BOUNDS FOR ELLIPTIC OPERATORS IN WEIGHTED SPACES

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Some estimates for solutions of the Dirichlet problem for second-order elliptic equations are obtained in this paper. Here the leading coefficients are locally VMO functions, while the hypotheses on the other coefficients and the boundary conditions involve a suitable weight function.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and let

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a(x) \tag{1.1}$$

be a uniformly elliptic operator with measurable coefficients in Ω. Several bounds for the solutions of the problem

$$\begin{aligned} Lu \geq f, \quad f \in L^p(\Omega),\\ u \in W^{2,p}(\Omega) \cap C^o(\bar{\Omega}),\\ u_{|\partial\Omega} \leq 0, \end{aligned} \tag{D}$$

($p \in]n/2,+\infty[$) have been given, and the application of such estimates allows to obtain certain uniqueness results for (D).

For instance, if $p \geq n$, a_i, $a \in L^p(\Omega)$ (with $a \leq 0$), a classical result of Pucci [4] shows that any solution u of the problem (D) verifies the bound

$$\sup_{\Omega} u \leq K\|f\|_{L^p(\Omega)}, \tag{1.2}$$

where $K \in \mathbb{R}_+$ depends on Ω, n, p, $\|a_i\|_{L^p(\Omega)}$ and on the ellipticity constant.

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The case $p < n$, where additional hypotheses on the leading coefficients are necessary, has been studied by several authors. Recently, a uniqueness result has been obtained in [3] under the assumption that the a_{ij}'s are of class VMO, $a_i = a = 0$ and $p \in]1, +\infty[$. This theorem has been extended to the case $a_i \neq 0$, $a \neq 0$ in [7].

If Ω is an arbitrary open subset of $\mathbb{R}^n$ and $p \in]n/2, +\infty[$, a bound of type (1.2) and a consequent uniqueness result can be found in [1]. In fact, it has been proved there that if the coefficients a_{ij} are bounded and locally VMO, the coefficients a_i, a satisfy suitable summability conditions and $\operatorname{ess\,sup}_\Omega a < 0$, then for any solution u of the problem

$$\begin{gathered} Lu \geq f, \quad f \in L^p_{\mathrm{loc}}(\Omega), \\ u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C^o(\bar{\Omega}), \\ u_{|\partial\Omega} \leq 0, \\ \limsup_{|x| \to +\infty} u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded}, \end{gathered} \tag{D'}$$

there exist a ball $B \subset\subset \Omega$ and a constant $c \in \mathbb{R}_+$ such that

$$\sup_\Omega u \leq c \left(\fint_B |f^-|^p dx \right)^{1/p}, \tag{1.3}$$

where f^- is the negative part of f,

$$\fint_B |f^-|^p dx = \frac{1}{|B|} \int_B |f^-|^p dx, \tag{1.4}$$

and c depends on n, p, on the ellipticity constant, and on the regularity of the coefficients of L.

The aim of this paper is to study a problem similar to that considered in [1], but with boundary conditions depending on an appropriate weight function. More precisely, fix a weight function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega)$ (see Section 2 for the definition of $\mathcal{A}(\Omega)$) and $s \in \mathbb{R}$, we consider a solution u of the problem

$$\begin{gathered} Lu \geq f, \quad f \in L^p_{\mathrm{loc}}(\Omega), \\ u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\ \limsup_{x \to x_o} \sigma^s(x) u(x) \leq 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \to +\infty} \sigma^s(x) u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded}. \end{gathered} \tag{1.5}$$

If the coefficients a_{ij} are bounded and locally VMO, the functions σa_i and $\sigma^2 a$ are bounded and $\operatorname{ess\,sup}_\Omega \sigma^2 a < 0$, we will prove that there exist a ball $B \subset\subset \Omega$ and a constant $c_o \in \mathbb{R}_+$ such that

$$\sup_\Omega \sigma^s u \leq c_o \left(\fint_B |\sigma^{s+2} f^-|^p dx \right)^{1/p}, \tag{1.6}$$

where c_o depends on n, p, s, σ, on the ellipticity constant, and on the regularity of the coefficients of L. As a consequence, some uniqueness results are also obtained.

2. Notation and function spaces

Let Ω be an open subset of $\mathbb{R}^n$ and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω. For each $E \in \Sigma(\Omega)$, we denote by $|E|$ the Lebesgue measure of E and put

$$E(x,r) = E \cap B(x,r) \quad \forall x \in \mathbb{R}^n,\ \forall r \in \mathbb{R}_+, \tag{2.1}$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ of radius r centered at x.

