CONTINUOUSLY DIFFERENTIABLE MEANS

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Received 3 March 2006; Revised 7 September 2006; Accepted 12 September 2006

We consider continuously differentiable means, say C^1-means. As for quasi-arithmetic means $Q_f(x_1,\dots,x_n)$, we need an assumption that f has no stationary points so that Q_f might be continuously differentiable. Introducing quasi-weights for C^1-means would give a satisfactory explanation for the necessity of this assumption. As a typical example of a class of C^1-means, we observe that a skew power mean M_t is a composition of power means if t is an integer.

1. Introduction

Let $M(x_1,\dots,x_n)$ be a continuously differentiable n-variable positive function on $(0,\infty)^n$. Then, throughout this paper, M is called a *continuously differentiable mean*, or shortly C^1-*mean* if M satisfies

(i) M is monotone increasing in each term;
(ii) $M(a,\dots,a) = a$ for all positive numbers a.

A mean M is called *homogeneous* if M satisfies

$$M(ax_1,\dots,ax_n) = aM(x_1,\dots,x_n) \tag{1.1}$$

for all $a,x_k > 0$. Almost all classical means are homogeneous C^1-ones. The Kubo-Ando (operator) means in [6] and chaotic ones in [2] are C^1-means. Here note that (numerical) Kubo-Ando means $K_f(a,b)$ are defined by

$$K_f(a,b) = af\left(\frac{b}{a}\right) \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 75941, Pages 1–15
DOI 10.1155/JIA/2006/75941

for positive operator monotone functions f, which form a special class of numerical means.

Let f be a continuously differentiable monotone function on $(0,\infty)$ with no stationary points, that is, $f'(x) \neq 0$ for all $x > 0$. In this case, f^{-1} is also continuously differentiable. Let $w = \{w_k\}$ be a weight, that is, a set of nonnegative numbers w_k with $\sum_k w_k = 1$. For such f and a weight w, it follows that a *quasi-arithmetic mean* $Q_{f,w}$ defined by

$$Q_{f,w}(x_1,\dots,x_n) = f^{-1}\left(\sum_{k=1}^{n} w_k f(x_k)\right) \tag{1.3}$$

is a typical C^1-mean. As we will see later in the next section, the assumption that f has no stationary points is necessary for continuous differentiability. Our main interest in this paper is when integral functions

$$\mathcal{M}_{f,P}(x_1,\dots,x_n) = f^{-1}\left(\int_0^{\infty} f(x)\,dP_{x_1,\dots,x_n}(x)\right) \tag{1.4}$$

are C^1-means, where $P_{x_1,\dots,x_n}$ is a probability measure on $(0,\infty)$ for each x_k. Note that these functions differ from the continuous quasi-arithmetic means, cf. [4, 5], but they include the above discrete quasi-arithmetic ones $Q_{f,w}$. In fact, for a convex combination for Dirac measures $P_{x_1,\dots,x_n} = \sum_{k=1}^{n} w_k \delta_{x_k}$, we have $M_{f,P} = Q_{f,w}$.

In this paper, we discuss continuous differentiability of such integral functions as means, and observe when $\mathcal{M}_{f,P}$ is a C^1-mean, particularly as 2-variable functions. Many mathematicians have been interested in means of positive numbers. But, even in a quasi-arithmetic mean, odd properties appear as we will see in some examples later. We noticed that the key in this problem is continuous differentiability for means. So we discuss continuously differentiable means and give some classes of such means. Finally, we discuss skew power means including logarithmic one as a path of C^1-means.

2. Quasi-weight for means

Power means defined by (see [3])

$$P_{r,w}(x_1,\dots,x_n) = \left(\sum_{k=1}^{n} w_k x_k^r\right)^{1/r} \tag{2.1}$$

are quasi-arithmetic C^1-means. Note that only homogeneous quasi-arithmetic means are the power means, which is shown in [4]. Then, we have

$$\frac{\partial P_{r,w}}{\partial x_k}(a,\dots,a) = \frac{1}{r}\left(\sum_{k=1}^{n} w_k a^r\right)^{1/r-1} \cdot r w_k a^{r-1} = w_k\left(a^r\right)^{1/r-1} a^{r-1} = w_k. \tag{2.2}$$

Moreover, we have the following property.

