ds \\
&\quad \le 2R_0 q_{\eta,\tau}(0)\int_0^{\delta_1}\frac{(\eta-s)s}{\eta^2}a(s)ds + 2R_0 p_{0,\tau}(\eta)\int_{\eta-\delta_1}^{\eta}\frac{(\eta-s)s}{\eta^2}a(s)ds < \frac{\varepsilon}{3}.
\end{aligned} \tag{2.24}$$

It is easy to see that there exists $\delta > 0$ such that for any $t_1,t_2 \in [0,\eta]$, $|t_1 - t_2| < \delta$,

$$\begin{gathered}
R_0\int_{\delta_1}^{\eta-\delta_1}\left|G^{(\tau)}_{[0,\eta]}(t_1,s) - G^{(\tau)}_{[0,\eta]}(t_2,s)\right|a(s)ds < \frac{\varepsilon}{3}, \\
R_0\frac{|p_{0,\tau}(t_2) - p_{0,\tau}(t_1)|}{p_{0,\tau}(\eta)} < \frac{\varepsilon}{3}.
\end{gathered} \tag{2.25}$$

By (2.24)–(2.25), we have for any $x \in B$ and $t_1,t_2 \in [0,\eta]$, $|t_1 - t_2| < \delta$,

$$\begin{aligned}
|(L_\tau x)(t_2) - (L_\tau x)(t_1)| &\le \int_0^{\eta}\left|G^{(\tau)}_{[0,\eta]}(t_2,s) - G^{(\tau)}_{[0,\eta]}(t_1,s)\right|a(s)x(s)ds \\
&\quad + (L_\tau x)(\eta)\frac{|p_{0,\tau}(t_2) - p_{0,\tau}(t_1)|}{p_{0,\tau}(\eta)} \\
&\le 2R_0\int_0^{\delta_1} G^{(\tau)}_{[0,\eta]}(s,s)a(s)ds \\
&\quad + 2R_0\int_{\eta-\delta_1}^{\eta} G^{(\tau)}_{[0,\eta]}(s,s)a(s)ds \\
&\quad + R_0\int_{\delta_1}^{\eta-\delta_1}\left|G^{(\tau)}_{[0,\eta]}(t_1,s) - G^{(\tau)}_{[0,\eta]}(t_2,s)\right|a(s)ds \\
&\quad + R_0\frac{|p_{0,\tau}(t_2) - p_{0,\tau}(t_1)|}{p_{0,\tau}(\eta)} < \varepsilon.
\end{aligned} \tag{2.26}$$

Thus, $L_\tau(B)$ is equicontinuous on $[0,\eta]$. Similarly, $L_\tau(B)$ is also equicontinuous on $[\eta,1]$. By the Arzela-Ascoli theorem, $L_\tau(B) \subset C[0,1]$ is a relatively compact set. Therefore, $L_\tau : P \mapsto Q_\tau$ is a completely continuous operator.

Finally, we show that $L_\tau : P \mapsto Q_\tau$ is increasing. For any $x_1,x_2 \in P$, $x_1 \le x_2 \in P$, let $y_1 = L_\tau x_1$ and $y_2 = L_\tau x_2$, $u = y_2 - y_1$. Then, by Lemma 2.7, we have

$$\begin{gathered}
-u''(t) + \tau a(t)u(t) = a(t)\big(x_2(t) - x_1(t)\big) \ge 0, \quad t \in (0,1), \\
u(0) = 0 = u(1) - \gamma u(\eta).
\end{gathered} \tag{2.27}$$

Then Lemma 2.9 implies that $u(t) \ge 0$ for $t \in [0,1]$, and so, $y_2 \ge y_1$. The proof is complete. □

LEMMA 2.11. *Assume (H_1) and (H_2) hold. Let $\lambda > 0$ be fixed. If there exists $R_\lambda > 0$ such that (2.1_n^λ) has at least one positive solution x_n with $\|x_n\| \le R_\lambda$ for each positive integer n, then there exist $\bar{x} \in C[0,1]$ and a subsequence $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $x_{n_k} \to \bar{x}$ as $k \to +\infty$. Moreover, $\bar{x}$ is a positive solution of the BVP (1.1_λ)*

