EXTENSIONS OF HARDY INEQUALITY

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We study extended Hardy inequalities using Littlewood-Paley theory and nonlinear estimates' method in Besov spaces. Our results improve and extend the well-known results of Cazenave (2003).

1. Introduction

A remarkable result of Hardy-type inequality comes from the following proposition, the proof of which is given by Cazenave [2].

PROPOSITION 1.1. *Let $1 \leqslant p < \infty$. If $q < n$ is such that $0 \leqslant q \leqslant p$, then $|u(\cdot)|^p/|\cdot|^q \in L^1(\mathbb{R}^n)$ for every $u \in W^{1,p}(\mathbb{R}^n)$. Furthermore,*

$$\int_{\mathbb{R}^n} \frac{|u(\cdot)|^p}{|\cdot|^q} dx \leqslant \left(\frac{p}{n-q}\right)^q \|u\|_{L^p}^{p-q} \|\nabla u\|_{L^p}^q, \tag{1.1}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$.

It is easy to see that the proposition fails when $s > 1$, where $s = q/p$. In this paper we are trying to find out what happens if $s > 1$. We show that it does not only become true but obtains better estimates.

The described result is stated and proved in Section 3. The method invoked is different from that by Cazenave in [2]; it relies on some Littlewood-Paley theory and Besov spaces' theory that are cited in Section 2.

2. Preliminaries

In this section we introduce some equivalent definitions and norms for Besov space needed in this paper. The reader is referred to the well-known books of Runst and Sickel [5], Triebel [6], and Miao [4] for details.

2 Extensions of Hardy inequality

We first introduce the following equivalent norms for the homogeneous Besov spaces $\dot{B}^s_{p,m}$:

$$\|u\|_{\dot{B}^s_{p,m}} \simeq \sum_{|\alpha|=[s]} \left(\int_0^{+\infty} t^{-m\sigma} \sup_{|y|\leqslant t} \|\triangle_y \partial^\alpha u\|_p^m \frac{dt}{t} \right)^{1/m}, \tag{2.1}$$

where

$$\triangle_y u \triangleq \tau_y u - u, \qquad \tau_y u(\cdot) = u(\cdot + y),$$
$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}, \qquad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $s = [s] + \sigma$ with $0 < \sigma < 1$, namely, $\sigma = s - [s]$, where $[s]$ denotes the largest integer not larger than s. In the case $m = \infty$, the norm $\|u\|_{\dot{B}^s_{p,\infty}}$ in the above definition should be modified as follows:

$$\|u\|_{\dot{B}^s_{p,\infty}} \simeq \sum_{|\alpha|=[s]} \sup_{t>0} t^{-\sigma} \sup_{|y|\leqslant t} \|\triangle_y \partial^\alpha u\|_p, \quad s \in \mathbb{R}^+. \tag{2.3}$$

We now introduce the Paley-Littlewood definition of Besov spaces.

Let $\hat{\varphi}_0 \in C_c^\infty(\mathbb{R}^n)$ with

$$\hat{\varphi}_0(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ 0, & |\xi| \geqslant 2, \end{cases} \tag{2.4}$$

be the real-valued bump function. It is easy to see that

$$\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi), \quad j \in \mathbb{Z},$$
$$\hat{\psi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) - \hat{\varphi}_0(2^{-j+1}\xi), \quad j \in \mathbb{Z}, \tag{2.5}$$

are also real-valued radial bump functions satisfying that

$$\sup_{\xi \in \mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\psi}_j(\xi)| < \infty, \quad j \in \mathbb{Z},$$
$$\sup_{\xi \in \mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\varphi}_j(\xi)| < \infty, \quad j \in \mathbb{Z}. \tag{2.6}$$

We have the Littlewood-Paley decomposition:

$$\hat{\varphi}_0(\xi) + \sum_{j=0}^\infty \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}} \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \backslash \{0\}, \tag{2.7}$$
$$\lim_{j\to+\infty} \hat{\varphi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