Lemma 2.1. *All C^1-means M satisty $(\partial M/\partial x_k)(a,\dots,a) \geqq 0$ and $\sum_{k=1}^{n}(\partial M/\partial x_k)(a,\dots,a) = 1$ for all $a > 0$.*

Proof. It follows from (ii) that

$$1=\frac{a+\varepsilon-a}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{M(a+\varepsilon,\dots,a+\varepsilon)-M(a,\dots,a)}{\varepsilon}=\sum_{k=1}^{n}\frac{\partial M}{\partial x_k}(a,\dots,a). \tag{2.3}$$

The assumption (i) implies $(\partial M/\partial x_k)(a,\dots,a)\geqq 0$. □

Thereby, we define a *kth quasi-weight* $w(M)_k(a)$ for a C^1-mean M at a by (cf. [1])

$$w(M)_k(a)=\frac{\partial M}{\partial x_k}(a,\dots,a). \tag{2.4}$$

Note that it is a constant for all a if M is homogeneous-like power means. In fact,

$$\begin{aligned} w(M)_1(a)&=\lim_{\varepsilon\to 0}\frac{M(a+\varepsilon,a,\dots,a)-M(a,\dots,a)}{\varepsilon}\\ &=\lim_{\varepsilon\to 0}\frac{M(1+\varepsilon/a,1,\dots,1)-M(1,\dots,1)}{\varepsilon/a}=w(M)_1(1). \end{aligned} \tag{2.5}$$

Moreover, even for a nonhomogeneous case, it can be constant and coincides with the weight.

Theorem 2.2. *If f has no stationary point, then the kth quasi-weight $w(Q_{f,w})_k(a)$ of a quasi-arithmetic mean $Q_{f,w}$ is the kth weight w_k.*

Proof. Note that $(\partial f\circ Q_{f,w}/\partial x_k)(a,\dots,a)=w_kf'(a)$. On the other hand, we have

$$\frac{\partial f\circ Q_{f,w}}{\partial x_k}(a,\dots,a)=f'\left(Q_{f,w}(a,\dots,a)\right)\frac{\partial Q_{f,w}}{\partial x_k}(a,\dots,a)=f'(a)w\left(Q_{f,w}\right)_k(a), \tag{2.6}$$

and hence $w(Q_{f,w})_k(a)=w_k$ by $f'(a)\neq 0$. □

When f has a stationary point, the following example shows that $Q_{f,w}$ is not always a C^1-mean.

Example 2.3. Let $Q_f=Q_{f,\{1/2,1/2\}}$. For a fixed $a>0$, put $f(x)=(x-a)^3+a^3$. Then we have $f^{-1}(x)=(x-a^3)^{1/3}+a$,

$$\begin{aligned} Q_f(x,y)&=\left(\frac{(x-a)^3+(y-a)^3}{2}\right)^{1/3}+a,\\ \frac{\partial Q_f}{\partial x}(x,y)&=\frac{(x-a)^2}{2^{1/3}\left((x-a)^3+(y-a)^3\right)^{2/3}}. \end{aligned} \tag{2.7}$$

Thus it is not continuously differentiable at (a,a). In fact, we cannot define the quasi-weights

$$\lim_{\varepsilon\to 0}\frac{\partial Q_f}{\partial x}(a+\varepsilon,a)=\lim_{\varepsilon\to 0}2^{-1/3}=2^{-1/3}, \tag{2.8}$$

while

$$\lim_{\varepsilon\to 0}\frac{\partial Q_f}{\partial x}(a+\varepsilon,a+\varepsilon)=\frac{1}{2^{1/3}\times 2^{2/3}}=\frac{1}{2}. \tag{2.9}$$

Therefore, Q_f is not a C^1-mean.

If a C^1-mean M satisfies

$$M(x_{\pi_1},\dots,x_{\pi_n})=M(x_1,\dots,x_n) \tag{2.10}$$

for all permutation π, then it is called *symmetric*. It is clear that all quasi-weights for all symmetric C^1-means are the same value $1/n$. But the converse is false by the following example.

Example 2.4. The first quasi-weight of the following arithmetic (resp., geometric) mean

$$A(a,b)=w_1a+w_2b \qquad (\text{resp., } G(a,b)=a^{w_2}b^{w_1}) \tag{2.11}$$

coincides with the first weight w_1 (resp., w_2). Putting $M(a,b)=(A(a,b)+G(a,b))/2$, we have

$$w(M)_1(a)=w(M)_2(a)=\frac{w_1+w_2}{2}=\frac{1}{2}, \tag{2.12}$$

while $M(a,b)$ is not symmetric if $w_1\neq w_2$.

3. Continuous differentiability

Since functions $\mathcal{M}_{f,P}$ include $Q_{f,w}$ as a case of singular measures, we should also assume that f has no stationary points to discuss $\mathcal{M}_{f,P}$. So we consider a case of absolutely continuous measures with the Radon-Nikodym derivative $\phi_{x_1,\dots,x_n}$:

$$\mathcal{M}_{f,\phi}(x_1,\dots,x_n)=f^{-1}\left(\int_0^\infty f(x)\phi_{x_1,\dots,x_n}(x)\,dx\right). \tag{3.1}$$

The following example shows that we need the condition $\phi(x)>0$ for all $x>0$ in order that $\mathcal{M}_{f,\phi}$ be a C^1-mean.

