GLOBAL SOLUTIONS FOR A NONLINEAR HYPERBOLIC EQUATION WITH BOUNDARY MEMORY SOURCE TERM

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Received 21 January 2005; Accepted 17 August 2005

We study a nonlinear hyperbolic equation with boundary memory source term. By the use of Galerkin procedure, we prove the global existence and the decay property of solution.

1. Introduction

This paper deals with a hyperbolic equation with boundary memory source terms:

$$\begin{gathered}
\rho(x)u'' - \Delta u' - \Delta u = g(u), \quad x \in \Omega,\ t > 0, \\
u = 0, \quad x \in \Gamma_0,\ t > 0, \\
\frac{\partial u'}{\partial \nu} + \frac{\partial u}{\partial \nu} + f(u') = \int_0^t K(t-\tau)h(\tau, u(\tau))\,d\tau, \quad x \in \Gamma_1,\ t > 0, \\
u(0,x) = u_0(x), \quad (\sqrt{\rho}u')(0,x) = (\sqrt{\rho}u_1)(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $u = u(t,x)$, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \varnothing$, where Γ_0 and Γ_1 have positive measures. $u' = \partial u/\partial t$, $u'' = \partial^2 u/\partial t^2$. Equations of type (1.1) are of interest in many applications such as in the theory of electromagnetic materials with memory which obey the Ohm's law. It can also describe the temperature evolution in a rigid conductor with a memory. We refer to [8, 9] to see the details. In many works concerned with equations of type (1.1), we cite Aassila et al. [1], where the following wave equation was considered:

$$\begin{gathered}
u'' - \Delta u + f_0(\nabla u) = 0, \quad x \in \Omega,\ t > 0, \\
u = 0, \quad x \in \Gamma_0,\ t > 0, \\
\frac{\partial u}{\partial \nu} + g(u') = \int_0^t K(t-\tau)h(u(\tau))\,d\tau, \quad x \in \Gamma_1,\ t > 0, \\
u(0,x) = u_0(x), \quad u'(0,x) = u_1(x), \quad x \in \Omega.
\end{gathered} \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 60734, Pages 1–16
DOI 10.1155/JIA/2006/60734

Under some conditions on nonlinear terms, they acquired the existence and uniform decay of solutions. Recently, Park and Park [12] generalized problem (1.2) by endowing with some discontinuous and multivalued terms. For more related works, we refer to [3, 4, 7, 11, 13] and the references therein. For problem (1.1) without memory source term, we point out the work [6] of Cavalcanti et al., where they investigated the following equation with boundary damping:

$$\begin{gathered}
\rho(x)u'' - \Delta u = 0, \quad x \in \Omega,\ t > 0, \\
u = 0, \quad x \in \Gamma_0,\ t > 0, \\
\frac{\partial u}{\partial \nu} + f(u') + g(u) = 0, \quad x \in \Gamma_1,\ t > 0, \\
u(0,x) = u_0(x), \quad \sqrt{\rho}u'(0,x) = \sqrt{\rho}u_1(x), \quad x \in \Omega.
\end{gathered} \tag{1.3}$$

Through a partition of boundary Γ and Galerkin procedures, they acquired the existence and decay behavior of the solution to problem (1.3). In another work of theirs [5], using similar method, they studied problem (1.3) with $\rho = 1$ and the source term $g(u) = |u|^p u$ coupled in the first equation. Motivated by the above works, we are devoted to study problem (1.1). By virtue of the potential well method, and through Galerkin procedures, we acquire the global existence and decay property of perturbed energy of solutions of problem (1.1). The organization of this paper is as follows. In Section 2, we make assumptions and introduce a potential well, and then state the main results. In Section 3, making use of Galerkin procedures, we study the existence of solution of problem (1.1). And in the last section, we derive the uniform decay by the perturbed energy method.

2. Assumptions and main results

In this section, we first give the notations used throughout this paper:

$$\begin{gathered}
(u,v) = \int_\Omega u(x)v(x)dx, \qquad (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma, \\
\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}, \qquad \|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \\
\|\cdot\|_{\Gamma_1,p} = \|\cdot\|_{L^p(\Gamma_1)}, \qquad \|\cdot\|_{\Gamma_1} = \|\cdot\|_{L^2(\Gamma_1)},
\end{gathered} \tag{2.1}$$

and r' denotes the conjugate exponent of $r > 1$.

Define

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}. \tag{2.2}$$

Since the measure of Γ_0 is positive, Poincarè inequality holds and trace embedding theorem holds (see [2]), we know that $\|\nabla u\|$ is equivalent to the norm on V. Let μ_1 and μ_2 be the optimal constants such that

$$\|u\| \le \mu_1 \|\nabla u\|, \quad \|u\|_{\Gamma_1} \le \mu_2 \|\nabla u\| \quad \forall u \in V. \tag{2.3}$$

Now we make the following assumptions.

(A_1) $f \in C(\mathbb{R})$, $f(s)s \geq 0$, and there exist positive constants k_1 and k_2 such that

$$k_1|s|^{q-1} \leq |f(s)| \leq k_2|s|^{q-1}, \tag{2.4}$$

where $2 < q < \infty$ if $N = 1,2$; $2 < q \leq 2(N-1)/(N-2)$ if $N \geq 3$.

(A_2) $g \in C(\mathbb{R})$, $g(s)\,s \geq 0$, and there exists positive constant k_3 such that

$$|g(s)| \leq k_3|s|^p, \tag{2.5}$$

where $1 < p < \infty$ if $N = 1,2$; $1 < p \leq N/(N-2)$ if $N \geq 3$.

(A_3) $K : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function verifying

$$K'(t) \leq -k_4K(t) \quad \forall t \geq 0, \qquad K(0) > 0, \qquad 1 - \mu_2^2\int_0^\infty K(s)ds \triangleq L > 0, \tag{2.6}$$

where $k_4 > 0$.