Proof. Let $z_0(t) = 1$ for $t \in [0,1]$, and $z_\lambda(t) = \lambda g(R_\lambda + 1)(L_\tau z_0)(t)$ for $t \in [0,1]$. Since L_0 is increasing and g is nonincreasing, then we have for any $n \in \mathbb{N}$,

$$x_n(t) = \lambda(L_0 F_n x_n)(t) \geq \lambda g(R_\lambda + 1)(L_0 z_0)(t) = z_\lambda(t), \quad t \in [0,1]. \tag{2.28}$$

Let us define the function F by

$$F(t) = \int_t^1 (1-s)a(s)ds, \quad t \in (0,1]. \tag{2.29}$$

Obviously, $F \in C(0,1]$, $F(1) = 0$, and F is nonincreasing on $(0,1]$. For each $n \in \mathbb{N}$, x_n is a concave function on [0,1]. Then there exists $t^n \in (0,1)$ such that $x_n'(t^n) = 0$. By (H_2), we have

$$-x_n''(t) \leq \lambda a(t) g(x_n(t)) \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right), \quad t \in (0,1), \tag{2.30}$$

where $\bar{h}(R_\lambda) = \max_{s \in [0,R_\lambda]} h(s)$. Integrate (2.30) from t^n to t ($t \in (t^n, 1)$) to obtain

$$\frac{-x_n'(t)}{g(x_n(t))} \leq \lambda \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right) \int_{t^n}^t a(s)ds. \tag{2.31}$$

Then integrate (2.31) from t^n to 1 to obtain

$$\int_{x_n(1)}^{x_n(t^n)} \frac{ds}{g(s)} \leq \lambda \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right) \int_{t^n}^1 (1-s)a(s)ds = \lambda \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right) F(t^n). \tag{2.32}$$

On the other hand, by (2.28), we have

$$\int_{x_n(1)}^{x_n(t^n)} \frac{ds}{g(s)} \geq \frac{x_n(t^n) - x_n(1)}{g(x_n(1))} \geq \frac{x_n(\eta)(1-\gamma)}{g(x_n(1))} \geq \frac{z_\lambda(\eta)(1-\gamma)}{g(z_\lambda(1))}. \tag{2.33}$$

By (2.32) and (2.33), we have

$$F(t^n) \geq \left[\lambda \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right)\right]^{-1} \frac{z_\lambda(\eta)(1-\gamma)}{g(z_\lambda(1))}. \tag{2.34}$$

Let $\beta_0 \in (0,1]$ be such that

$$F(\beta_0) = \left[\lambda \left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right)\right]^{-1} \frac{z_\lambda(\eta)(1-\gamma)}{g(z_\lambda(1))}. \tag{2.35}$$

Then (2.34) implies that $t^n \leq \beta_0$. Similarly, we can show that there exists $\alpha_0 > 0$ such that $t^n \geq \alpha_0$ for each $n \in \mathbb{N}$. Let us define the function $I : \mathbb{R}^+ \mapsto \mathbb{R}^+$ by $I(x) = \int_0^x ds/g(s)$ for

$x \in \mathbb{R}^+$. For any $t_1, t_2 \in [\beta_0, 1]$, $t_1 < t_2$, by (2.31), we have

$$\begin{aligned} I(x_n(t_1)) - I(x_n(t_2)) &= \int_{x_n(t_2)}^{x_n(t_1)} \frac{ds}{g(s)} = \int_{t_1}^{t_2} -\frac{x_n'(s)ds}{g(x_n(s))} \\ &\le \lambda\left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right) \int_{t_1}^{t_2} dt \int_0^t a(s)ds \\ &\le \lambda\left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right)\left(\int_{t_1}^{t_2} (t_2 - s)a(s)ds + (t_2 - t_1)\int_0^{t_1} a(s)ds\right) \\ &\le \lambda\left(1 + \frac{\bar{h}(R_\lambda)}{g(R_\lambda + 1)}\right)\left(\int_{t_1}^{t_2} (1 - s)a(s)ds + (t_2 - t_1)\int_0^{1-(t_2-t_1)} a(s)ds\right). \end{aligned} \tag{2.36}$$

This and the inequalities (2.21) in [11] imply that the set $I(\{x_n\}_{n=1}^{+\infty})$ is equicontinuous on $[\beta_0, 1]$. It is easy to see that I^{-1}, the inverse function of I, is uniformly continuous on $[0, I(R_\lambda)]$. Therefore, the set $\{x_n\}_{n=1}^{+\infty}$ is equcontinuous on $[\beta_0, 1]$. Similarly, $\{x_n\}_{n=1}^{+\infty}$ is equcontinuous on $[0, \alpha_0]$.