Example 3.1. Consider the derivative

$$\phi_{a,b}(x)=\frac{3(x-1)^2\chi_{[a,b]}(x)}{(b-1)^3-(a-1)^3} \tag{3.2}$$

(for convenience's sake, $\chi_{[a,b]}=-\chi_{[b,a]}$ if $a>b$ and $\phi_{a,a}(x)=a$). Then we have $\phi_{a,b}(1)=0$. Since

$$\mathcal{M}_{x,\phi}(a,b)=\frac{3}{(b-1)^3-(a-1)^3}\int_a^b x(x-1)^2\,dx=\frac{(1+3b/4)(b-1)^3-(1+3a/4)(a-1)^3}{(b-1)^3-(a-1)^3}, \tag{3.3}$$

it follows that $\mathcal{M}_{x,\phi}(a,b)$ is symmetric. So quasi-weights are 1/2 if it is a C^1-mean, while

$$\lim_{\varepsilon\to 0}\frac{\mathcal{M}_{x,\phi}(1+\varepsilon,1)-\mathcal{M}_{x,\phi}(1,1)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{1+(3/4)\varepsilon-1}{\varepsilon}=\frac{3}{4}. \tag{3.4}$$

Now let p be a polynomial with a degree $m>0$ which is monotone and convex on $(0,\infty)$. Then $p'(x)>0$ for all $x>0$. For a continuously differentiable monotone function f with no stationary points, we define

$$M_{f,p}(a,b)=f^{-1}\left(\int_a^b\frac{f(x)p'(x)}{p(b)-p(a)}dx\right) \tag{3.5}$$

for $a\neq b$ and $M_{f,p}(a,a)=a$. We will prove that $M_{f,p}$ defined above is a C^1-mean. To do this, we first show the continuity of $M_{f,p}$.

Lemma 3.2. *$M_{f,p}$ is continuous.*

Proof. For the interval $I_{a,b}$ between a and b, there exists $\xi_{a,b}\in I_{a,b}$ with

$$\int_a^b f(t)p'(t)dt=f(\xi_{a,b})p'(\xi_{a,b})(b-a) \tag{3.6}$$

by the mean value theorem. Moreover, Cauchy's mean-value theorem says that there exists $c_{a,b}$ with

$$\frac{1}{p'(c_{a,b})}=\frac{b-a}{p(b)-p(a)} \tag{3.7}$$

for $a\neq b$. Put $B_\varepsilon(c)=(c-\varepsilon,c+\varepsilon)$ for a fixed $c>0$. Then, for $a\neq b\in B_\varepsilon(c)$,

$$M_{f,p}(a,b)=f^{-1}\left(\int_a^b\frac{f(x)p'(x)}{p(b)-p(a)}dx\right)=f^{-1}\left(\frac{f(\xi_{a,b})p'(\xi_{a,b})}{p'(c_{a,b})}\right), \tag{3.8}$$

so that, as $\varepsilon\to 0$, we have $\xi_{a,b},c_{a,b}\to c$ and $M_{f,p}(a,b)$ converges to

$$f^{-1}\left(\frac{f(c)p'(c)}{p'(c)}\right)=c, \tag{3.9}$$

which implies $M_{f,p}$ is continuous. □

To verify that $M_{f,p}$ is a C^1-mean, we first show that it satisfies (i).

Lemma 3.3. *$M_{f,p}(a,b)$ is invariant for every affine transform of $f\mapsto tf+s$ $(t\neq 0)$ and $M_{f,p}(a,b)$ is monotone increasing in each term.*

Proof. Since $(tf+s)^{-1}(x)=f^{-1}((1/t)(x-s))$, we have

$$\begin{aligned}M_{tf+s,p}(a,b)&=(tf+s)^{-1}\left(\int_a^b\frac{(tf(x)+s)p'(x)}{p(b)-p(a)}dx\right)\\&=(tf+s)^{-1}\left(t\int_a^b\frac{f(x)p'(x)}{p(b)-p(a)}dx+s\int_a^b\frac{p'(x)}{p(b)-p(a)}dx\right)\\&=(tf+s)^{-1}\left(t\int_a^b\frac{f(x)p'(x)}{p(b)-p(a)}dx+s\right)\\&=f^{-1}\left(\int_a^b\frac{f(x)p'(x)}{p(b)-p(a)}dx\right)=M_{f,p}(a,b).\end{aligned}\tag{3.10}$$

Thus we may assume that f (and hence f^{-1}) is monotone increasing. Since

$$\begin{aligned}\frac{\partial}{\partial y}\int_a^y\frac{f(x)p'(x)}{p(y)-p(a)}dx&=\frac{\left(f(y)(p(y)-p(a))-\int_a^y f(x)p'(x)dx\right)p'(y)}{(p(y)-p(a))^2}\\&\geqq\frac{\left(f(y)(p(y)-p(a))-f(y)\int_a^y p'(x)dx\right)p'(y)}{(p(y)-p(a))^2}=0,\end{aligned}\tag{3.11}$$

we have $\int_a^y(f(x)p'(x)/(p(y)-p(a)))dx$ is monotone increasing for y. Therefore, $M_{f,p}(a,b)$ satisfies (i). □

Next, to show the differentiability, we cite the following fundamental lemma (for the sake of completeness, here we give a proof).