(A_4) $h(\tau,s)$ is measurable with τ and continuous with s, and it satisfies

$$|h(\tau,s) - s| \leq \sqrt{\frac{K(\tau)}{K(0)}}|s| \quad \forall s \in \mathbb{R},\ \tau \geq 0. \tag{2.7}$$

(A_5) $\rho(x) \geq 0$, $\rho \not\equiv 0$ and $\rho \in L^\infty(\Omega)$.

(A_6) Assume that the initial data

$$u_0, u_1 \in V \cap H^{3/2}(\Omega) \tag{2.8}$$

and satisfy the compatibility conditions

$$\begin{gathered} -\Delta(u_0 + u_1) = g(u_0), \quad x \in \Omega, \\ u_0 = 0, \quad u_1 = 0, \quad x \in \Gamma_0, \\ \frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + f(u_1) = 0, \quad x \in \Gamma_1. \end{gathered} \tag{2.9}$$

Remark 2.1. (i) The assumptions (A_3) and (A_4) imply that $h(\tau,s) \approx (1 + \sqrt{K(\tau)})s$.

(ii) Given $u_1 \in V \cap H^{3/2}(\Omega)$, by the assumption ($A_2$) and the theory of elliptic problems, we see that problem (2.9) admits a weak solution $u_0 \in V \cap H^{3/2}(\Omega)$.

Let $B_* > 0$ be the optimal constant such that

$$\|v\|_{p+1} \leq B_*\|\nabla v\| \quad \forall v \in V, \tag{2.10}$$

where p is the number given in the assumption (A_2).

If we define

$$B_\infty \triangleq \sup_{v\in V, v\neq 0}\left(\frac{(1/(p+1))\|v\|_{p+1}^{p+1}}{\|\nabla v\|^{p+1}}\right), \tag{2.11}$$

then

$$B_\infty \le \frac{B_*^{p+1}}{p+1}, \qquad \frac{1}{p+1}\|v\|_{p+1}^{p+1} \le B_\infty \|\nabla v\|^{p+1} \quad \forall v \in V. \tag{2.12}$$

Now for some function u, we define

$$J(u) = \frac{L}{2}\|\nabla u\|^2 - \frac{k_3}{p+1}\|u\|_{p+1}^{p+1},$$
$$E(t) = \frac{1}{2}\left\|\sqrt{\rho}u'\right\|^2 + \frac{1}{2}\|\nabla u\|^2 - \int_\Omega G(u)dx - \frac{1}{2}\left(\int_0^t K(\tau)d\tau\right)\left\|u(t)\right\|_{\Gamma_1}^2 + \frac{1}{2}(K\square u)(t), \tag{2.13}$$

where

$$G(s) = \int_0^s g(\eta)d\eta, \qquad (K\square u)(t) = \int_0^t K(t-\tau)\left\|h\big(\tau,u(\tau)\big) - u(t)\right\|_{\Gamma_1}^2 d\tau. \tag{2.14}$$

Putting

$$d \triangleq \inf_{u\in V, u\neq 0}\left\{\sup_{\lambda>0} J(\lambda u)\right\}, \quad H(\lambda) \triangleq \frac{L}{2}\lambda^2 - k_3 B_\infty \lambda^{p+1}, \quad \lambda > 0. \tag{2.15}$$

We have the following result.

Proposition 2.2. *Let the assumptions (A_2)–(A_4) be fulfilled. It holds that*

$$d = \max_{\lambda>0} H(\lambda) = H(\lambda_\infty) = \frac{(p-1)L}{2(p+1)}\lambda_\infty^2, \tag{2.16}$$

where $\lambda_\infty = (L/(p+1)k_3B_\infty)^{1/(p-1)}$.

If $\|\nabla u\| < \lambda_\infty$, then

$$J(u) \ge 0, \qquad \|\nabla u\|^2 \le \frac{2(p+1)}{(p-1)L}E(t). \tag{2.17}$$

Proof. From

$$H'(\lambda) = L\lambda - (p+1)k_3 B_\infty \lambda^p = \left[L - (p+1)k_3 B_\infty \lambda^{p-1}\right]\lambda, \tag{2.18}$$

we see that $\lambda_\infty = [L/((p+1)k_3 B_\infty)]^{1/(p-1)}$ is the maximum point of H. Hence,

$$\max_{\lambda>0} H(\lambda) = H(\lambda_\infty) = \frac{(p-1)L}{2(p+1)}\lambda_\infty^2. \tag{2.19}$$

Note the definition of B_∞, by the direct computation, we have

$$\begin{aligned} d &= \inf_{u\in V, u\neq 0}\left\{\sup_{\lambda>0} J(\lambda u)\right\} \\ &= \left[\frac{L}{2}\left(\frac{L}{k_3}\right)^{2/(p-1)} - \frac{k_3}{p+1}\left(\frac{L}{k_3}\right)^{(p+1)/(p-1)}\right]\inf_{u\in V, u\neq 0}\left(\frac{\|\nabla u\|^{p+1}}{\|u\|_{p+1}^{p+1}}\right)^{2/(p-1)} \\ &= \frac{(p-1)L}{2(p+1)}\left(\frac{L}{(p+1)k_3 B_\infty}\right)^{2/(p-1)} = \frac{(p-1)L}{2(p+1)}\lambda_\infty^2. \end{aligned} \tag{2.20}$$

Thus the first conclusion is valid.

