CONTINUITY OF MULTILINEAR OPERATORS ON TRIEBEL-LIZORKIN SPACES

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The continuity of some multilinear operators related to certain convolution operators on the Triebel-Lizorkin space is obtained. The operators include Littlewood-Paley operator and Marcinkiewicz operator.

1. Introduction

Let T be the Calderón-Zygmund singular integral operator, a well-known result of Coifman et al. (see [6]) states that the commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in$ BMO) is bounded on $L^p(R^n)$ $(1 < p < \infty)$; Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator; in [8, 9], these results on the Triebel-Lizorkin spaces and the case $b \in \mathrm{Lip}\,\beta$ (where $\mathrm{Lip}\,\beta$ is the homogeneous Lipschitz space) are obtained. The main purpose of this paper is to study the continuity of some multilinear operators related to certain convolution operators on the Triebel-Lizorkin spaces. In fact, we will obtain the continuity on the Triebel-Lizorkin spaces for the multilinear operators only under certain conditions on the size of the operators. As the applications, the continuity of the multilinear operators related to the Littlewood-Paley operator and Marcinkiewicz operator on the Triebel-Lizorkin spaces are obtained.

2. Notations and results

Throughout this paper, Q will denote a cube of R^n with side parallel to the axes, and for a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $1 \le r < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\delta r/n}}\int_Q |f(y)|^r dy\right)^{1/r}, \tag{2.1}$$

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we denote $M_{\delta,r}(f) = M_r(f)$ if $\delta = 0$, which is the Hardy-Littlewood maximal function when $r = 1$ (see [10]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space, and let the Lipschitz space $\dot{\wedge}_\beta$ be the space of functions f such that

$$\|f\|_{\dot{\wedge}_\beta} = \sup_{x,\, h \in R^n,\, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta < \infty}, \tag{2.2}$$

where Δ_h^k denotes the kth difference operator (see [9]).

We are going to study the multilinear operator as follows.

Let m be a positive integer and let A be a function on R^n. We denote

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha. \tag{2.3}$$

Definition 2.1. Let $F(x,t)$ define on $R^n \times [0,+\infty)$, denote

$$\begin{gathered} F_t(f)(x) = \int_{R^n} F(x-y,t) f(y)\,dy, \\ F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} F(x-y,t) f(y)\,dy. \end{gathered} \tag{2.4}$$

Let H be the Hilbert space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||; \tag{2.5}$$

and also define $T(f)(x) = \|F_t(f)(x)\|$.

In particular, consider the following two sublinear operators.

Definition 2.2. Fix $\varepsilon > 0$, $n > \delta \ge 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int \psi(x)\,dx = 0$;

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$;

(3) $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$g_\delta^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{2.6}$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y) f(y)\,dy \tag{2.7}$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Denote that $F_t(f) = \psi_t * f$, and also define that

$$g_\delta(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{2.8}$$

which is the Littlewood-Paley g function when $\delta = 0$ (see [11]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$, then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\delta(f)(x) = ||F_t(f)(x)||, \qquad g_\delta^A(f)(x) = ||F_t^A(f)(x)||. \tag{2.9}$$

Definition 2.3. Let $0 \le \delta < n$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x - y|^\gamma$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\delta^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{2.10}$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y)dy; \tag{2.11}$$

denote

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy, \tag{2.12}$$

and also define that

$$\mu_\delta(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{2.13}$$

which is the Marcinkiewicz operator when $\delta = 0$ (see [12]).

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that

$$\mu_\delta(f)(x) = ||F_t(f)(x)||, \qquad \mu_\delta^A(f)(x) = ||F_t^A(f)(x)||. \tag{2.14}$$

It is clear that Definitions 2.2 and 2.3 are the particular examples of Definition 2.1. Note that when $m = 0$, T^A is just the commutator of F_t and A, while when $m > 0$, it is nontrivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5, 7]). The main purpose of this paper is to study the continuity for the multilinear operators on the Triebel-Lizorkin spaces. We will prove the following theorems in Section 3.

Theorem 2.4. *Let g_δ^A be the multilinear Littlewood-Paley operator as in Definition 2.2. If $0 < \beta < \min(1, \varepsilon)$ and $D^\alpha A \in \dot{\wedge}_\beta$ for $|\alpha| = m$, then*

(a) g_δ^A *maps* $L^p(R^n)$ *continuously into* $\dot{F}_q^{\beta,\infty}(R^n)$, *for* $1<p<n/\delta$ *and* $1/q=1/p-\delta/n$;
(b) g_δ^A *maps* $L^p(R^n)$ *continuously into* $L^q(R^n)$ *for* $1<p<n/(\delta+\beta)$ *and* $1/p-1/q=(\delta+\beta)/n$.

Theorem 2.5. *Let* μ_δ^A *be the multilinear Marcinkiewicz operator as in Definition 2.3. If* $0<\beta<\min(1/2,\gamma)$ *and* $D^\alpha A\in\dot{\Lambda}_\beta$ *for* $|\alpha|=m$, *then*
(a) μ_δ^A *maps* $L^p(R^n)$ *continuously into* $\dot{F}_q^{\beta,\infty}(R^n)$ *for* $1<p<n/\delta$ *and* $1/q=1/p-\delta/n$,
(b) μ_δ^A *maps* $L^p(R^n)$ *continuously into* $L^q(R^n)$ *for* $1<p<n/(\delta+\beta)$ *and* $1/p-1/q=(\delta+\beta)/n$.