Lemma 3.4. *Let g be a C^2-function on $(0,\infty)$ and*

$$G(x,y)=\begin{cases}\dfrac{g(y)-g(x)}{y-x} & \text{if } x\neq y,\\ g'(x) & \text{if } x=y.\end{cases}\tag{3.12}$$

Then G is a C^1-function on $(0,\infty)^2$.

Proof. Since G is symmetric, it suffices to show that G_x is continuous at the diagonal set $\{(c,c)\mid c>0\}$. By the l'Hospital theorem,

$$\lim_{h\to0}\frac{G(c+h,c)-G(c,c)}{h}=\lim_{h\to0}\frac{g(c+h)-g(c)-hg'(c)}{h^2}=\lim_{h\to0}\frac{g'(c+h)-g'(c)}{2h}=\frac{g''(c)}{2},\tag{3.13}$$

that is, $G_x(c,c) = g''(c)/2$. Since g'' is continuous,

$$\lim_{h\to 0} G_x(c+h,c+h) = \lim_{h\to 0} \frac{g''(c+h)}{2} = \frac{g''(c)}{2}. \tag{3.14}$$

On the other hand, for $x \neq y$, we have

$$G_x(x,y) = \frac{\partial}{\partial x}\left(\frac{g(y)-g(x)}{y-x}\right) = \frac{g(y)-g(x)-g'(x)(y-x)}{(y-x)^2}. \tag{3.15}$$

Now what we must show is $\lim_{h\neq k\to 0} G_x(c+h,c+k) = G_x(c,c)$. Let $h \neq k \in B_\varepsilon(0)$. By Taylor's expansion theorem, there exists $\xi_{h,k} \in I_{c+h,c+k} \subset B_\varepsilon(c)$ with

$$g(c+h) = g(c+k) + g'(c+k)(h-k) + \frac{1}{2}g''(\xi_{h,k})(h-k)^2. \tag{3.16}$$

The mean-value theorem says that there exists $c_{h,k} \in I_{c+h,c+k} \subset B_\varepsilon(c)$ with

$$g'(c+k) - g'(c+h) = g''(c_{h,k}v)(k-h). \tag{3.17}$$

Thereby, as $\varepsilon \to 0$, we have $\xi_{h,k}, c_{h,k} \to c$ and

$$\begin{aligned} G_x(c+h,c+k) &= \frac{g(c+k)-g(c+h)-g'(c+h)(k-h)}{(k-h)^2} \\ &= \frac{g'(c+k)(k-h)-g''(\xi_{h,k})(k-h)^2/2-g'(c+h)(k-h)}{(k-h)^2} \\ &= \frac{g'(c+k)-g'(c+h)}{k-h} - \frac{g''(\xi_{h,k})}{2} \\ &= g''(c_{h,k}) - \frac{g''(\xi_{h,k})}{2} \longrightarrow g''(c) - \frac{g''(c)}{2} = \frac{g''(c)}{2}, \end{aligned} \tag{3.18}$$

which implies G is continuously differentiable. □

Now we have the following theorem.

THEOREM 3.5. *The above function $M_{f,p}(a,b)$ defines a symmetric C^1-mean.*

Proof. Let F_k be primitive functions defined inductively

$$F'_{k+1} = F_k, \qquad F_0 = f. \tag{3.19}$$

Since $p^{(m)}(x)$ is a constant function, we have

$$\begin{aligned}\int_a^b f(x)p'(x)dx &= [F_1(x)p'(x)]_a^b - \int_a^b F_1(x)p''(x)dx \\ &= F_1(b)p'(b) - F_1(a)p'(a) - [F_2(x)p''(x)]_a^b + \int_a^b F_2(x)p^{(3)}(x)dx \\ &= \cdots \\ &= \sum_{k=1}^{m-1}(-1)^{k+1}\big(F_k(b)p^{(k)}(b) - F_k(a)p^{(k)}(a)\big) + (-1)^{m+1}\int_a^b F_{m-1}(x)p^{(m)}(x)dx \\ &= \sum_{k=1}^{m}(-1)^{k+1}\big(F_k(b)p^{(k)}(b) - F_k(a)p^{(k)}(a)\big).\end{aligned} \tag{3.20}$$