Denote by $\mathcal{A}(\Omega)$ the class of measurable functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\beta^{-1}\rho(y) \leq \rho(x) \leq \beta\rho(y) \quad \forall y \in \Omega,\ \forall x \in \Omega(y,\rho(y)), \tag{2.2}$$

where $\beta \in \mathbb{R}_+$ is independent of x and y. For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \Big\{z \in \partial\Omega : \lim_{x\to z}\rho(x) = 0\Big\}. \tag{2.3}$$

It is known that

$$\rho \in L^\infty_{\text{loc}}(\bar{\Omega}), \qquad \rho^{-1} \in L^\infty_{\text{loc}}(\bar{\Omega} \setminus S_\rho), \tag{2.4}$$

and, if $S_\rho \neq \emptyset$,

$$\rho(x) \leq \operatorname{dist}(x, S_\rho) \quad \forall x \in \Omega \tag{2.5}$$

(see [2, 6]). Having fixed $\rho \in \mathcal{A}(\Omega)$ such that $S_\rho = \partial\Omega$, it is possible to find a function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to ρ and such that

$$\sigma \in L^\infty_{\text{loc}}(\bar{\Omega}), \qquad \sigma^{-1} \in L^\infty_{\text{loc}}(\Omega), \tag{2.6}$$

$$\sigma(x) \leq \operatorname{dist}(x,\partial\Omega) \quad \forall x \in \Omega, \tag{2.7}$$

$$|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega,\ \forall \alpha \in \mathbb{N}^n_o, \tag{2.8}$$

$$\gamma^{-1}\sigma(y) \leq \sigma(x) \leq \gamma\sigma(y) \quad \forall y \in \Omega,\ \forall x \in \Omega(y,\sigma(y)), \tag{2.9}$$

where $c_\alpha, \gamma \in \mathbb{R}_+$ are independent of x and y (see [6]). For more properties of functions of $\mathcal{A}(\Omega)$ we refer to [2, 6].

If Ω has the property

$$|\Omega(x,r)| \geq Ar^n \quad \forall x \in \Omega,\ \forall r \in]0,1], \tag{2.10}$$

where A is a positive constant independent of x and r, it is possible to consider the space $\text{BMO}(\Omega,t)$, $t \in \mathbb{R}_+$, of functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{\text{BMO}(\Omega,t)} = \sup_{\substack{x\in\Omega \\ r\in]0,t]}} ⨍_{\Omega(x,r)} \left| g - ⨍_{\Omega(x,r)} g \right| dy < +\infty, \tag{2.11}$$

where $⨍_{\Omega(x,r)} g\,dy = 1/|\Omega(x,r)| \int_{\Omega(x,r)} g\,dy$. If $g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, t_A)$, where

$$t_A = \sup\left\{ t \in \mathbb{R}_+ : \sup_{\substack{x\in\Omega \\ r\in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \right\}, \tag{2.12}$$

we will say that $g \in \mathrm{VMO}(\Omega)$ if $[g]_{\mathrm{BMO}(\Omega,t)} \to 0$ for $t \to 0^+$. A function $\eta[g] : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *modulus of continuity* of g in $\mathrm{VMO}(\Omega)$ if

$$
\begin{gathered}
{}_{\mathrm{BMO}(\Omega,t)} \le \eta[g](t) \quad \forall t \in \mathbb{R}_+, \\
\lim_{t \to 0^+} \eta[g](t) = 0.
\end{gathered}
\tag{2.13}
$$

We say that $g \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$ if $(\zeta g)_o \in \mathrm{VMO}(\mathbb{R}^n)$ for any $\zeta \in C_o^\infty(\Omega)$, where $(\zeta g)_o$ denotes the zero extension of ζg outside of Ω. A more detailed account of properties of the above defined spaces $\mathrm{BMO}(\Omega)$ and $\mathrm{VMO}(\Omega)$ can be found in [5].

3. An a priori bound

Fix $p \in]n/2, +\infty[$. Let B be an open ball of $\mathbb{R}^n$, $n \ge 3$, of radius δ. We consider in B the differential operator

$$
L_B = \sum_{i,j=1}^n \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i} + \alpha(x),
\tag{3.1}
$$

with the following condition on the coefficients:

$$
\begin{gathered}
\alpha_{ij} = \alpha_{ji} \in L^\infty(B) \cap \mathrm{VMO}(B), \quad i,j = 1,\dots,n, \\
\exists \mu \in \mathbb{R}_+ : \sum_{i,j=1}^n \alpha_{ij} \zeta_i \zeta_j \ge \mu |\zeta|^2 \quad \text{a.e. in } B,\ \forall \zeta \in \mathbb{R}^n, \\
\alpha_i \in L^\infty(B), \quad i = 1,\dots,n,\ \alpha \in L^\infty(B),\ \alpha \le 0 \text{ a.e. in } B.
\end{gathered}
\tag{h_B}
$$