From (2.30), we have for any $t \in [\alpha_0, \beta_0]$,

$$|x_n'(t)| \le \lambda\left(g\left(\min_{t\in[\alpha_0,\beta_0]} z_\lambda(t)\right) + \bar{h}(R_\lambda)\right)\int_{\alpha_0}^{\beta_0} a(s)ds. \tag{2.37}$$

Thus, $\{x_n\}_{n=1}^{+\infty}$ is equcontinuous on $[\alpha_0, \beta_0]$. Then, by the Arzela-Ascoli theorem, we see that $\{x_n\}_{n=1}^{+\infty} \subset C[0,1]$ is a relatively compact set. Thus, there exist $\bar{x} \in C[0,1]$ and a subsequence $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $x_{n_k} \to \bar{x}$. By a standard argument (see [11]), we have that $\bar{x}$ is a positive solution of the BVP (1.1_λ). The proof is complete. □

Lemma 2.12. *Assume that* (H_1) *and* (H_2) *hold. Then for small enough* $\lambda > 0$, *the BVP* (1.1_λ) *has at least one positive solution.*

Proof. Let $R_0 > 0$ and λ_0 be such that

$$0 < \lambda_0 < \frac{1}{2}\int_{\gamma R_0}^{R_0} \frac{ds}{g(s)}\left(\int_0^1 s(1-s)a(s)ds\right)^{-1}\left(1 + \frac{\bar{h}(R_0)}{g(R_0+1)}\right)^{-1}. \tag{2.38}$$

By Lemma 2.10, $\lambda_0 L_0 F_n : P \mapsto Q_0$ is a completely continuous operator for each $n \in \mathbb{N}$. Now we will show that

$$\mu\lambda_0 L_0 F_n u \ne u, \quad \mu \in [0,1],\ u \in \partial B(\theta, R_0),\ n \in \mathbb{N}, \tag{2.39}$$

where $B(\theta, R_0) = \{x \in Q_0 \mid \|x\| < R_0\}$. Suppose (2.39) is not true. Then there exist $\mu_0 \in [0,1]$, $u_0 \in \partial B(\theta, R_0)$, and $n_0 \in \mathbb{N}$ such that $\mu_0\lambda_0 L_0 F_{n_0} u_0 = u_0$. Obviously, $\mu_0 > 0$.

By Lemma 2.7, we have

$$u_0''(t)+\mu_0\lambda_0 a(t)\left(g\left(u_0+\frac{1}{n_0}\right)+h(u_0)\right)=0,\quad 0<t<1,$$
$$u_0(0)=0=u_0(1)-\gamma u_0(\eta). \tag{2.40}$$

Thus u_0 is a concave function on [0,1], and there exists $t_0\in(0,1)$ such that $u_0'(t_0)=0$.

A similar argument as in the proof of (2.32) guarantees that

$$\begin{aligned}\int_{u_0(1)}^{u_0(t_0)}\frac{ds}{g(s)}&\le\lambda_0\mu_0\left(1+\frac{\bar h(R_0)}{g(R_0+1)}\right)\int_{t_0}^1(1-s)a(s)ds\\&\le\frac{\lambda_0\mu_0}{t_0}\left(1+\frac{\bar h(R_0)}{g(R_0+1)}\right)\int_0^1 s(1-s)a(s)ds,\\\int_{u_0(0)}^{u_0(t_0)}\frac{ds}{g(s)}&\le\frac{\lambda_0\mu_0}{1-t_0}\left(1+\frac{\bar h(R_0)}{g(R_0+1)}\right)\int_0^1 s(1-s)a(s)ds.\end{aligned}\tag{2.41}$$