It follows that

$$M_{f,p}(a,b) = f^{-1}\left(\frac{\sum_{k=1}^{m}(-1)^{k+1}\big(F_k(b)p^{(k)}(b) - F_k(a)p^{(k)}(a)\big)}{p(b)-p(a)}\right). \tag{3.21}$$

Here we put

$$H_k(a,b) \equiv \frac{F_k(b)p^{(k)}(b) - F_k(a)p^{(k)}(a)}{p(b)-p(a)}. \tag{3.22}$$

Note that a polynomial $P(a,b) = (p(b)-p(a))/(b-a)$ no longer have a divisor $b-a$ by $p'(x) > 0$, and hence $P(a,b) \neq 0$ for all $a,b > 0$. By Lemma 3.3, P is continuously differentiable (by putting $P(a,a) \equiv p'(a)$). Setting such functions

$$P[k](a,b) = \frac{p^{(k)}(b) - p^{(k)}(a)}{b-a}, \qquad G[k](a,b) = \frac{F_k(b) - F_k(a)}{b-a} \tag{3.23}$$

(where $P[k](a,a) \equiv p^{(k+1)}(a)$, $G[k](a,a) \equiv F_{k-1}(a)$), we have

$$H_k(a,b) = \frac{F_k(b)P[k](a,b) + G[k](a,b)p^{(k)}(a)}{P(a,b)}, \tag{3.24}$$

and hence it is continuously differentiable by Lemma 3.3. Therefore, since f^{-1} is also continuously differentiable, $M_{f,p}(a,b)$ is a symmetric C^1-mean by Lemma 3.2. □

4. Skew power means

For positive numbers a and b, consider the following special case for $\mathcal{M}_{f,\phi}$ (see [2]):

$$M_t(a,b) = \left(\frac{1}{b-a}\int_a^b x^t\,dx\right)^{1/t} = \left(\frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right)^{1/t}. \tag{4.1}$$

Table 4.1

t	$-\infty$	-2	-1	0	1	2	∞
	Minimum	Geometric	Logarithmic	Identric	Arithmetic	—	Maximum
	$\min\{a,b\}$	$\sqrt{ab}$	$\dfrac{b-a}{\log b-\log a}$	$\dfrac{a^{a/(a-b)}b^{b/(b-a)}}{e}$	$\dfrac{a+b}{2}$	$\left(\dfrac{a^2+ab+b^2}{3}\right)^{1/2}$	$\max\{a,b\}$

By definition, we call them *skew power means* which include various classical means, for example, the logarithmic and identric ones as shown in Table 4.1.

Here we use 2-variable power means

$$P_{r,w}(a,b)=\left((1-w)a^r+wb^r\right)^{1/r}, \tag{4.2}$$

and, in particular,

$$P_{0,w}(a,b)\equiv\lim_{r\to 0}P_{r,w}(a,b)=a^{1-w}b^w \tag{4.3}$$

is a weighted geometric mean. For the convenience's sake, we omit w for $w=1/2$ which is a symmetric mean.

Let α_n be the smallest number N such that M_n is a composition of N power means ($\alpha_n=\infty$ if not). Putting

$$L_{n,m}(a,b)=\left(M_n(a^m,b^m)\right)^{1/m}, \tag{4.4}$$

we define $\beta_{n,m}$ as the smallest number N such that $L_{n,m}$ is a composition of N power means ($\beta_{n,m}=\infty$ if not). Now we have the following theorem.

THEOREM 4.1. *The mean function $M_n(a,b)$ is represented by a composition for finitely many power means as follows.*

(i) *For all positive integers n, it is a composition of at most $2n-1$ power means.*

(ii) *For all integers $n\leqq -2$, it is a composition of at most $-2n-3$ power means.*

Proof

Case 1. Let n be a fixed positive integer and k a fixed nonzero integer. By $M_1=P_1$, we have

$$\alpha_1=1, \tag{4.5}$$

and also

$$\alpha_2\leqq 3 \tag{4.6}$$

by

$$M_2(a,b)=\left(\frac{a^2+ab+b^2}{3}\right)^{1/2}=\left(\frac{2}{3}\frac{a^2+b^2}{2}+\frac{1}{3}ab\right)^{1/2}=P_{2,1/3}(P_2(a,b),P_0(a,b)). \tag{4.7}$$