Let $\mu_0, \mu_1, \mu_2 \in \mathbb{R}_+$ such that

$$
\sum_{i,j=1}^n \|\alpha_{ij}\|_{L^\infty(B)} \le \mu_0, \qquad \delta \sum_{1=1}^n \|\alpha_i\|_{L^\infty(B)} \le \mu_1, \qquad \delta^2 \|\alpha\|_{L^\infty(B)} \le \mu_2.
\tag{3.2}
$$

Note that under the assumption (h_B), the operator L_B from $W^{2,p}(B)$ into $L^p(B)$ is bounded and the estimate

$$
\|L_B u\|_{L^p(B)} \le c_1 \|u\|_{W^{2,p}(B)} \quad \forall u \in W^{2,p}(B)
\tag{3.3}
$$

holds, where $c_1 \in \mathbb{R}_+$ depends on n, p, μ_0, μ_1, μ_2.

Lemma 3.1. *Suppose that condition (h_B) is verified, and let u be a solution of the problem*

$$
\begin{gathered}
u \in W^{2,p}(B), \\
L_B u \ge \phi, \quad \phi \in L^p(B), \\
u_{|\partial B} \le 0.
\end{gathered}
\tag{3.4}
$$

Then there exists $c \in \mathbb{R}_+$ such that

$$
\sup_B u \le c\delta^{2-n/p} \|\phi^-\|_{L^p(B)},
\tag{3.5}
$$

where c depends on n, p, μ, μ_0, μ_1, μ_2, $[p(\alpha_{ij})]_{\mathrm{BMO}(\mathbb{R}^n,\cdot)}$, *and where* $p(\alpha_{ij})$ *is an extension of* α_{ij} *to* $\mathbb{R}^n$ *in* $L^\infty(\mathbb{R}^n)\cap \mathrm{VMO}(\mathbb{R}^n)$.

Proof. Put $B = B(y,\delta)$, where y is the centre of B, and $B^* = B(y,1)$.

Consider the function $T : B \to B^*$ defined by the position

$$T(x) = y + \frac{x-y}{\delta} = z, \tag{3.6}$$

and for each function g defined on B, put $g^* = g\circ T^{-1}$.

We observe that

$$L_B^* u^* = \delta^2 (L_B u)^*, \tag{3.7}$$

where

$$L_B^* = \sum_{i,j=1}^n \alpha_{ij}^*(z)\frac{\partial^2}{\partial z_i \partial z_j} + \delta \sum_{i=1}^n \alpha_i^*(z)\frac{\partial}{\partial z_i} + \delta^2 \alpha^*(z). \tag{3.8}$$

Denote by $p(\alpha_{ij})$ an extension of α_{ij} to $\mathbb{R}^n$ such that

$$p(\alpha_{ij}) \in L^\infty(\mathbb{R}^n) \cap \mathrm{VMO}(\mathbb{R}^n) \tag{3.9}$$

(for the existence of such function see [5, Theorem 5.1]). Since

$$p(\alpha_{ij})^* \in L^\infty(\mathbb{R}^n) \cap \mathrm{VMO}(\mathbb{R}^n), \quad p(\alpha_{ij})^*_{|_{B^*}} = \alpha_{ij}^*, \tag{3.10}$$

it follows that

$$\alpha_{ij}^* \in L^\infty(B^*) \cap \mathrm{VMO}(B^*). \tag{3.11}$$

Moreover, the condition (h_B) yields that

$$\begin{gathered}\alpha_{ij}^* = \alpha_{ji}^*, \quad i,j = 1,\dots,n,\\ \sum_{i,j=1}^n \alpha_{ij}^*\zeta_i\zeta_j \ge \mu|\zeta|^2 \quad \text{a.e. in } B^*,\ \forall \zeta\in\mathbb{R}^n,\\ \alpha_i^* \in L^\infty(B^*), \quad i=1,\dots,n, \qquad \alpha^*\in L^\infty(B^*), \quad \alpha^*\le 0 \quad \text{a.e. in } B^*.\end{gathered} \tag{3.12}$$

We observe that the condition (3.12) implies that for r, $s \in]1,+\infty[$ the modulus of continuity of $\delta\alpha_i^*$ in $L^r(B^*)$ and that of $\delta^2\alpha^*$ in $L^s(B^*)$ depend only on $\|\delta\alpha_i^*\|_{L^\infty(B^*)}$ and $\|\delta^2\alpha^*\|_{L^\infty(B^*)}$, respectively.

Thus, applying (3.10), (3.12), and [7, Theorem 2.1], it follows that the problem

$$\begin{gathered}L_B^* v = \psi \in L^p(B^*),\\ v \in W^{2,p}(B^*) \cap \overset{o}{W}{}^{1,p}(B^*)\end{gathered} \tag{3.13}$$

has a unique solution v satisfying the estimate

$$\|v\|_{W^{2,p}(B^*)} \le K\|\psi\|_{L^p(B^*)}, \tag{3.14}$$

where K depends on $n,\ p,\ \mu,\ \mu_0,\ \mu_1,\ \mu_2,\ [p(\alpha_{ij})^*]_{\mathrm{BMO}(R^n,\cdot)}$.