Since $u_0(t_0)=R_0$ and $u_0(1)=\gamma u_0(\eta)\le\gamma R_0$, by (2.41), we have

$$\lambda_0\ge\frac{1}{2}\left(\left(1+\frac{\bar h(R_0)}{g(R_0+1)}\right)\int_0^1 s(1-s)a(s)ds\right)^{-1}\int_{\gamma R_0}^{R_0}\frac{ds}{g(s)},\tag{2.42}$$

which contradicts (2.38). Therefore, (2.39) holds, and so

$$i(\lambda_0L_0F_n,B(\theta,R_0),Q_0)=1,\quad n\in\mathbb{N}.\tag{2.43}$$

This means that for each $n\in\mathbb{N}$, the operator $\lambda_0L_0F_n$ has at least one positive fixed point x_n such that $\|x_n\|\le R_0$. By Lemma 2.7, the BVP (2.1_n^λ) has a positive solution x_n such that $\|x_n\|\le R_0$. Then by Lemma 2.11, the BVP (1.1_λ) has at least one positive solution. The proof is complete. □

LEMMA 2.13. *Let $\alpha(t)$ and $\beta(t)$ be lower and upper solutions of (2.1_n^λ) for some $n\in\mathbb{N}$ and $\lambda>0$, $0\le\alpha(t)\le\beta(t)$. Then (2.1_n^λ) has at least one positive solution $u_{n,\lambda}$ such that*

$$\alpha(t)\le u_{n,\lambda}(t)\le\beta(t),\quad t\in[0,1].\tag{2.44}$$

Proof. Let us define the function F_n^* by

$$(F_n^*x)(t)=\begin{cases}g\left(\beta(t)+\dfrac{1}{n}\right)+h(\beta(t)), & x\ge\beta(t),\\ g\left(x+\dfrac{1}{n}\right)+h(x), & \alpha(t)<x<\beta(t),\\ g\left(\alpha(t)+\dfrac{1}{n}\right)+h(\alpha(t)), & \alpha(t)<x,\end{cases}\tag{2.45}$$

for $x \in P$. Then there exists a constant $C_n > 0$ such that $0 \le (F_n^* x)(t) \le C_n$ for $x \in P$. Now Lemma 2.10 and Schauder's fixed point theorem guarantees that the operator $\lambda L_0 F_n^*$ has at least one fixed point. Then the BVP

$$\begin{gathered} u''(t) + \lambda a(t)(F_n^* u)(t) = 0, \quad t \in (0,1), \\ u(0) = 0 = u(1) - \gamma u(\eta) \end{gathered} \tag{2.46}$$

has at least one solution $u_{n,\lambda}(t)$. Now, we will show that $\alpha(t) \le u_{n,\lambda}(t) \le \beta(t)$ for $t \in [0,1]$. Suppose that $\varepsilon_0 = \max_{t\in[0,1]}\{u_{n,\lambda}(t) - \beta(t)\} > 0$. Let $y_{n,\lambda}(t) = u_{n,\lambda}(t) - \beta(t)$. Then, $y_{n,\lambda}(t) \le \varepsilon_0$ for $t \in [0,1]$. Let $t_0 \in [t_1,t_2] \subset [0,1]$ be such that

(a) $y_{n,\lambda}(t_0) = \varepsilon_0$,
(b) $y_{n,\lambda}(t) > 0$ for $t \in (t_1,t_2)$,
(c) $[t_1,t_2]$ is the maximal interval which has the properties (a) and (b).

Then we have the following three cases.

(1) If $t_0 \in (0,1)$, then $t_0 \in (t_1,t_2)$, $y'_{n,\lambda}(t_0) = 0$. Also

$$-y''_{n,\lambda}(t) \le \lambda a(t)\left[g\left(\beta(t) + \frac{1}{n}\right) + h(\beta(t)) - g\left(\beta(t) + \frac{1}{n}\right) - h(\beta(t))\right] = 0 \tag{2.47}$$

for $t \in [t_1,t_2]$. Then $y'_{n,\lambda}(t) \le 0$ for $t \in (t_1,t_0)$, and $y'_{n,\lambda}(t) \ge 0$ for $t \in (t_0,t_2)$. Since $y_{n,\lambda}(t_0) = \max_{t\in[0,1]} y_{n,\lambda}(t)$, then $y_{n,\lambda}(t) = \varepsilon_0$ for $t \in [t_1,t_2]$, contradicting the properties (b) and (c).