For convenience, we introduce the following notations:

$$\triangle_j f = \mathscr{F}^{-1} \hat{\psi}_j \mathscr{F} f = \psi_j * f, \quad j \in \mathbb{Z},$$
$$S_j f = \mathscr{F}^{-1} \hat{\varphi}_j \mathscr{F} f = \varphi_j * f, \quad j \in \mathbb{Z}. \tag{2.8}$$

Then we have the following Littlewood-Paley definition of Besov spaces and Triebel spaces:

$$\dot{B}^s_{p,m} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{\dot{B}^s_{p,m}} = \left(\sum_{j \in \mathbb{Z}} 2^{jsm} \|\triangle_j f\|_p^m \right)^{1/m} \right.$$
$$= \left. \left(\sum_{j \in \mathbb{Z}} 2^{jsm} \|\psi_j * f\|_p^m \right)^{1/m} < \infty \right\},$$

$$\dot{F}^s_{p,m} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{\dot{F}^s_{p,m}} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jsm} |\triangle_j f|^m \right)^{1/m} \right\|_p \right.$$
$$= \left. \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jsm} |\psi_j * f|^m \right)^{1/m} \right\|_p < \infty \right\}, \tag{2.9}$$

$$\dot{B}^s_{p,\infty} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{\dot{B}^s_{p,\infty}} = \sup_{j \in \mathbb{Z}} 2^{js} \|\triangle_j f\|_p = \sup_{j \in \mathbb{Z}} 2^{js} \|\psi_j * f\|_p < \infty \right\},$$

$$\dot{F}^s_{p,\infty} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{\dot{F}^s_{p,\infty}} = \left\| \sup_{j \in \mathbb{Z}} 2^{js} |\triangle_j f| \right\|_p = \left\| \sup_{j \in \mathbb{Z}} 2^{js} |\psi_j * f| \right\|_p < \infty \right\}.$$

Remark 2.1. We have the identities (equivalent quasinorms) $L_p = F^0_{p,2}$, $\dot{H}^s = \dot{F}^s_{2,2} = \dot{B}^s_{2,2}$.

3. Main result

THEOREM 3.1. *Let* $1 \leqslant p < \infty$. *If* $0 \leqslant s < n/p$, *a constant* C *exits such that for any* $u \in \dot{B}^s_{p,1}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \leqslant C \|u\|^p_{\dot{B}^s_{p,1}}. \tag{3.1}$$

Remark 3.2. (i) If $s = 0$, the result will be more precise replacing $\dot{B}^0_{p,1}$ by $\dot{F}^0_{p,2}$.

(ii) Noting interpolation inequality in [1] by Bergh and Löfström between $\dot{H}^{0,p}$ and $\dot{H}^{1,p}$, the theorem implies the proposition when $0 < s < 1$.

(iii) If $s = 1$, the result will be more precise replacing $\dot{B}^1_{p,1}$ by $\dot{F}^1_{p,2} = \dot{H}^{1,p}$.

(iv) If $p = 2$, we have more precise proposition substituting $\dot{F}^s_{2,2} = \dot{B}^s_{2,2}$ for $\dot{B}^s_{2,1}$.

(v) The Hardy-type inequality will be excellent substituting $\dot{F}^s_{p,2}$ for $\dot{B}^s_{p,1}$, but it fails using this method, in fact we obtain this estimate:

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \leqslant C \|u\|^{p-1}_{\dot{F}^s_{p,2}} \|u\|_{\dot{B}^s_{p,1}}, \tag{3.2}$$

where $\dot{F}^s_{p,2}$ is a Triebel space.

In order to prove the theorem, we need the following two lemmas, the first of which was easily proved using Littlewood-Paley theory in Lemarié-Rieusset [3] and the other will be proved here.