If $\|\nabla u\| < \lambda_\infty$, then we obtain

$$\begin{aligned} E(t) \geq J(u(t)) &\geq \frac{L}{2}\|\nabla u\|^2 - k_3 B_\infty \|\nabla u\|^{p+1} > \|\nabla u\|^2\left(\frac{L}{2} - k_3 B_\infty \lambda_\infty^{p-1}\right) \\ &= \|\nabla u\|^2\left(\frac{L}{2} - \frac{L}{p+1}\right) = \frac{(p-1)L}{2(p+1)}\|\nabla u\|^2. \end{aligned} \tag{2.21}$$

Thus the second conclusion is valid. □

Remark 2.3. The number d defined in Proposition 2.2 is the Mountain Pass level related to the elliptic problem

$$\begin{gathered} -L\Delta u = k_3 |u|^{p-1} u, \qquad x \in \Omega, \\ u = 0, \qquad x \in \Gamma_0, \\ \frac{\partial u}{\partial \nu} = 0, \qquad x \in \Gamma_1, \end{gathered} \tag{2.22}$$

see [5] or [14]. In fact, d is equal to the number

$$\inf_{\alpha\in\Lambda}\sup_{t\in[0,1]} J(\alpha(t)), \tag{2.23}$$

where

$$\Lambda = \{\alpha \in C([0,1];V);\ \alpha(0) = 0,\ J(\alpha(1)) < 0\}. \tag{2.24}$$

Now we are in a position to state the main results of this paper.

Theorem 2.4. *Let the assumptions (A_1)–(A_6) hold. If in addition, the initial data satisfy*

$$||\nabla u_0|| < \lambda_\infty, \qquad E(0) < d, \tag{2.25}$$

then for any $T > 0$, problem (1.1) admits a solution $u \in L^\infty(0,T;V)$ and satisfies $\sqrt{\rho}u' \in L^\infty(0,T;L^2(\Omega))$, $u' \in L^2(0,T;V)$, $\rho u'' \in L^{q'}(0,T;L^2(\Omega))$.

Theorem 2.5. *Let u be the solution obtained in Theorem 2.4. If $q = 2$, then the solution u verifies the following decay estimate:*

$$E(t) \leq 3d \exp\left(-\frac{2}{3}Ct\right) \quad \forall t \geq 0 \tag{2.26}$$

for some positive constant C.

3. Proof of Theorem 2.4

In this section, we will use Faedo-Galerkin procedure to prove Theorem 2.4.
Step 1. Let $\{\omega_k\}_{k=1}^\infty$ be a basis in V, which is orthogonal in $L^2(\Omega)$. For fixed n, let

$$V_n = [\omega_1, \dots, \omega_n] \tag{3.1}$$

be the linear span of $\{\omega_k\}_{k=1}^n$, and let

$$\rho_\varepsilon = \rho + \varepsilon \quad (\varepsilon > 0), \quad u_{\varepsilon n}(t) = \sum_{k=1}^n q_{\varepsilon kn}(t)\omega_k \in V_n. \tag{3.2}$$

Consider the Cauchy problem:

$$\begin{aligned} &(\rho_\varepsilon u''_{\varepsilon n}, \omega) + (\nabla u'_{\varepsilon n}(t), \nabla\omega) + (\nabla u_{\varepsilon n}(t), \nabla\omega) + (f(u'_{\varepsilon n}), \omega)_{\Gamma_1} \\ &\quad = (g(u_{\varepsilon n}), \omega) + \int_0^t K(t-\tau)(h(\tau, u_{\varepsilon n}(\tau)), \omega)_{\Gamma_1} d\tau, \quad \forall \omega \in V_n, \end{aligned} \tag{3.3}$$

$$u_{\varepsilon n}(0) = \sum_{k=1}^n q_{\varepsilon kn}(0)\omega_k \longrightarrow u_0 \quad \text{strongly in } V, \tag{3.4}$$

$$u'_{\varepsilon n}(0) = \sum_{k=1}^n q'_{\varepsilon kn}(0)\omega_k \longrightarrow u_1 \quad \text{strongly in } L^2(\Omega). \tag{3.5}$$

By the standard method of ordinary differential equations, system (3.3)-(3.4) has a local solution $u_{\varepsilon n}(t)$ on interval $(0, t_{\varepsilon n})$ with $q_{\varepsilon kn}(t) \in W^{2,1}(0, t_{\varepsilon n})$. The extension of this solution to the whole interval $[0, \infty)$ will be deduced by a series a priori estimates.

Using the method exploited in the paper [15], we can construct the energy function and the energy identity associated to problem (3.3)-(3.4) as follows:

$$E_{\varepsilon n}(t) = \frac{1}{2}||\sqrt{\rho_\varepsilon} u'_{\varepsilon n}|^2 + \frac{1}{2}||\nabla u_{\varepsilon n}||^2 - \int_\Omega G(u_{\varepsilon n})dx \\ - \frac{1}{2}\left(\int_0^t K(\tau)d\tau\right)||u_{\varepsilon n}(t)||^2_{\Gamma_1} + \frac{1}{2}(K\Box u_{\varepsilon n})(t), \tag{3.6}$$

$$E_{\varepsilon n}(t) - E_{\varepsilon n}(s) = -\int_s^t \int_{\Gamma_1} f(u'_{\varepsilon n}(\eta))u'_{\varepsilon n}(\eta)d\Gamma\, d\eta \\ - \int_s^t \|\nabla u'_{\varepsilon n}(\eta)\|^2 d\eta + \frac{1}{2}\int_s^t (K'\Box u_{\varepsilon n})(\eta)d\eta \\ + \frac{K(0)}{2}\int_s^t ||h(\eta, u_{\varepsilon n}(\eta)) - u_{\varepsilon n}(\eta)||^2_{\Gamma_1} d\eta - \frac{1}{2}\int_s^t K(\eta)||u_{\varepsilon n}(\eta)||^2_{\Gamma_1} d\eta \tag{3.7}$$

for $0 \le s \le t < t_{\varepsilon n}$.