(2) If $t_0 = 1$, then $y_{n,\lambda}(1) = u_{n,\lambda}(1) - \beta(1) \le \gamma(u_{n,\lambda}(\eta) - \beta(\eta)) = \gamma y_{n,\lambda}(\eta) \le \gamma y_{n,\lambda}(1)$, and so $y_{n,\lambda}(1) = 0$, a contradiction.

(3) If $t_0 = 0$, then $y_{n,\lambda}(0) = u_{n,\lambda}(0) - \beta(0) < 0$, a contradiction.

Therefore, $\beta(t) \ge u_{n,\lambda}(t)$ for $t \in [0,1]$. Similarly, we can show that $\alpha(t) \le u_{n,\lambda}(t)$ for $t \in [0,1]$. Thus, $u_{n,\lambda}(t)$ is a positive solution of (2.1_n^λ). The proof is complete. □

3. Proof of the main results

Proof of Theorem 2.1. Let

$$\Lambda = \{\lambda \in (0,+\infty) \mid (1.1_\lambda) \text{ has at least one positive solution}\}. \tag{3.1}$$

By Lemma 2.12, $\Lambda \ne \emptyset$. Assume that $\lambda_0 \in \Lambda$. Then we can show that

(1) $\lambda' \in \Lambda$ for any $0 < \lambda' \le \lambda_0$,
(2)

$$\lambda_0 \le \frac{p_{0,0}(1) - \gamma p_{0,0}(\eta)}{q_{1,0}(\eta)}\left(\int_{(1/2)\eta}^{\eta} s^2 a(s)\,ds\right)^{-1} \max\left\{\frac{1}{b_0\theta_0^\omega}, \frac{1}{g(2)}\right\}. \tag{3.2}$$

Assume that (1.1_λ) has a positive solution $z_0(t)$. It is easy to see that $z_0(t)$ and 0 are upper and lower solutions of $(2.1_{\lambda'}^n)$ for each $n \in \mathbb{N}$, respectively. By Lemma 2.13, for each $n \in \mathbb{N}$, $(2.1_{\lambda'}^n)$ has a positive solution $x_{n,\lambda'}$ such that $0 \le x_{n,\lambda'} \le z_0$. Thus, by Lemma 2.11, there exist $\bar{x}_{\lambda'} \in C[0,1]$ and a subsequence $\{x_{n_k,\lambda'}\}_{k=1}^{+\infty}$ of $\{x_{n,\lambda'}\}_{n=1}^{+\infty}$ such that $x_{n_k,\lambda'} \to \bar{x}_{\lambda'}$ as $k \to +\infty$ and $\bar{x}_{\lambda'}$ is a positive solution of $(1.1_{\lambda'})$. Thus, $\lambda' \in \Lambda$.

From Lemma 2.7, we have $x_{n_k,\lambda'} = \lambda' L_0 F_{n_k} x_{n_k,\lambda'}$. Then by Lemma 2.10,

$$x_{n_k,\lambda'}(t) \geq \theta_0 ||x_{n_k,\lambda'}|| t, \quad t \in [0,1]. \tag{3.3}$$

If $\|x_{n_k,\lambda'}\| \leq 1$, then by (H_2), we have

$$\begin{aligned}
1 \geq ||x_{n_k,\lambda'}|| \geq x_{n_k,\lambda'}(\eta) &\geq \frac{g(2)\lambda' p_{0,0}(1)}{p_{0,0} - \gamma p_{0,0}(\eta)} \int_0^1 G^{(0)}_{[0,1]}(\eta,s)a(s)ds \\
&= \frac{g(2)\lambda' p_{0,0}(1)}{p_{0,0}(1) - \gamma p_{0,0}(\eta)} \left[\int_0^\eta q_{1,0}(\eta) \frac{p_{0,0}(s)}{p_{0,0}(1)} a(s)ds + \int_\eta^1 p_{0,0}(\eta) \frac{q_{1,0}(s)}{q_{1,0}(0)} a(s)ds \right] \\
&\geq \frac{g(2)\lambda' q_{1,0}(\eta)}{p_{0,0}(1) - \gamma p_{0,0}(\eta)} \int_{(1/2)\eta}^\eta s a(s)ds,
\end{aligned} \tag{3.4}$$

and so

$$\lambda' \leq \frac{p_{0,0}(1) - \gamma p_{0,0}(\eta)}{g(2) q_{1,0}(\eta)} \left(\int_{(1/2)\eta}^\eta s a(s)ds \right)^{-1}. \tag{3.5}$$