Since

$$\begin{aligned}M_{2n+1}(a,b)&=\left(\frac{a^{2n+1}+a^{2n}b+\cdots+b^{2n+1}}{2n+2}\right)^{1/(2n+1)}=\left(\frac{a^n+\cdots+b^n}{n+1}\frac{a^{n+1}+b^{n+1}}{2}\right)^{1/(2n+1)}\\&=M_n(a,b)^{n/(2n+1)}P_{n+1}(a,b)^{(n+1)/(2n+1)}=P_{0,(n+1)/(2n+1)}(M_n(a,b),P_{n+1}(a,b)),\end{aligned} \tag{4.8}$$

we have

$$\alpha_{2n+1}\leqq\alpha_n+2. \tag{4.9}$$

Moreover,

$$\alpha_{2n}\leqq\beta_{n,2}+\beta_{n-1,2}+3, \tag{4.10}$$

since

$$\begin{aligned}M_{2n}(a,b)&=\left(\frac{a^{2n}+a^{2n-1}b+\cdots+b^{2n}}{2n+1}\right)^{1/2n}\\&=\left(\frac{a^{2n}+a^{2(n-1)}b^2+\cdots+b^{2n}+(a^{2n-2}+a^{2n-4}b^2+\cdots+b^{2n-2})ab}{2n+1}\right)^{1/2n}\\&=\left(\frac{(n+1)M_n(a^2,b^2)^n+nM_{n-1}(a^2,b^2)^{n-1}ab}{2n+1}\right)^{1/2n}\\&=P_{2n,n/(2n+1)}\left(\sqrt{M_n(a^2,b^2)},P_{0,1/n}\left(\sqrt{M_{n-1}(a^2,b^2)},\sqrt{ab}\right)\right)\\&=P_{2n,n/(2n+1)}\left(L_{n,2}(a,b),P_{0,1/n}\left(\sqrt{M_{n-1}(a^2,b^2)},\sqrt{ab}\right)\right)\\&=P_{2n,n/(2n+1)}(L_{n,2}(a,b),P_{0,1/n}(L_{n-1,2}(a,b),P_0(a,b))).\end{aligned} \tag{4.11}$$

It follows from $L_{1,2k}(a,b)=M_1(a^{2k},b^{2k})^{1/(2k)}=P_{2k}(a,b)$ that

$$\beta_{1,2k}=1. \tag{4.12}$$

Moreover we have

$$\beta_{2,2k}\leqq 3 \tag{4.13}$$

by

$$L_{2,2k}(a,b) = M_2\left(a^{2k},b^{2k}\right)^{1/2k} = \left(\frac{a^{4k}+(ab)^{2k}+b^{4k}}{3}\right)^{1/4k} = P_{4k,1/3}\left(P_{4k}(a,b),P_0(a,b)\right). \tag{4.14}$$

Since

$$\begin{aligned} L_{2n+1,2k}(a,b) &= M_{2n+1}\left(a^{2k},b^{2k}\right)^{1/2k} \\ &= P_{0,(n+1)/(2n+1)}\left(M_n\left(a^{2k},b^{2k}\right),P_{n+1}\left(a^{2k},b^{2k}\right)\right)^{1/2k} \quad \text{by (4.8)} \\ &= P_{0,(n+1)/(2n+1)}\left(M_n\left(a^{2k},b^{2k}\right)^{1/2k},P_{n+1}\left(a^{2k},b^{2k}\right)^{1/2k}\right) \\ &= P_{0,(n+1)/(2n+1)}\left(L_{n,2k}(a,b),P_{2k(n+1)}(a,b)\right), \end{aligned} \tag{4.15}$$

we have

$$\beta_{2n+1,2k} \leqq \beta_{n,2k}+2. \tag{4.16}$$

Also we have

$$\beta_{2n,2k} \leqq \beta_{n,4k}+\beta_{n-1,4k}+3 \tag{4.17}$$

by the following relation:

$$\begin{aligned} L_{2n,2k}(a,b) &= M_{2n}\left(a^{2k},b^{2k}\right)^{1/2k} \\ &= P_{2n,n/(2n+1)}\left(\sqrt{M_n\left(a^{4k},b^{4k}\right)},P_{0,1/n}\left(\sqrt{M_{n-1}\left(a^{4k},b^{4k}\right)},\sqrt{a^{2k}b^{2k}}\right)\right)^{1/2k} \quad (\text{by (4.11)}) \\ &= \left(\frac{n+1}{2n+1}\sqrt{M_n\left(a^{4k},b^{4k}\right)}^{2n}+\frac{n}{2n+1}P_{0,1/n}\left(\sqrt{M_{n-1}\left(a^{4k},b^{4k}\right)},a^kb^k\right)^{2n}\right)^{1/4kn} \\ &= \left(\frac{n+1}{2n+1}M_n\left(a^{4k},b^{4k}\right)^{4kn/4k}+\frac{n}{2n+1}P_{0,1/n}\left(\sqrt{M_{n-1}\left(a^{4k},b^{4k}\right)},a^kb^k\right)^{4kn/2k}\right)^{1/4kn} \\ &= P_{4kn,n/(2n+1)}\left(L_{n,4k}(a,b),P_{0,1/n}\left(\sqrt{M_{n-1}\left(a^{4k},b^{4k}\right)},a^kb^k\right)^{1/2k}\right) \\ &= P_{4kn,n/(2n+1)}\left(L_{n,4k}(a,b),P_{0,1/n}\left(M_{n-1}\left(a^{4k},b^{4k}\right)^{1/4k},\sqrt{ab}\right)\right) \\ &= P_{4kn,n/(2n+1)}\left(L_{n,4k}(a,b),P_{0,1/n}\left(L_{n-1,4k}(a,b),P_0(a,b)\right)\right). \end{aligned} \tag{4.18}$$