Using the assumption (A_4), it is easily known that

$$\frac{K(0)}{2}\int_s^t ||h(\eta, u_{\varepsilon n}(\eta)) - u_{\varepsilon n}(\eta)||^2_{\Gamma_1} d\eta - \frac{1}{2}\int_s^t K(\eta)||u_{\varepsilon n}(\eta)||^2_{\Gamma_1} d\eta \le 0. \tag{3.8}$$

Then using the assumption (A_3), (3.7) and (3.8) imply that $E_{\varepsilon n}(t)$ is a decreasing function. By the assumption (A_2), we see that

$$|G(u)| \le C_1|u|^{p+1}, \tag{3.9}$$

Here and in the sequel C_i, $i = 1, 2, \dots$, will denote various constants independent of ε and n. Exploiting the continuity of the Nemyskii operator and (3.4), it follows that

$$\int_\Omega G(u_{0\varepsilon n})dx \longrightarrow \int_\Omega G(u_0)dx \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

Therefore, using (3.4) and (3.5), it entails

$$E_{\varepsilon n}(0) \longrightarrow E(0) \quad \text{as } n \longrightarrow \infty,\ \varepsilon \longrightarrow 0. \tag{3.11}$$

Define

$$B_n \triangleq \sup_{u\in V_n, u\neq 0}\left(\frac{(1/(p+1))\|u\|^{p+1}_{p+1}}{\|\nabla u\|^{p+1}}\right), \qquad \lambda_n \triangleq \left(\frac{L}{(p+1)k_3 B_n}\right)^{1/(p-1)}, \\ d_n \triangleq \frac{(p-1)L}{2(p+1)}\lambda_n^2. \tag{3.12}$$

By the assumption (A_2), it follows that

$$0 < B_n \le B_{n+1} \le \cdots \le B_\infty, \quad \lambda_\infty \le \cdots \le \lambda_{n+1} \le \lambda_n, d \le \cdots \le d_{n+1} \le d_n, \quad n \ge 1. \tag{3.13}$$

By (2.25), (3.4), (3.5), (3.11), and (3.13), we know that, for sufficiently large n_0 and sufficiently small ε_0,

$$||\nabla u_{\varepsilon n}(0)|| < \lambda_n, \quad E_{\varepsilon n}(0) < d_n, \quad n \ge n_0, \ \varepsilon \le \varepsilon_0. \tag{3.14}$$

From now on, we may assume that $n \ge n_0$ and $\varepsilon \le \varepsilon_0$. By (3.6) and the assumptions (A_2) and (A_3), we deduce that

$$E_{\varepsilon n}(t) \ge \frac{L}{2}||\nabla u_{\varepsilon n}(t)||^2 - k_3 B_n ||\nabla u_{\varepsilon n}(t)||^{p+1} = H_n(||\nabla u_{\varepsilon n}(t)||), \tag{3.15}$$

where $H_n(\lambda) = (L/2)\lambda^2 - k_3 B_n \lambda^{p+1}$ has the similar property of the function H defined in Proposition 2.2. It is easy to verify that H_n is increasing for $0 < \lambda < \lambda_n$ and decreasing for $\lambda > \lambda_n$, $H_n(\lambda_n) = d_n$, and $H_n(\lambda) \to -\infty$ as $\lambda \to +\infty$. Since $E_{\varepsilon n}(0) < d_n$, there exist $\lambda_n^1 < \lambda_n < \lambda_n^2$ such that $H_n(\lambda_n^1) = H_n(\lambda_n^2) = E_{\varepsilon n}(0)$. From (3.7) and (3.8), we have

$$E_{\varepsilon n}(t) \le E_{\varepsilon n}(0) \quad \forall t \in [0, t_{\varepsilon n}). \tag{3.16}$$

Denote $\lambda_n^0 = \|\nabla u_{\varepsilon n}(0)\|$, so $\lambda_n^0 < \lambda_n$. By (3.15), we have $H_n(\lambda_n^0) \le E_{\varepsilon n}(0)$, thus $\lambda_n^0 < \lambda_n^1$.

We claim that $\|\nabla u_{\varepsilon n}(t)\| \le \lambda_n^1$ for all $t \in [0, t_{\varepsilon n})$. Suppose, by contradiction, that $\|\nabla u_{\varepsilon n}(t_0)\| > \lambda_n^1$ for some $t_0 \in (0, t_{\varepsilon n})$. From the continuity of $\|\nabla u_{\varepsilon n}(\cdot)\|$, we can suppose that $\|\nabla u_{\varepsilon n}(t_0)\| < \lambda_n$. Then by (3.15), $E_{\varepsilon n}(t_0) \ge H_n(\|\nabla u_{\varepsilon n}(t_0)\|) > H_n(\lambda_n^1) = E_{\varepsilon n}(0)$, which contradicts (3.16). From (3.14) and (3.16), it yields $E_{\varepsilon n}(t) < d_n$ for $t \in [0, t_{\varepsilon n})$. Then using (3.13), one gets

$$||\nabla u_{\varepsilon n}(t)|| \le \lambda_1, \qquad E_{\varepsilon n}(t) < d_1 \tag{3.17}$$

for $t \in [0, t_{\varepsilon n})$. By (3.17), the assumption (A_2), and the Sobolev embedding theorem, we deduce that

$$\int_\Omega G(u_{\varepsilon n})\,dx \le \frac{k_3}{p+1}||u_{\varepsilon n}||_{p+1}^{p+1} \le C_2. \tag{3.18}$$

Therefore, from (3.6), (3.17), and (3.18), it follows that

$$||\sqrt{\rho_\varepsilon}\, u'_{\varepsilon n}(t)|| \le C_3. \tag{3.19}$$

Estimates (3.17) and (3.19) imply that $t_{\varepsilon n} = \infty$.