If $\|x_{n_k,\lambda'}\| \geq 1$, then by (H_2) and (3.3), we have

$$\begin{aligned}
||x_{n_k,\lambda'}|| &\geq x_{n_k,\lambda'}(\eta) \\
&\geq \frac{b_0 \lambda' p_{0,0}(1)}{p_{0,0}(1) - \gamma p_{0,0}(\eta)} \int_0^1 G^{(0)}_{[0,1]}(\eta,s)a(s)[x_{n_k,\lambda'}]^w ds \\
&\geq \frac{b_0 \lambda' q_{1,0}(\eta)\theta_0^\omega}{p_{0,0}(1) - \gamma p_{0,0}(\eta)} \int_{(1/2)\eta}^\eta s^2 a(s)ds ||x_{n_k,\lambda'}||^w \\
&\geq \frac{b_0 \lambda' q_{1,0}(\eta)\theta_0^\omega}{p_{0,0}(1) - \gamma p_{0,0}(\eta)} \int_{(1/2)\eta}^\eta s^2 a(s)ds ||x_{n_k,\lambda'}||,
\end{aligned} \tag{3.6}$$

and so

$$\lambda' \leq \frac{p_{0,0}(1) - \gamma p_{0,0}(\eta)}{b_0 \theta_0^\omega q_{1,0}(\eta)} \left(\int_{(1/2)\eta}^\eta s^2 a(s)ds \right)^{-1}. \tag{3.7}$$

Then, (3.2) follows from (3.5) and (3.7), and (3.2) implies that Λ is a bounded set. Let $\lambda^* = \sup \Lambda$. Therefore, (1.1_λ) has at least one positive solution for $0 < \lambda < \lambda^*$.

Finally, we will show that $\lambda^* \in \Lambda$ if $\omega > 1$. Let $\{\lambda_n\} \subset \Lambda$ be an increasing number sequence such that $\lambda_n \to \lambda^*$ as $n \to +\infty$, and $\lambda_n \geq \lambda^*/2$ for $n = 1,2,\ldots$. Assume that (1.1_{λ_n}) has positive solution z_n for each $n \in \mathbb{N}$. Then z_n is an upper solution of $(2.1^k_{\lambda_n})$ and 0 is a lower solution of $(2.1^k_{\lambda_n})$ for each $k \in \mathbb{N}$. By Lemma 2.13, $(2.1^k_{\lambda_n})$ has a positive solution $z_{n,k}$ such that $0 \leq z_{n,k} \leq z_n$. Then, by Lemma 2.7,

$$z_{n,k} = \lambda_n L_0 F_k z_{n,k}. \tag{3.8}$$

Let $k \in \mathbb{N}$ be fixed. Now we will show that $\{z_{n,k}\}_{n=1}^{+\infty}$ is bounded. In fact, by (3.8) and Lemmas 2.6 and 2.10, we have

$$\begin{aligned}
||z_{n,k}|| &\geq (\lambda_n L_0 F_k z_{n,k})(\eta) \\
&\geq \frac{\lambda^* p_{0,0}(1)}{2(p_{0,0}(1)-\gamma p_{0,0}(\eta))} \int_{(1/2)\eta}^{\eta} q_{1,0}(\eta) \frac{p_{0,0}(s)}{p_{0,0}(1)} a(s) h(z_{n,k}(s))\,ds \\
&\geq \frac{\lambda^* b_0 q_{1,0}(\eta)}{2(p_{0,0}(1)-\gamma p_{0,0}(\eta))} \int_{(1/2)\eta}^{\eta} p_{0,0}(s) a(s) [z_{n,k}(s)]^w\,ds \\
&\geq \frac{\lambda^* b_0 \theta_0^w q_{1,0}(\eta)}{2(p_{0,0}(1)-\gamma p_{0,0}(\eta))} \int_{(1/2)\eta}^{\eta} s^2 a(s) ||z_{n,k}(s)||^w\,ds,
\end{aligned} \tag{3.9}$$

and so

$$||z_{n,k}|| \leq \left[\frac{2(p_{0,0}(1)-\gamma p_{0,0}(\eta))}{\lambda^* q_{1,0}(\eta) b_0 \theta_0^w} \left(\int_{(1/2)\eta}^{\eta} s^2 a(s)\,ds \right)^{-1} \right]^{1/(w-1)}. \tag{3.10}$$