Now, we show that

$$A(n): \beta_{n,2k} \leqq 2n-1 \quad (\forall k \in \mathbb{Z} \setminus \{0\}) \tag{4.19}$$

holds for all positive integers n. In fact, both $A(1)$ and $A(2)$ are true by (4.12) and (4.13). Assume that $A(n)$ holds for all $1 \leqq n \leqq N$ ($N \geqq 2$). If $N = 2\ell$, then

$$\beta_{N+1,2k} = \beta_{2\ell+1,2k} \leqq \beta_{\ell,2k} + 2 \leqq 2\ell - 1 + 2 = N + 1 \leqq 2(N+1) - 1 \tag{4.20}$$

holds by (4.16) and the assumption, so that $A(N+1)$ holds. Also if $N = 2\ell + 1$, then

$$\beta_{N+1,2k} = \beta_{2(\ell+1),2k} \leqq \beta_{\ell+1,4k} + \beta_{\ell,4k} + 3 \leqq 2(\ell+1) - 1 + 2\ell - 1 + 3 = 4\ell + 3 = 2(N+1) - 1 \tag{4.21}$$

holds by (4.17) and the assumption, so that $A(N+1)$ holds. Therefore, $A(n)$ holds for all positive integers n by induction. We next show that the required inequality

$$B(n) : \alpha_n \leqq 2n - 1 \tag{4.22}$$

holds for all positive integers n. In fact, both $B(1)$ and $B(2)$ are true by (4.5) and (4.6). Assume that $B(n)$ holds for all $1 \leqq n \leqq N$ ($N \geqq 2$). If $N = 2\ell$, then

$$\alpha_{N+1} = \alpha_{2\ell+1} \leqq \alpha_\ell + 2 \leqq 2\ell - 1 + 2 = N + 1 \leqq 2(N+1) - 1 \tag{4.23}$$

holds by (4.9) and the assumption, and hence $B(N+1)$ holds. Also if $N = 2\ell + 1$, then

$$\alpha_{N+1} = \alpha_{2(\ell+1)} \leqq \beta_{\ell+1,2} + \beta_{\ell,2} + 3 \leqq 2(\ell+1) - 1 + 2\ell - 1 + 3 = 4\ell + 3 = 2(N+1) - 1 \tag{4.24}$$

holds by (4.10), $A(n)$, and the assumption, so that $B(N+1)$ holds. Therefore, $B(n)$ holds for all positive integers n by induction. Thus we have Case 1.

Case 2. Let n be a fixed positive integer. By $M_{-2} = P_0$, we have

$$\alpha_{-2} = 1. \tag{4.25}$$

It follows from $L_{1,-1} = P_1$ that

$$\beta_{1,-1} = 1. \tag{4.26}$$

Since

$$L_{2,-1}(a,b) = M_2\left(\frac{1}{a}, \frac{1}{b}\right)^{-1} = \left(\frac{a^{-2} + (ab)^{-1} + b^{-2}}{3}\right) = P_{-2,1/3}\left(P_{-2}(a,b), P_0(a,b)\right), \tag{4.27}$$

we have

$$\beta_{2,-1} \leqq 3. \tag{4.28}$$

Moreover, we have

$$\beta_{2n+1,-1} \leqq \beta_{n,-1} + 2 \tag{4.29}$$

by the following equation:

$$\begin{aligned} L_{2n+1,-1}(a,b) &= M_{2n+1}\left(\frac{1}{a},\frac{1}{b}\right)^{-1} \\ &= \left(M_n\left(\frac{1}{a},\frac{1}{b}\right)^{-1}\right)^{n/(2n+1)} P_{-(n+1)}(a,b)^{(n+1)/(2n+1)} \quad \text{by (4.8)} \\ &= P_{0,(n+1)/(2n+1)}\left(L_{n,-1}(a,b),P_{-(n+1)}(a,b)\right). \end{aligned} \tag{4.30}$$