The estimate (3.5) follows now from (3.14) using the same arguments of the proof of Lemma 3.2 [1] in order to obtain there (e_B) from [1, (3.23)]. □

4. Hypotheses and preliminary results

Let Ω be an open subset of $\mathbb{R}^n$, $n \ge 3$. Fix $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ such that $S_\rho = \partial\Omega$.

Consider a function $g \in C_o^\infty(\bar{\mathbb{R}}_+)$ satisfying the condition

$$0 \le g \le 1, \qquad g(t) = 1 \quad \text{if } t \ge 1, \qquad g(t) = 0 \quad \text{if } t \le \frac{1}{2}. \tag{4.1}$$

For any $k \in \mathbb{N}$, we put

$$\eta_k(x) = \frac{1}{k}\zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \quad x \in \Omega, \tag{4.2}$$

where $\zeta_k(x) = g(k\sigma(x))$, $x \in \Omega$. Clearly, $\eta_k \in C^\infty(\Omega)$ for any $k \in \mathbb{N}$ and

$$\eta_k(x) = \begin{cases} \dfrac{1}{k} & \text{if } x \in \bar{\Omega}_k, \\ \sigma(x) & \text{if } x \in \Omega \setminus \Omega_{2k}, \end{cases} \tag{4.3}$$

where

$$\Omega_k = \left\{x \in \Omega : \sigma(x) > \frac{1}{k}\right\}, \quad k \in \mathbb{N}. \tag{4.4}$$

In the following we will use the notation

$$f_x = \left(\sum_{i=1}^n f_{x_i}^2\right)^{1/2}, \qquad f_{xx} = \left(\sum_{i,j=1}^n f_{x_i x_j}^2\right)^{1/2}. \tag{4.5}$$

It is easy to show that for each $k \in \mathbb{N}$,

$$\sigma(x) \le \eta_k(x) \le 2\sigma(x), \quad x \in \Omega \setminus \bar{\Omega}_k, \tag{4.6}$$

$$c_k'\sigma(x) \le \eta_k(x) \le \sigma(x), \quad x \in \Omega_k, \tag{4.7}$$

$$(\eta_k(x))_x \le c_1(\sigma(x))_x, \quad x \in \Omega, \tag{4.8}$$

$$(\eta_k(x))_{xx} \le c_2\frac{(\sigma(x))_x^2 + \sigma(x)(\sigma(x))_{xx}}{\sigma(x)}, \quad x \in \Omega, \tag{4.9}$$

where $c'_k \in \mathbb{R}_+$ depends on k and σ, and $c_1, c_2 \in \mathbb{R}_+$ depend only on n. Moreover, for any $s \in \mathbb{R}$, we have

$$\frac{(\eta_k^s(x))_x}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x}{\sigma(x)}, \quad x \in \Omega, \tag{4.10}$$

$$\frac{(\eta_k^s(x))_{xx}}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x^2 + \eta_k(x)(\eta_k(x))_{xx}}{\sigma^2(x)}, \quad x \in \Omega, \tag{4.11}$$

where $c_3 \in \mathbb{R}_+$ depends on s and n.

We consider in Ω the differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + a(x), \tag{4.12}$$

and put

$$L_o = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \tag{4.13}$$

We will make the following assumption on the coefficients of L:

$$\begin{gathered}
a_{ij} = a_{ji} \in L^\infty(\Omega) \cap \mathrm{VMO}_{\mathrm{loc}}(\Omega), \quad i,j = 1,\dots,n, \\
\exists \nu, \nu_0 \in \mathbb{R}_+ : \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \leq \nu_0, \qquad \sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j \geq \nu|\zeta|^2 \quad \text{a.e. in } \Omega,\ \forall \zeta \in \mathbb{R}^n, \\
\exists \nu_1, \nu_2 \in \mathbb{R}_+ : \operatorname*{ess\,sup}_{\Omega} \left(\sigma(x) \sum_{i=1}^{n} |a_i(x)| \right) \leq \nu_1, \quad \operatorname*{ess\,sup}_{\Omega} (\sigma^2(x)|a(x)|) \leq \nu_2, \\
\exists a_o \in \mathbb{R}_+ : \operatorname*{ess\,sup}_{\Omega} (\sigma^2(x) a(x)) = -a_o.
\end{gathered} \tag{h_1}$$