For any $T > 0$ and for all $t \in [0,T]$, by the assumptions (A_1), (A_3), and (A_4), we get from (3.7), (3.14), and (3.16) that

$$\int_0^t \|\nabla u'_{\varepsilon n}(\tau)\|^2 d\tau \le C_4,$$
$$\int_0^t \|u'_{\varepsilon n}(\tau)\|^q_{\Gamma_1,q} d\tau \le k_1^{-1} \int_0^t \left(f(u'_{\varepsilon n}(\tau)), u'_{\varepsilon n}(\tau)\right)_{\Gamma_1} d\tau \le C_5. \tag{3.20}$$

Then using the assumptions (A_1)-(A_2), the Sobolev embedding theorem, (3.13), and (3.20), we derive that, for all $t \in [0,T]$,

$$\int_0^t \|f(u'_{\varepsilon n})\|^{q'}_{\Gamma_1,q'} d\tau \le C_6, \qquad \|g(u_{\varepsilon n})\|^{(p+1)'}_{(p+1)'} \le C_7. \tag{3.21}$$

Similarly, by the assumptions (A_3)-(A_4) and the Sobolev embedding theorem, it leads to

$$\int_0^t \|h(\tau, u_{\varepsilon n}(\tau))\|_{\Gamma_1} d\tau \le C_8, \qquad \left\| \int_0^t K(t-\tau) h(\tau, u_{\varepsilon n}(\tau)) d\tau \right\|_{\Gamma_1} \le C_9 \quad \forall t \in [0,T]. \tag{3.22}$$

Replacing ω in (3.3) with $v \in V$, and exploiting the Hölder inequality, the Sobolev embedding theorem, (3.17), (3.21), and (3.22), it entails

$$|(\rho_\varepsilon u''_{\varepsilon n}, v)| \le C_{10} \Bigg(\|\nabla u'_{\varepsilon n}\| + \|\nabla u_{\varepsilon n}\| + \|f(u'_{\varepsilon n})\|_{\Gamma_1,q'} + \|g(u_{\varepsilon n})\|_{(p+1)'}$$
$$+ \left\| \int_0^t K(t-\tau) h(\tau, u_{\varepsilon n}(\tau)) d\tau \right\|_{\Gamma_1} \Bigg) \|\nabla v\|, \tag{3.23}$$
$$\|\rho_\varepsilon u''_{\varepsilon n}\| \le C_{11} \left(1 + \|\nabla u'_{\varepsilon n}\| + \|f(u'_{\varepsilon n})\|_{\Gamma_1,q'}\right)$$

for all $t \in [0,T]$.

Integrating the above inequality over $[0,t]$, using (3.20)-(3.21) and the Hölder inequality, we get

$$\int_0^t \|\rho_\varepsilon u''_{\varepsilon n}\|^{q'} d\tau \le C_{12} \int_0^t \left(1 + \|\nabla u'_{\varepsilon n}\|^{q'} + \|f(u'_{\varepsilon n})\|^{q'}_{\Gamma_1,q'}\right) d\tau \le C_{13}. \tag{3.24}$$

Step 2. The limiting process. From the estimates (3.17), (3.19)–(3.24), using the standard arguments, it yields that, up to a subsequence, as $n \to \infty$,

$$\begin{gathered} u_{\varepsilon n} \longrightarrow u_\varepsilon \quad \text{weakly}^* \text{ in } L^\infty(0,T;V), \\ \sqrt{\rho_\varepsilon} u'_{\varepsilon n} \longrightarrow \sqrt{\rho_\varepsilon} u'_\varepsilon \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \end{gathered} \tag{3.25}$$

$$u'_{\varepsilon n} \longrightarrow u'_\varepsilon \quad \text{weakly in } L^2(0,T;V), \tag{3.26}$$

$$f(u'_{\varepsilon n}) \longrightarrow \gamma \quad \text{weakly}^* \text{ in } L^{q'}((0,T)\times\Gamma_1), \tag{3.27}$$

$$g(u_{\varepsilon n}) \longrightarrow \chi \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^{(p+1)'}(\Omega)), \tag{3.28}$$

$$h(t,u_{\varepsilon n}) \longrightarrow \xi \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.29}$$

$$\rho_\varepsilon u''_{\varepsilon n} \longrightarrow \rho_\varepsilon u''_\varepsilon \quad \text{weakly}^* \text{ in } L^{q'}(0,T;L^2(\Omega)). \tag{3.30}$$

Since $V \hookrightarrow L^2(\Omega)$ and $V \hookrightarrow L^2(\Gamma_1)$, then by the Aubin-Lions compactness lemma [10, Theorem 5.1], we get from (3.25) and (3.26) that, as $n \to \infty$,

$$u_{\varepsilon n} \longrightarrow u_\varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. on } (0,T)\times\Omega, \tag{3.31}$$

$$\begin{gathered} u_{\varepsilon n} \longrightarrow u_\varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)) \text{ and a.e. on } (0,T)\times\Gamma_1, \\ u'_{\varepsilon n} \longrightarrow u'_\varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. on } (0,T)\times\Omega, \end{gathered} \tag{3.32}$$

$$u'_{\varepsilon n} \longrightarrow u'_\varepsilon \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)) \text{ and a.e. on } (0,T)\times\Gamma_1, \tag{3.33}$$

From (3.25), (3.26), and (3.30), we acquire, as $n \to \infty$,

$$(u_{\varepsilon n},\omega) \longrightarrow (u_\varepsilon,\omega) \quad \text{weakly in } L^2[0,T], \tag{3.34}$$

$$\begin{gathered} (u'_{\varepsilon n},\omega) \longrightarrow (u'_\varepsilon,\omega) \quad \text{weakly in } L^2[0,T], \\ (\rho_\varepsilon u''_{\varepsilon n},\omega) \longrightarrow (\rho_\varepsilon u''_\varepsilon,\omega) \quad \text{weakly in } L^{q'}[0,T]. \end{gathered} \tag{3.35}$$