Since

$$\begin{aligned} L_{2n,-1}(a,b) &= M_{2n}\left(\frac{1}{a},\frac{1}{b}\right)^{-1} \\ &= P_{2n,n/(2n+1)}\left(L_{n,2}\left(\frac{1}{a},\frac{1}{b}\right),P_{0,1/n}\left(L_{n-1,2}\left(\frac{1}{a},\frac{1}{b}\right),P_0\left(\frac{1}{a},\frac{1}{b}\right)\right)\right)^{-1} \\ &= \left(\frac{n+1}{2n+1}L_{n,2}\left(\frac{1}{a},\frac{1}{b}\right)^{2n}+\frac{n}{2n+1}P_{0,1/n}\left(L_{n-1,2}\left(\frac{1}{a},\frac{1}{b}\right),P_0\left(\frac{1}{a},\frac{1}{b}\right)\right)^{2n}\right)^{-1/2n} \\ &= \left(\frac{n+1}{2n+1}L_{n,-2}(a,b)^{-2n}+\frac{n}{2n+1}\left(\left(M_{n-1}\left(a^{-2},b^{-2}\right)\right)^{(n-1)/2n}(ab)^{-1/2n}\right)^{2n}\right)^{-1/2n} \\ &= \left(\frac{n+1}{2n+1}L_{n,-2}(a,b)^{-2n}+\frac{n}{2n+1}\left(\left(M_{n-1}\left(a^{-2},b^{-2}\right)\right)^{-(n-1)/2n}P_0(a,b)^{1/n}\right)^{-2n}\right)^{-1/2n} \\ &= \left(\frac{n+1}{2n+1}L_{n,-2}(a,b)^{-2n}+\frac{n}{2n+1}\left(\left(L_{n-1,-2}(a,b)\right)^{(n-1)/n}P_0(a,b)^{1/n}\right)^{-2n}\right)^{-1/2n} \\ &= P_{-2n,n/(2n+1)}\left(L_{n,-2}(a,b),P_{0,1/n}\left(L_{n-1,-2}(a,b),P_0(a,b)\right)\right), \end{aligned} \tag{4.31}$$

it follows from $A(n)$ that

$$\beta_{2n,-1} \leqq \beta_{n,-2}+\beta_{n-1,-2}+3 \leqq 2n-1+2(n-1)-1+3 = 4n-1. \tag{4.32}$$

Then we have

$$\alpha_{-(n+2)} \leqq \beta_{n,-1}+2, \tag{4.33}$$

since

$$\begin{aligned} M_{-(n+2)}(a,b) &= \left(\frac{b^{-(n+1)}-a^{-(n+1)}}{-(n+1)(b-a)}\right)^{-1/(n+2)} = \left(\frac{b^{-(n+1)}-a^{-(n+1)}}{(n+1)(1/b-1/a)ab}\right)^{-1/(n+2)} \\ &= \left(\frac{(1/a)^n+(1/a)^{n-1}(1/b)+\cdots+(1/b)^n}{n+1}\right)^{-1/(n+2)}(ab)^{1/(n+2)} \\ &= \left(M_n\left(\frac{1}{a},\frac{1}{b}\right)^{-1}\right)^{n/(n+2)}\sqrt{ab}^{2/(n+2)} = P_{0,2/(n+2)}\left(L_{n,-1}(a,b),P_0(a,b)\right). \end{aligned} \tag{4.34}$$

Now, we show that

$$C(n):\beta_{n,-1}\leqq 2n-1 \tag{4.35}$$

holds for all positive integers n. In fact, both $C(1)$ and $C(2)$ are true by (4.26) and (4.28). Assume that $C(n)$ holds for all $1\leqq n\leqq N$ $(N\geqq 2)$. If $N=2\ell$, then

$$\beta_{N+1,-1}=\beta_{2\ell+1,-1}\leqq\beta_{\ell,-1}+2\leqq 2\ell-1+2=N+1\leqq 2(N+1)-1 \tag{4.36}$$

holds by (4.29) and the assumption, so that $C(N+1)$ holds. Also if $N=2\ell+1$, then

$$\beta_{N+1,-1}=\beta_{2(\ell+1),-1}\leqq 4(\ell+1)-1=4\ell+3=2(N+1)-1 \tag{4.37}$$

holds by (4.32) and the assumption, so that $C(N+1)$ holds. Therefore, $C(n)$ holds for all positive integers n by induction. Thereby, let m be an integer with $m\leqq -2$. If $m\leqq -3$, then $1\leqq -m-2$, and hence

$$\alpha_m=\alpha_{-(-m-2+2)}\leqq\beta_{-m-2,-1}+2\leqq 2(-m-2)-1+2=-2m-3 \tag{4.38}$$

by $C(n)$. Also if $m=-2$, then the above inequality holds by (4.25). Thus we have Case 2, which completes the proof. □

Finally we conjecture that the above relations hold for all rational numbers $t\neq 0,-1$.

Acknowledgments

The authors would like to thank the referees for the careful reading and useful suggestions which improved the paper. The authors also thank Fumio Hiai and Shizuo Miyajima for their valuable comments. The fourth and fifth authors are partially supported by the Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science, Japan.

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