Fixed $s \in \mathbb{R}$, let u be a solution of the problem

$$\begin{gathered}
Lu \geq f, \quad f \in L^p_{\mathrm{loc}}(\Omega), \quad u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\
\limsup_{x \to x_o} \sigma^s(x) u(x) \leq 0 \quad \forall x_o \in \partial\Omega, \\
\limsup_{|x| \to +\infty} \sigma^s(x) u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded}.
\end{gathered} \tag{P}$$

For any $k \in \mathbb{N}$, we put

$$w_k(x) = \eta_k^s(x) u(x), \quad x \in \Omega. \tag{4.14}$$

Lemma 4.1. *Suppose that condition* (h_1) *holds. Then, for any* $k \in \mathbb{N}$ *there exist functions* b_i^k ($i = 1,\dots,n$), b^k, g^k *and positive constants* β_1 *and* β_2 *such that*

$$\operatorname*{ess\,sup}_{\Omega}\left(\sigma(x)\sum_{i=1}^{n}|\,b_i^k(x)\,|\right) \leq \beta_1, \tag{4.15}$$

$$\operatorname*{ess\,sup}_{\Omega}\left(\sigma^2(x)\,|\,b^k(x)\,|\right) \leq \beta_2, \tag{4.16}$$

$$g^k \in L^p_{\mathrm{loc}}(\Omega), \tag{4.17}$$

where β_1 *depends on* s, n, ν_0, ν_1 *and* β_2 *depends on* s, n, ν_0, ν_2. *Moreover, the function* w_k, $k \in \mathbb{N}$, *satisfies the following conditions:*

$$\begin{gathered} w_k \in W^{2,p}_{\mathrm{loc}}(\Omega), \quad \limsup_{x\to x_o} w_k(x) \leq 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x|\to+\infty} w_k(x) \leq 0 \quad \textit{if } \Omega \textit{ is unbounded}, \end{gathered} \tag{4.18}$$

$$L_o w_k + \sum_{i=1}^{n} b_i^k (w_k)_{x_i} + b^k w_k \geq g^k \quad \textit{in } \Omega. \tag{4.19}$$

Proof. Fix $k \in \mathbb{N}$. From (4.6)–(4.11) and from (2.6), (2.8), it easily follows that the function w_k, defined by (4.14), verifies (4.18).

Moreover, observe that

$$\begin{gathered} L_o w_k - u L_o \eta_k^s - 2\sum_{i,j=1}^{n} a_{ij}(\eta_k^s)_{x_j} u_{x_i} + \sum_{i=1}^{n} a_i(\eta_k^s u)_{x_i} \\ -u\sum_{i=1}^{n} a_i(\eta_k^s)_{x_i} + a\eta_k^s u = \eta_k^s L u, \quad x \in \Omega. \end{gathered} \tag{4.20}$$

Since

$$(\eta_k^s)_{x_j} u_{x_i} = (\eta_k^s u)_{x_i} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} - \frac{(\eta_k^s)_{x_i}(\eta_k^s)_{x_j}}{(\eta_k^s)^2}(\eta_k^s u), \tag{4.21}$$

from (4.20), (4.19) follows, where we have put

$$\begin{aligned} b_i^k &= a_i - 2\sum_{j=1}^{n} a_{ij}\frac{(\eta_k^s)_{x_j}}{\eta_k^s}, \quad i = 1,\dots,n, \\ b^k &= a + 2\sum_{i,j=1}^{n} a_{ij}\frac{(\eta_k^s)_{x_i}(\eta_k^s)_{x_j}}{(\eta_k^s)^2} - \sum_{i,j=1}^{n} a_{ij}\frac{(\eta_k^s)_{x_i x_j}}{\eta_k^s}, \\ g^k &= \eta_k^s f + \sum_{i=1}^{n} a_i\frac{(\eta_k^s)_{x_i}}{\eta_k^s} w_k. \end{aligned} \tag{4.22}$$

On the other hand, using the hypothesis (h_1), (4.6)–(4.11), and (2.8) it is easy to show that there exist $\beta_1 \in \mathbb{R}_+$ depending on s, n, ν_0, ν_1 and $\beta_2 \in \mathbb{R}_+$ depending on s, n, ν_0, ν_2, such that (4.15), (4.16), (4.17) hold. □

Now we suppose that the following hypothesis on ρ holds:

$$\lim_{k\to+\infty}\left(\sup_{\Omega\setminus\Omega_k}\left((\sigma(x))_x+\sigma(x)(\sigma(x))_{xx}\right)\right)=0. \tag{h_2}$$

An example of function ρ such that σ satisfies (h_2) is provided in [2].