Note that $W^{1,2}[0,T] \hookrightarrow C[0,T]$ and $W^{1,q'} \hookrightarrow C[0,T]$, from (3.35), we get that

$$(u_{\varepsilon n}(0),\omega) \longrightarrow (u_\varepsilon(0),\omega), (u'_{\varepsilon n}(0),\omega) \longrightarrow (u'_\varepsilon(0),\omega), \tag{3.36}$$

and hence

$$u_\varepsilon(0) = u_0 \quad \text{in } V, \qquad u'_\varepsilon(0) = u_1 \quad \text{in } L^2(\Omega). \tag{3.37}$$

Now letting $n \to \infty$ in (3.3) and using (3.25)–(3.37), we acquire

$$\begin{aligned} &\int_0^T \left[(\rho_\varepsilon u''_\varepsilon,v) + (\nabla u'_\varepsilon,\nabla v) + (\nabla u_\varepsilon,\nabla v) + (\gamma,v)_{\Gamma_1} - (\chi,v)\right]dt \\ &\quad = \int_0^T \int_0^t K(t-\tau)(\xi,v)_{\Gamma_1}\,d\tau\,dt \end{aligned} \tag{3.38}$$

for any $v \in V$.

Step 3. We first prove $\gamma = f(u'_\varepsilon)$. By (3.33), the continuity of $f(s)$, we see that $f(u'_{\varepsilon n}) \to f(u'_\varepsilon)$ and a.e. on $(0,T)\times\Gamma_1$. Thus by use of (3.21), and of Lions' [10, Lemma 1.3], we have $f(u'_{\varepsilon n}) \to f(u'_\varepsilon)$ weakly in $L^{q'}((0,T)\times\Gamma_1)$. Then (3.27) and the uniqueness of weak* limit give $\gamma = f(u'_\varepsilon)$ in $L^{q'}((0,T)\times\Gamma_1)$. By analogous analysis, one can get $\xi = h(t,u_\varepsilon(t))$ in $L^2((0,T)\times\Gamma_1)$.

Below we show $\chi = g(u_\varepsilon)$ along the line of the paper [16]. By (3.31) and the continuity of $g(s)$, we have $g(u_{\varepsilon n}) \to g(u_\varepsilon)$ a.e. on $Q_T = (0,T)\times\Omega$. Therefore, exploiting Lusin and Egoroff's theorem, for any $\delta > 0$, there exists a measurable set $Q \subset Q_T$ such that $|Q| < \delta$, and $g(u_{\varepsilon n}) \to g(u_\varepsilon)$ uniformly on $Q_T \setminus Q$ as $n \to \infty$. By the Sobolev embedding theorem, we know from (3.25) that $u_\varepsilon \in L^{p+1}(Q_T)$. And hence the assumption (A_2) implies that $g(u_\varepsilon) \in L^{(p+1)'}(Q_T)$. For any $\hat{p} > p$, by the use of (3.28), we get

$$\begin{aligned} \|g(u_{\varepsilon n}) - g(u_\varepsilon)\|_{L^{(\hat{p}+1)'}(Q)} &\le \|g(u_{\varepsilon n}) - g(u_\varepsilon)\|_{L^{(p+1)'}(Q)}\delta^{(\hat{p}-p)/(\hat{p}+1)(p+1)} \\ &\le C_{14}\delta^{(\hat{p}-p)/(\hat{p}+1)(p+1)}. \end{aligned} \tag{3.39}$$

So, as $n \to \infty$,

$$\begin{aligned} &\|g(u_{\varepsilon n}) - g(u_\varepsilon)\|_{L^{(\hat{p}+1)'}(Q_T)} \\ &\quad\le \|g(u_{\varepsilon n}) - g(u_\varepsilon)\|_{L^{(\hat{p}+1)'}(Q)} + \|g(u_{\varepsilon n}) - g(u_\varepsilon)\|_{L^{(\hat{p}+1)'}(Q_T\setminus Q)} \\ &\quad\le C_{15}\delta^{(\hat{p}-p)/(\hat{p}+1)(p+1)}. \end{aligned} \tag{3.40}$$

By the arbitrariness of δ, we get that, as $n \to \infty$,

$$g(u_{\varepsilon n}) \longrightarrow g(u_\varepsilon) \quad \text{strongly in } L^{(\hat{p}+1)'}(Q_T). \tag{3.41}$$

Using (3.28), (3.41), and the uniqueness of weak* limit, we acquire that $\chi = g(u_\varepsilon)$ in $L^{(\hat{p}+1)'}(Q_T)$.

Because $C_0^\infty(Q_T)$ is dense in $L^{p+1}(Q_T)$, for any $\phi \in L^{p+1}(Q_T)$, we can choose a sequence $\{\phi_n\}$, ϕ_n in $C_0^\infty(Q_T)$ $(n = 1,2,\ldots)$, such that $\phi_n \to \phi$ strongly in $L^{p+1}(Q_T)$, and

$$\left|\int_0^T (g(u_\varepsilon) - \chi, \phi_n - \phi)\,dt\right| \le \|g(u_\varepsilon) - \chi\|_{L^{(p+1)'}(Q_T)}\|\phi_n - \phi\|_{L^{p+1}(Q_T)} \longrightarrow 0 \tag{3.42}$$

as $n \to \infty$. From (3.42), it follows that

$$\begin{gathered} \int_0^T (g(u_\varepsilon) - \chi, \phi)\,dt = \lim_{n\to\infty}\int_0^T (g(u_\varepsilon) - \chi, \phi_n)\,dt \longrightarrow 0, \\ g(u_\varepsilon) = \chi \quad \text{in } L^{(p+1)'}(Q_T). \end{gathered} \tag{3.43}$$

Note that all the estimates above are independent of ε, then using the similar arguments as above and letting $\varepsilon \to 0$, there exists a limit function u of u_ε being the solution of problem (1.1). The proof of Theorem 2.4 is completed.