LEMMA 3.3. *Let s be in $]0,n[$. Then for any p in $[1,\infty]$, $|\cdot|^{-s} \in \dot{B}^{n/p-s}_{p,\infty}$.*

LEMMA 3.4. *Let $1 \leqslant p < \infty$. If $0 \leqslant s < n/p$, then $u^p \in \dot{B}^0_{q',1}$ for every $u \in \dot{B}^s_{p,1}$, where $q' = q/(q-1)$ and $q = n/sp$.*

Proof of Lemma 3.4. By equivalent definition and norms for Besov space, it is sufficient to establish that

$$||u^p||_{\dot{B}^0_{q',1}} \leqslant ||u||^p_{\dot{B}^s_{p,1}}. \tag{3.3}$$

Hence

$$||F||_{\dot{B}^0_{q',1}} \simeq \int_0^{+\infty} \sup_{|y| \leqslant t} ||\triangle_y F||_{q'} \frac{dt}{t}. \tag{3.4}$$

Let $F(u) = |u(x)|^p$. Using Newton-Leibniz formula and inequality $(|a|+|b|)^p \leqslant 2^p(|a|^p + |b|^p)$, we deduce that

$$\left| \tau_y F(u) - F(u) \right| = \left| \int_0^1 dF(\theta \tau_y |u| + (1-\theta)|u|) \right| \leqslant C(|\tau_y u|^{p-1} + |u|^{p-1}) |\tau_y u - u|, \tag{3.5}$$

where C is a constant.

By definition of $\triangle_y$ and thanks to the Hölder inequality, we have that

$$||\triangle_y F||_{q'} \leqslant C||u||^{p-1}_{(p-1)\chi_1} ||\tau_y u - u||_{\chi_2}, \tag{3.6}$$

where $1/\chi_1 = (p-1)(1/p - s/n)$ and $1/\chi_2 = 1/p - s/n$.

Note that

$$\begin{aligned}
\dot{B}^s_{p,1}(\mathbb{R}^n) &\hookrightarrow L^{(p-1)\chi_1}(\mathbb{R}^n), \\
\dot{B}^s_{p,1}(\mathbb{R}^n) &\hookrightarrow \dot{B}^0_{\chi_2,1}(\mathbb{R}^n).
\end{aligned} \tag{3.7}$$

Thus we infer that

$$||u^p||_{\dot{B}^0_{q',1}} \leqslant ||u||^{p-1}_{(p-1)\chi_1} ||u||_{\dot{B}^0_{\chi_2,1}} \leqslant C||u||^p_{\dot{B}^s_{p,1}} \tag{3.8}$$

implying the lemma. $\square$

Proof of Theorem 3.1. Let us define

$$I_{s,p}(u) \triangleq \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx = \langle |\cdot|^{-sp}, |u|^p \rangle. \tag{3.9}$$

Using Littlewood-Paley decomposition, we can write

$$
\begin{aligned}
I_{s,p}(u) &= \sum_{|j-j'|\leqslant 2} \left\langle \triangle_j | \cdot |^{-sp}, \triangle_{j'} |u|^p \right\rangle \\
&\leqslant C \sup_j \left\| \triangle_j | \cdot |^{-sp} \right\|_q \sum_{j'\in\mathbb{Z}} \left\| \triangle_{j'} |u|^p \right\|_{q'} \\
&\leqslant C \left\| | \cdot |^{-sp} \right\|_{\dot{B}^0_{q,\infty}} \left\| u^p \right\|_{\dot{B}^0_{q',1}},
\end{aligned}
\tag{3.10}
$$

where $q = n/sp > 1$. Lemma 3.3 claims that $|\cdot|^{-sp}$ belongs to $\dot{B}^0_{q,\infty}$ and Lemma 3.4 claims in particular that $\|u^p\|_{\dot{B}^0_{q',1}} \leqslant \|u\|^p_{\dot{B}^s_{p,1}}$. Thus $I_{s,p}(u) \leqslant C \|u\|^p_{\dot{B}^s_{p,1}}$, which implies the theorem.

$\square$

Acknowledgment

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