Lemma 4.2. *Suppose that conditions (h_1) and (h_2) hold. Then there exists $k_o\in\mathbb{N}$ such that*

$$\begin{gathered}\operatorname*{ess\,sup}_{\Omega}\left(\sigma(x)\sum_{i=1}^{n}|b_i^{k_o}(x)|\right)\leq\nu_1+\frac{a_o}{2},\\ \operatorname*{ess\,sup}_{\Omega}(\sigma^2(x)b^{k_o}(x))\leq-\frac{a_o}{2},\\ g^{k_o}(x)\geq\eta_{k_o}^s(x)f(x)-\frac{a_o}{8}\sigma^{-2}(x)|w_{k_o}(x)|,\quad x\in\Omega.\end{gathered}\tag{4.23}$$

Proof. From (4.10), (4.11), and hypothesis (h_1), we deduce that

$$\begin{gathered}\sigma\left|\sum_{i,j=1}^{n}a_{ij}\frac{(\eta_k^s)_{x_j}}{\eta_k^s}\right|\leq c_4(\eta_k)_x,\\ \sigma^2\left|\sum_{i,j=1}^{n}a_{ij}\frac{(\eta_k^s)_{x_i}(\eta_k^s)_{x_j}}{(\eta_k^s)^2}\right|+\sigma^2\left|\sum_{i,j=1}^{n}a_{ij}\frac{(\eta_k^s)_{x_ix_j}}{\eta_k^s}\right|\leq c_5\left((\eta_k)_x^2+\eta_k(\eta_k)_{xx}\right),\\ \sigma^2\left|\sum_{i=1}^{n}a_i\frac{(\eta_k^s)_{x_i}}{\eta_k^s}\right|\leq c_6(\eta_k)_x,\end{gathered}\tag{4.24}$$

where $c_4,c_5\in\mathbb{R}_+$ depend on s, n, ν_0 and $c_6\in\mathbb{R}_+$ depends on s, n, ν_1. Observing that $(\eta_k)_x=(\eta_k)_{xx}=0$ in $\bar{\Omega}_k$, the statement follows now from (4.8), (4.9), (h_1), (h_2), and (4.24). □

5. Main results

It is well know that there exists a function $\tilde{\alpha}\in C^\infty(\Omega)\cap C^{0,1}(\bar{\Omega})$ which is equivalent to $\operatorname{dist}(\cdot,\partial\Omega)$ (see, e.g., [8]). For every positive integer m, we define the function

$$\psi_m: x\in\bar{\Omega}\longrightarrow g(m\tilde{\alpha}(x))\left(1-g\left(\frac{|x|}{2m}\right)\right), \tag{5.1}$$

where $g\in C^\infty(\bar{\mathbb{R}}_+)$ verifies (4.1). It is easy to show that ψ_m belongs to $C_o^\infty(\Omega)$ for every $m\in\mathbb{N}$ and

$$0\leq\psi_m\leq 1,\quad \operatorname{supp}\psi_m\subseteq E_{2m},\quad \psi_{m|_{\bar{E}_m}}=1, \tag{5.2}$$

where

$$E_m=\left\{x\in\Omega:|x|<m,\ \tilde{\alpha}(x)>\frac{1}{m}\right\}. \tag{5.3}$$

Remark 5.1. It follows from hypothesis (h_1) and from [5, Lemma 4.2] that for any $m \in \mathbb{N}$ the functions $(\psi_m a_{ij})_o$ (obtained as extensions of $\psi_m a_{ij}$ to $\mathbb{R}^n$ with zero values out of Ω) belong to $\mathrm{VMO}(\mathbb{R}^n)$ and

$$[(\psi_m a_{ij})_o]_{\mathrm{BMO}(R^n,t)} \leq [\psi_m a_{ij}]_{\mathrm{BMO}(\Omega,t)}, \tag{5.4}$$

for t small enough.

In the following we denote by w, b_i, b, and g the functions defined by (4.14), (4.22), respectively, corresponding to $k = k_o$, where k_o is the positive integer of Lemma 4.2

We can now prove the main result of the paper.

Theorem 5.2. *Suppose that conditions (h_1) and (h_2) hold, and let u be a solution of the problem (P). Then there exist an open ball $B \subset\subset \Omega$ and a constant $c_o \in \mathbb{R}_+$ such that*

$$\sup_{\Omega} \sigma^s(x)u(x) \leq c_o \left(\fint_B |\sigma^{s+2} f^-|^p \, dx \right)^{1/p}, \tag{5.5}$$

where c_o depends only on n, p, s, γ, ν, ν_0, ν_1, ν_2, a_o, $\eta[\psi_m a_{ij}]$ $(m \in \mathbb{N})$.

Proof. It can be assumed that $\sup_\Omega \sigma^s(x)\, u(x) > 0$. Thus it follows from (4.14) and (4.18) that there exists $y \in \Omega$ such that $\sup_\Omega w(x) = w(y)$; moreover, there exists $R_o \in]0, \mathrm{dist}(y, \partial\Omega)[$ such that $w(x) > 0$ for all $x \in B(y, R_o)$.