4. Proof of Theorem 2.5

In order to derive the decay property of the solution given by Theorem 2.4, we divide the proof into two cases.
Case 1. Suppose that

$$\int_0^t \left(\|\sqrt{\rho}u'(\tau)\|^2 + \|\nabla u(\tau)\|^2\right) d\tau \le E(t) \tag{4.1}$$

for all $t \ge 0$. In this situation, we first prove two lemmas, then based on them, we complete the proof.

Define

$$m(t) = (\rho u'(t), u(t)) - k\int_0^t [\|\sqrt{\rho}u'(\tau)\|^2 + \|\nabla u(\tau)\|^2] d\tau, \qquad E^\sigma(t) = E(t) + \sigma m(t), \tag{4.2}$$

where $k = \mu_2^2\|K\|_{L^1(0,\infty)} + (p+1)L/2$.

Lemma 4.1. *There exists $K_1 > 0$ such that for each $\sigma > 0$,*

$$|E^\sigma(t) - E(t)| \le \sigma K_1 E(t) \quad \forall t \ge 0. \tag{4.3}$$

Proof. By the definition of B_n and B_∞, it is easy to get $\lim_{n\to\infty} B_n = B_\infty$, and so

$$\lim_{n\to\infty} \lambda_n = \lambda_\infty, \qquad \lim_{n\to\infty} d_n = d_\infty. \tag{4.4}$$

Since $\|\nabla u_{\varepsilon n}(t)\| \le \lambda_n$ and $E_{\varepsilon n}(t) \le d_n$, by letting $n \to \infty$ and $\varepsilon \to 0$, we know that $\|\nabla u(t)\| \le \lambda_\infty$ and $E(t) \le d_\infty$. Therefore, by the conclusion of Proposition 2.2, we know that

$$\|\nabla u(t)\|^2 \le \frac{2(p+1)}{(p-1)L}E(t), \tag{4.5}$$

$$\frac{1}{2}\|\nabla u(t)\|^2 - \int_\Omega G(u(t))\,dx \ge J(u(t)) \ge 0. \tag{4.6}$$

Then by the assumptions (A_3)-(A_4) and the second inequality of (2.3), it is easy to show

$$E(t) \ge \frac{1}{2}\|\sqrt{\rho}u'(t)\|^2 + \frac{L}{2}\|\nabla u(t)\|^2 - \int_\Omega G(u(t))\,dx. \tag{4.7}$$

Applying the Hölder inequality, the Young inequality, (2.3), and (4.5)–(4.7), we have

$$\begin{aligned} |(\rho u'(t), u(t))| &\le \|\sqrt{\rho}\|_\infty \|\sqrt{\rho}u'(t)\|\|u(t)\| \\ &\le \frac{\|\sqrt{\rho}\|_\infty^2}{2}\|\sqrt{\rho}u'(t)\|^2 + \frac{\mu_1^2}{2}\|\nabla u(t)\|^2 \\ &\le \left(\|\sqrt{\rho}\|_\infty^2 + \frac{p+1}{p-1}L^{-1}\mu_1^2\right)E(t). \end{aligned} \tag{4.8}$$

Take $K_1 = k + \|\sqrt{\rho}\|_\infty^2 + ((p+1)/(p-1))L^{-1}\mu_1^2$, we conclude the result. □

Lemma 4.2. *There exist $K_2 > 0$ and $\sigma_1 > 0$ such that*

$$(E^\sigma)'(t) \le -\sigma K_2 E(t) \quad \forall t \ge 0,\ \forall \sigma \in (0,\sigma_1]. \tag{4.9}$$

Proof. Using problem (1.1), we get that

$$\begin{aligned}
m'(t) &= (\rho u''(t),u(t)) + (\rho u'(t),u'(t)) - k\Big[\|\sqrt{\rho}u'(t)\|^2 + \|\nabla u(t)\|^2\Big] \\
&= -(\nabla u'(t),\nabla u(t)) - \|\nabla u(t)\|^2 - (f(u'(t)),u(t))_{\Gamma_1} + (g(u(t)),u(t)) \\
&\quad + \int_0^t K(t-\tau)(h(\tau,u(\tau)),u(t))_{\Gamma_1}d\tau + (\rho u'(t),u'(t)) - k\Big[\|\sqrt{\rho}u'(t)\|^2 + \|\nabla u(t)\|^2\Big] \\
&\le -E(t) + \frac{3}{2}\|\rho\|_\infty\|u'(t)\|^2 - (\nabla u'(t),\nabla u(t)) - \Big(\frac{1}{2}+k\Big)\|\nabla u(t)\|^2 - \int_\Omega G(u(t))dx \\
&\quad - \frac{1}{2}\Big(\int_0^t K(\tau)d\tau\Big)\|u\|_{\Gamma_1}^2 + \frac{1}{2}(K\Box u)(t) - (f(u'(t)),u(t)) \\
&\quad + (g(u(t)),u(t)) + \int_0^t K(t-\tau)(h(\tau,u(\tau)),u(t))_{\Gamma_1}d\tau.
\end{aligned} \tag{4.10}$$