This means that $\{z_{n,k}\}_{n=1}^{+\infty}$ is a bounded set. Using the fact that $L_0 : P \mapsto Q_0$ is a completely continuous operator and $\{\lambda_n\}_{n=1}^{+\infty}$ is a bounded set, we see that $\{z_{n,k}\}$ is a relatively compact set. Without loss of generality, we assume that $z_{n,k} \to z_{0,k}$ as $n \to +\infty$. Now the Lebesgue dominant convergence theorem guarantees that $z_{0,k} = \lambda^* L_0 F_k z_{0,k}$. Then, by Lemma 2.7, $z_{0,k}$ is a positive solution of $(1.1_{\lambda^*}^k)$. By Lemma 2.11, (1.1_{λ^*}) has a positive solution u^*. The proof is complete. □

Proof of Theorem 2.2. Let λ^* be defined as in Theorem 2.1 and let $\lambda \in (0, \lambda^*)$ be fixed. Let us define the nonlinear operators F and T_λ by

$$(Fx)(t) = f(x(t)) + Mx(t),\ t \in [0,1], \quad x \in P, \tag{3.11}$$

and $(T_\lambda x)(t) = (\lambda L_{\lambda M} F x)(t)$ for all $x \in P$ and $t \in [0,1]$. It follows from Lemma 2.7 that to show that (1.1_λ) has at least two positive solutions, we only need to show that the operator T_λ has at least two fixed points.

Let $z_0(t) = 1$ for $t \in [0,1]$ and $\Omega_\lambda = \{x \in Q_{\lambda M} \mid \exists \tau > 0 \text{ such that } T_\lambda x \leq u^* - \tau (L_{\lambda M} z_0)(t)\}$. Since u^* is a positive solution of (1.1_{λ^*}), then

$$\begin{gathered}
-(u^*)''(t) + \lambda M a(t) u^*(t) = \lambda a(t)(Fu^*)(t) + (\lambda^* - \lambda) a(t) f(u^*(t)), \quad 0 < t < 1, \\
u^*(0) = 0, \qquad u^*(1) = \gamma u^*(\eta).
\end{gathered} \tag{3.12}$$

By Lemma 2.7, we have $u^* = T_\lambda u^* + (\lambda^* - \lambda) L_{\lambda M} f(u^*)$. Since $L_{\lambda M}$ is increasing and $f(u^*) \geq c$, then we have

$$T_\lambda u^* \leq u^* - c(\lambda^* - \lambda)(L_{\lambda M} z_0)(t). \tag{3.13}$$

This means that $u^* \in \Omega_\lambda$, and so $\Omega_\lambda \neq \emptyset$.

For any $x_0 \in \Omega_\lambda$, by Lemma 2.10, we have

$$\begin{aligned} ||u^*|| &\geq (T_\lambda x)(\eta) \geq \frac{\lambda p_{0,\lambda M}(1)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{(1/2)\eta}^{\eta} q_{1,\lambda M}(\eta) \frac{p_{0,\lambda M}(s)}{p_{0,\lambda M}(1)} a(s) h(x(s)) ds \\ &\geq \frac{\lambda q_{1,\lambda M}(\eta)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{(1/2)\eta}^{\eta} s a(s) b_0 [x_0(s)]^w ds \\ &\geq \frac{b_0 \lambda \theta_{\lambda M}^{\omega} q_{1,\lambda M}(\eta)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{(1/2)\eta}^{\eta} s^2 a(s) ||x_0(s)||^w ds, \end{aligned} \tag{3.14}$$

and so

$$||x_0|| \leq \left[\frac{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)}{b_0 \lambda \theta_{\lambda M}^{\omega} q_{1,\lambda M}(\eta)} \left(\int_{(1/2)\eta}^{\eta} s^2 a(s) ds \right)^{-1} ||u^*|| \right]^{1/w} =: R_0. \tag{3.15}$$

This means that Ω_λ is a bounded set.