Let $\lambda, \alpha, \alpha_o \in \mathbb{R}_+$, with $\alpha_o > 1$ (that will be chosen late), such that

$$\lambda\alpha \leq \min\{R_o, \sigma(y)\}, \qquad \alpha = \alpha_o \sigma(y). \tag{5.6}$$

In the following we denote by B the open ball $B(y, \alpha\lambda)$.

We put

$$\varphi(x) = 1 + \lambda^2 - \frac{|x-y|^2}{\alpha^2}, \quad x \in \bar{B}, \tag{5.7}$$

and observe that

$$1 \leq \varphi(x) \leq 1 + \lambda^2 \leq 2, \quad x \in \bar{B}, \tag{5.8}$$

$$\varphi_{x_i} \leq \frac{2\lambda}{\alpha}, \qquad \varphi_{x_i}\varphi_{x_j} \leq \frac{4\lambda^2}{\alpha^2}, \quad i, j = 1, \dots, n, \tag{5.9}$$

$$\varphi_{x_i x_j} = 0 \quad \text{if } i \neq j, \qquad \varphi_{x_i x_j} = -\frac{2}{\alpha^2} \quad \text{if } i = j. \tag{5.10}$$

Consider now the function v defined by

$$v(x) = \varphi(x)w(x) - w(y), \quad x \in \bar{B}. \tag{5.11}$$

Obviously,

$$v_{|\partial\Omega} = w_{|\partial\Omega} - w(y) \leq 0, \qquad v(y) = \lambda^2 w(y). \tag{5.12}$$

It is easy to show that

$$\begin{aligned} &L_o(\varphi w) - wL_o\varphi - 2\sum_{i,j=1}^{n} a_{ij}\varphi_{x_j}w_{x_i} + \sum_{i=1}^{n} b_i(\varphi w)_{x_i} \\ &\quad - \sum_{i=1}^{n} b_i\varphi_{x_i}w + b\varphi w = \varphi\left(L_o w + \sum_{i=1}^{n} b_i w_{x_i} + bw\right) \geq \varphi g \quad \text{in } B. \end{aligned} \tag{5.13}$$

Thus

$$L_o(\varphi w) + \sum_{i=1}^{n} d_i(\varphi w)_{x_i} + d\varphi w \geq \varphi g + \sum_{i=1}^{n} b_i\varphi_{x_i}w \quad \text{in } B, \tag{5.14}$$

where

$$d_i = b_i - 2\sum_{j=1}^{n} a_{ij}\frac{\varphi_{x_j}}{\varphi}, \quad i = 1,\dots,n, \tag{5.15}$$

$$d = b + 2\sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2} - \sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_i x_j}}{\varphi}. \tag{5.16}$$

Therefore we obtain from (5.14) that

$$L_o v + \sum_{i=1}^{n} d_i v_{x_i} + dv \geq h, \tag{5.17}$$

where

$$h = \varphi g + w\sum_{i=1}^{n} b_i\varphi_{x_i} - dw(y). \tag{5.18}$$

Clearly, (2.9), (5.6), and (5.9) yield that

$$\left|\varphi_{x_i}\right| \leq 2\gamma\frac{\sigma}{\alpha_o^2\sigma^2(y)} \quad \text{in } B, \tag{5.19}$$

and hence it follows from Lemma 4.2 that

$$\begin{aligned} h &\geq \varphi\eta_{k_o}^s f - \frac{a_o}{8}\sigma^{-2}\varphi w(y) - 2\gamma w(y)\left(\nu_1 + \frac{a_o}{2}\right)\frac{1}{\alpha_o^2}\sigma^{-2}(y) - dw(y) \\ &\geq \varphi\eta_{k_o}^s f + \left[-d - \left(\frac{a_o}{4}\gamma^2 + 2\frac{\gamma\nu_1}{\alpha_o^2} + \frac{\gamma a_o}{\alpha_o^2}\right)\sigma^{-2}(y)\right]w(y). \end{aligned} \tag{5.20}$$

The constant α_o can be chosen in such a way that $d < -d_o\sigma^{-2}(y)$ in B, where

$$d_o = \frac{a_o}{4}\gamma^2 + 2\frac{\gamma\nu_1}{\alpha_o^2} + \frac{\gamma a_o}{\alpha_o^2}. \tag{5.21}$$