Exploiting assumptions (A_3), (A_4), the Hölder inequality, the Young inequality, and (2.3), we have

$$\begin{aligned}
&\Big|\int_0^t K(t-\tau)(h(\tau,u(\tau)),u(t))_{\Gamma_1}d\tau\Big| \\
&\qquad \le \mu_2^2\|K\|_{L^1(0,\infty)}\|\nabla u(t)\|^2 + \frac{1}{4}\int_0^t K(t-\tau)\|h(\tau,u(\tau))\|_{\Gamma_1}^2 d\tau \\
&\qquad \le \mu_2^2\|K\|_{L^1(0,\infty)}\|\nabla u(t)\|^2 + \frac{1}{2}(K\Box u)(t) + \frac{1}{2}\Big(\int_0^t K(\tau)d\tau\Big)\|u(t)\|_{\Gamma_1}^2,
\end{aligned} \tag{4.11}$$

$$|-(\nabla u'(t),\nabla u(t))| \le \frac{1}{4}\|\nabla u(t)\|^2 + \|\nabla u'(t)\|^2. \tag{4.12}$$

By the assumption (A_2) and noting (4.6), we get

$$|(g(u(t)),u(t))| \le k_3\|u(t)\|_{p+1}^{p+1} \le \frac{(p+1)L}{2}\|\nabla u(t)\|^2. \tag{4.13}$$

Applying the assumption (A_1) with $q = 2$, the Hölder inequality, the trace embedding theorem, and the Young inequality, we have

$$|-(f(u'(t)),u(t))_{\Gamma_1}| \le k_2\|u'(t)\|_{\Gamma_1}\|u(t)\|_{\Gamma_1} \le k_2^2\mu_2^4\|\nabla u'(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^2. \tag{4.14}$$

Combining (4.5), (4.10)–(4.14), and the assumptions (A_2) and (A_4), we get

$$m'(t) \le -E(t)\Big(\frac{3}{2}\|\rho\|_\infty\mu_1^2 + k_2^2\mu_2^4 + 1\Big)\|\nabla u'(t)\|^2 + (K\Box u)(t). \tag{4.15}$$

Moreover, by the assumptions (A_1), (A_3), and (A_4), we get from (3.7) and (3.8) that

$$\begin{aligned} E'(t) &\leq -\|\nabla u'(t)\|^2 + \frac{1}{2}(K'\Box u)(t) \\ &\leq -\|\nabla u'(t)\|^2 - \frac{k_4}{2}(K\Box u)(t). \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$\begin{aligned} (E^\sigma)'(t) = E'(t) + \sigma m'(t) \leq -\sigma E(t) - \left(\frac{k_4}{2} - \sigma\right)(K\Box u)(t) \\ - \left[1 - \sigma\left(\frac{3}{2}\|\rho\|_\infty \mu_1^2 + k_2^2\mu_2^4 + 1\right)\right]\|\nabla u'(t)\|^2. \end{aligned} \tag{4.17}$$

Now we choose σ sufficiently small such that

$$\frac{k_4}{2} - \sigma > 0, \qquad 1 - \sigma\left(\frac{3}{2}\|\rho\|_\infty \mu_1^2 + k_2^2\mu_2^4 + 1\right) > 0. \tag{4.18}$$

Then take $K_2 = \sigma$ in (4.17) to complete the proof. □

Completion of the proof of Case 1. Let $\sigma_0 = \min\{1/(2K_1), \sigma_1\}$. Then for $\sigma \in (0, \sigma_0]$, we know from Lemma 4.1 that

$$\frac{1}{2}E(t) \leq E^\sigma(t) \leq \frac{3}{2}E(t). \tag{4.19}$$

Set $K_3 = \sigma K_2 = \sigma^2$, by Lemma 4.2 and (4.19), it yields

$$(E^\sigma)'(t) \leq -K_3 E(t) \leq -\frac{2}{3}K_3 E^\sigma(t). \tag{4.20}$$

Hence

$$\frac{d}{dt}\left[E^\sigma(t)\exp\left(\frac{2}{3}K_3 t\right)\right] \leq 0 \quad \forall t \geq 0. \tag{4.21}$$

Integrating (4.21) and using (4.19), we have

$$E(t) \leq 3E(0)\exp\left(-\frac{2}{3}K_3 t\right) \quad \forall t \geq 0. \tag{4.22}$$

The proof of Case 1 is completed. □

Case 2. Suppose that there exists a $t_0 \geq 0$ such that

$$\int_0^{t_0} \left(\|\sqrt{\rho}u'(\tau)\|^2 + \|\nabla u(\tau)\|^2\right)d\tau > E(t_0). \tag{4.23}$$

Without loss of generality, we suppose that $t_0 > 0$ and is the first one such that the above inequality holds. This fells out that

$$\int_0^{t_0} \left(||\sqrt{\rho}u'(\tau)||^2 + ||\nabla u(\tau)||^2 \right) d\tau \le E(t) \quad \forall t \in [0, t_0). \tag{4.24}$$

Then along the line of proofs for Case 1, we deduce that (4.22) holds for each $t \in [0, t_0)$.

On the other hand, by (4.5)-(4.6) and the nonincreasing property of $E(t)$, we easily get that for all $t \ge t_0$,

$$\begin{aligned} ||\sqrt{\rho}u'(t)||^2 + ||\nabla u(t)||^2 &\le CE(t) \le CE(t_0) \\ &\le C\int_0^{t_0} \left[||\sqrt{\rho}u'(\tau)||^2 + ||\nabla u(\tau)||^2 \right] d\tau \\ &\le C\int_0^{t} \left[||\sqrt{\rho}u'(\tau)||^2 + ||\nabla u(\tau)||^2 \right] d\tau, \end{aligned} \tag{4.25}$$

where C is a positive constant. Then by the Gronwall inequality and noting that $\rho \not\equiv 0$, we infer from (4.25) that $u = 0$ as $t \ge t_0$. Therefore, the decay property of u is trivial for $t \ge t_0$. Case 2 is proved.

Combining Case 1 and Case 2, we complete the proof Theorem 2.5.

Acknowledgments

This work was supported by the National Natural Science Foundation of China 10471022, and the Ministry of Education of China, Science and Technology Major Projects, Grant 104090. The authors thank the referee for his/her valuable comments and suggestions.

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