For any $x_0 \in \Omega_\lambda$, there exists $\tau_0 > 0$ such that $T_\lambda x_0 \leq u^* - \tau_0 (L_{\lambda M} z_0)(t)$. For any $x \in Q_{\lambda M}$, by Lemma 2.10, we have for $t \in [0,1]$,

$$(T_\lambda x)(t) - (T_\lambda x_0)(t) = (\lambda L_{\lambda M}(Fx - Fx_0))(t) \leq \lambda ||Fx - Fx_0|| (L_{\lambda M} z_0)(t), \tag{3.16}$$

and since F is continuous on $Q_{\lambda M}$, then there exists $\delta > 0$ such that

$$\lambda \|Fx - Fx_0\| \leq \frac{\tau_0}{2} \tag{3.17}$$

for any $x \in Q_{\lambda M}$ with $\|x - x_0\| < \delta$.

By (3.16) and (3.17), we have

$$(T_\lambda x)(t) \leq T_\lambda x_0(t) + \frac{\tau_0}{2} (L_{\lambda M} z_0)(t) \leq u^*(t) - \frac{\tau_0}{2} (L_{\lambda M} z_0)(t), \quad t \in [0,1], \tag{3.18}$$

for any $x \in Q_{\lambda M}$ with $\|x - x_0\| < \delta$. This implies that $x \in \Omega_\lambda$, and so Ω_λ is an open set.

Now we will show that

$$\mu T_\lambda x \neq x, \quad x \in \partial \Omega_\lambda, \ \mu \in [0,1]. \tag{3.19}$$

Suppose (3.19) is not true. Then there exist $x_0 \in \partial \Omega_\lambda$, $\mu_0 \in [0,1]$ such that $\mu_0 T_\lambda x_0 = x_0$. Obviously, $T_\lambda x_0 \leq u*$, and so $x_0 = \mu_0 T_\lambda x_0 \leq u^*$. Since T_λ is increasing, we have

$$T_\lambda x_0 \leq T_\lambda u^* \leq u^* - c(\lambda^* - \lambda)(L_{\lambda M} z_0)(t). \tag{3.20}$$

This implies that $x_0 \in \Omega_\lambda$, a contradiction. Thus, (3.19) holds, and so

$$i(T_\lambda, \Omega_\lambda, Q_{\lambda M}) = i(\theta, \Omega_\lambda, Q_{\lambda M}) = 1. \tag{3.21}$$

Let

$$R_0' = \left[\frac{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)}{b_0 \theta_M^w \lambda q_{1,\lambda M}(\eta)} \left(\int_{(1/2)\eta}^{\eta} s^2 a(s) ds \right)^{-1} \right]^{1/(w-1)}, \tag{3.22}$$

and $R_1 > \max\{R_0, R_0'\}$. For any $x \in \partial(B(\theta,R_1) \cap Q_{\lambda M})$, we have

$$\begin{aligned}
||T_\lambda x|| \geq (\lambda T x)(\eta) &\geq \frac{\lambda p_{0,\lambda M}(1)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{\eta/2}^{\eta} q_{1,\lambda M}(\eta) \frac{p_{0,\lambda M}(s)}{p_{0,\lambda M}(1)} a(s) h(x(s))\, ds \\
&\geq \frac{\lambda b_0 q_{1,\lambda M}(\eta)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{\eta/2}^{\eta} s a(s)[x(s)]^w\, ds \\
&\geq \frac{\theta_M^w \lambda b_0 q_{1,\lambda M}(\eta)}{p_{0,\lambda M}(1) - \gamma p_{0,\lambda M}(\eta)} \int_{\eta/2}^{\eta} s^2 a(s)||x(s)||^w\, ds > R_1.
\end{aligned} \tag{3.23}$$

Then, we have

$$i(T_\lambda, B(\theta,R_1) \cap Q_{\lambda M}, Q_{\lambda M}) = 0. \tag{3.24}$$

By (3.21) and (3.24), we have

$$i(\lambda, (B(\theta,R_1) \cap Q_{\lambda M}) \backslash \bar{\Omega}_\lambda, Q_{\lambda M}) = 0 - 1 = -1. \tag{3.25}$$

It follows from (3.21) and (3.25) that T_λ has at least two fixed points in $(B(\theta,R_1) \cap Q_{\lambda M}) \backslash \bar{\Omega}_\lambda$ and Ω_λ, respectively. Thus (1.1_λ) has at least two positive solutions for $0 < \lambda < \lambda^*$. □

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