In fact, by Lemma 4.2, (5.9) and (5.10), we have

$$\begin{aligned} d+d_o\sigma^{-2}(y) &= b+2\sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}-\sum_{i,j=1}^{n} a_{ij}\frac{\varphi_{x_ix_j}}{\varphi}+d_o\sigma^{-2}(y) \\ &\le -\frac{a_o}{2}\sigma^{-2}+8\nu_o\frac{\lambda^2}{\alpha^2}+2\nu_o\frac{1}{\alpha^2}+d_o\sigma^{-2}(y) \\ &\le \left[-\gamma^2\frac{a_o}{4}+(10\nu_o+2\gamma\nu_1+\gamma a_o)\frac{1}{\alpha_o^2}\right]\sigma^{-2}(y), \end{aligned} \tag{5.22}$$

and hence, fixed α_o such that

$$\frac{1}{\alpha_o^2}\le\frac{\gamma^2 a_o}{4(10\nu_o+2\gamma\nu_1+\gamma a_o)}, \tag{5.23}$$

it follows that

$$d<-d_o\sigma^{-2}(y)\quad \text{in } B. \tag{5.24}$$

By (5.11), (5.12), and (5.15)–(5.17), we deduce that the problem

$$\begin{gathered} v\in W^{2,p}(B), \\ L_ov+\sum_{i=1}^{n} d_iv_{x_i}+dv\ge\varphi\eta_{k_o}^s f,\quad f\in L^p(B), \\ v_{|\partial B}\le 0 \end{gathered} \tag{5.25}$$

satisfies the hypotheses of Lemma 3.1. Therefore, it follows from (5.6), (4.15), and (4.16) that there exists a constant $c_1\in\mathbb{R}_+$, depending on $n, p, s, \gamma, \nu, \nu_0, \nu_1, \nu_2, [p(a_{ij_{|B}})]_{\mathrm{BMO}(R^n,\cdot)}$, such that

$$v(x)\le c_1(\lambda\alpha)^{2-n/p}\left\|(\varphi\eta_{k_o}^s f)^-\right\|_{L^p(B)}\quad \forall x\in B. \tag{5.26}$$

So it follows from (5.8) and from (5.26) with $x=y$ that

$$\lambda^2 w(y)\le c_1(\lambda\alpha)^{2-n/p}\left\|(\varphi\eta_{k_o}^s f)^-\right\|_{L^p(B)}\le 2c_1(\lambda\alpha)^{2-n/p}\left\|\eta_{k_o}^s f^-\right\|_{L^p(B)}. \tag{5.27}$$

Thus by (5.6) and (5.27) we have

$$w(y)\le c_2(\lambda\alpha)^{-n/p}\alpha_o^2\sigma^2(y)\|\eta_{k_o}^s f^-\|_{L^p(B)}\le c_3(\lambda\alpha)^{-n/p}\alpha_o^2\|\sigma^2\eta_{k_o}^s f^-\|_{L^p(B)}, \tag{5.28}$$

where $c_2, c_3\in\mathbb{R}_+$ depend on the same parameters as c_1. Finally from (4.6), (4.7), (4.14), and (5.28) we obtain

$$\sup_{\Omega}\sigma^s u\le c_4(\lambda\alpha)^{-n/p}\left(\int_B|\sigma^{2+s}f^-|^p dx\right)^{1/p}\le c_5\left(⨍_B|\sigma^{s+2}f^-|\,dx\right)^{1/p}, \tag{5.29}$$

where $c_4, c_5 \in \mathbb{R}_+$ depend on the same parameters as c_1 and on a_o. Then, if we choose

$$p\left(a_{ij_{|B}}\right) = \left(\psi_{m_1} a_{ij}\right)_o, \tag{5.30}$$

where m_1 is a positive integer such that $\psi_{m_{1|B}} = 1$, (5.5) follows from (5.29), (5.30), and from Remark 5.1. □

Corollary 5.3. *Suppose that conditions (h_1) and (h_2) hold, and let u be a solution of the problem*

$$\begin{gathered} Lu = f, \quad \sigma^{s+2} f \in L^\infty(\Omega), \quad u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\ \limsup_{x \to x_o} \sigma^s(x) u(x) = 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \to +\infty} \sigma^s(x) u(x) = 0 \quad \text{if } \Omega \text{ is unbounded.} \end{gathered} \tag{p'}$$

Then

$$\sup_\Omega \sigma^s |u| \le c_o \|\sigma^{s+2} f\|_{L^\infty(\Omega)}, \tag{5.31}$$

where $c_o \in \mathbb{R}_+$ is the constant of the statement of Theorem 5.2.

Proof. The result can be obtained applying Theorem 5.2 to the functions u and $-u$. □

The following uniqueness result is an obvious consequence of Corollary 5.3.

Corollary 5.4. *If the hypotheses (h_1) and (h_2) hold, then the problem*

$$\begin{gathered} Lu = 0, \quad u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\ \limsup_{x \to x_o} \sigma^s(x) u(x) = 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \to +\infty} \sigma^s(x) u(x) = 0 \quad \text{if } \Omega \text{ is unbounded} \end{gathered} \tag{p''}$$

has only the zero solution.

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