3. Main theorem and proof

We first prove a general theorem.

Theorem 3.1 (main theorem). *Let* $0\le\delta<n$, $0<\beta<1$, *and* $D^\alpha A\in\dot{\Lambda}_\beta$ *for* $|\alpha|=m$. *Suppose* F_t, T, *and* T^A *are the same as in Definition 2.1, if* T *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1<p<n/\delta$ *and* $1/q=1/p-\delta/n$, *and* T *satisfies the following size condition:*

$$\left|\left|F_t^A(f)(x)-F_t^A(f)(x_0)\right|\right|\le C\sum_{|\alpha|=m}\left|\left|D^\alpha A\right|\right|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}f(x)\tag{3.1}$$

for any cube Q *with* $\operatorname{supp} f\subset(2Q)^c$ *and* $x\in Q$, *then*
(a) T^A *is bounded from* $L^p(R^n)$ *to* $\dot{F}_q^{\beta,\infty}(R^n)$ *for* $1<p<n/\delta$ *and* $1/q=1/p-\delta/n$,
(b) T^A *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1<p<n/(\delta+\beta)$ *and* $1/q=1/p-(\delta+\beta)/n$.

To prove the theorem, we need the following lemmas.

Lemma 3.2 (see [9]). *For* $0<\beta<1$, $1<p<\infty$,

$$\begin{aligned}\|f\|_{\dot{F}_p^{\beta,\infty}}&\approx\left|\left|\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-f_Q|\,dx\right|\right|_{L^p}\\&\approx\left|\left|\sup_{\cdot\in Q}\inf_c\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-c|\,dx\right|\right|_{L^p}.\end{aligned}\tag{3.2}$$

Lemma 3.3 (see [9]). *For* $0<\beta<1$, $1\le p\le\infty$,

$$\begin{aligned}\|f\|_{\dot{\Lambda}_\beta}&\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-f_Q|\,dx\\&\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|f(x)-f_Q|^p\,dx\right)^{1/p}.\end{aligned}\tag{3.3}$$

Lemma 3.4 (see [1, 2]). *Suppose that* $1\le r<p<n/\delta$ *and* $1/q=1/p-\delta/n$. *Then*

$$\left|\left|M_{\delta,r}(f)\right|\right|_{L^q}\le C\|f\|_{L^p}.\tag{3.4}$$

Lemma 3.5 (see [5]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$\left|R_m(A;x,y)\right| \leq C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha A(z)\right|^q dz\right)^{1/q}, \tag{3.5}$$

where $\tilde{Q}(x,y)$ is the cube centered at x and has side length $5\sqrt{n}|x-y|$.

Proof of Theorem 3.1 (main theorem). Fix a cube $Q = Q(x_0,l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}(1/\alpha!)(D^\alpha A)_{\tilde{Q}}x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
F_t^A(f)(x) &= \int_{R^n}\frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m}F(x-y,t)f(y)dy\\
&= \int_{R^n}\frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m}F(x-y,t)f_2(y)dy\\
&\quad + \int_{R^n}\frac{R_m(\tilde{A};x,y)}{|x-y|^m}F(x-y,t)f_1(y)dy\\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{F(x-y,t)(x-y)^\alpha}{|x-y|^m}D^\alpha\tilde{A}(y)f_1(y)dy,
\end{aligned} \tag{3.6}$$

then

$$\begin{aligned}
\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| &= \left|\|F_t^A(f)(x)\| - \|F_t^{\tilde{A}}(f_2)(x_0)\|\right|\\
&\leq \left\|F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right\|\\
&\quad + \sum_{|\alpha|=m}\frac{1}{\alpha!}\left\|F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right\|\\
&\quad + \left\|F_t^{\tilde{A}}(f_2)(x) - F_t^{\tilde{A}}(f_2)(x_0)\right\| = A(x)+B(x)+C(x),
\end{aligned} \tag{3.7}$$

thus,

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^A(f)(x) - T^{\tilde{A}}(f)(x_0)\right|dx\\
&\quad\leq \frac{1}{|Q|^{1+\beta/n}}\int_Q A(x)dx + \frac{1}{|Q|^{1+\beta/n}}\int_Q B(x)dx\\
&\quad\quad + \frac{1}{|Q|^{1+\beta/n}}\int_Q C(x)dx := I + II + III.
\end{aligned} \tag{3.8}$$

Now, let us estimate I, II, and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemmas 3.3 and 3.5, we get

$$\begin{aligned} |R_m(\tilde{A};x,y)| &\le C|x-y|^m \sum_{|\alpha|=m} \sup_{x\in\tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}| \\ &\le C|x-y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta}, \end{aligned} \tag{3.9}$$

thus, taking r, s such that $1 \le r < p$ and $1/s = 1/r - \delta/n$, by the (L^r, L^s) boundedness of T and Holder' inequality, we obtain

$$\begin{aligned} I &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \frac{1}{|Q|} \int_Q |T(f_1)(x)|\,dx \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||T(f_1)||_{L^s} |Q|^{-1/s} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||f_1||_{L^r} |Q|^{-1/s} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \end{aligned} \tag{3.10}$$

Secondly, using the following inequality (see [9]):

$$||(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{\tilde{Q}}||_{L^r} \le C|Q|^{1/s+\beta/n} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(x), \tag{3.11}$$

and similar to the proof of I, we gain

$$II \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \tag{3.12}$$

For III, using the size condition of T, we have

$$III \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \tag{3.13}$$

We now put these estimates together; and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemmas 3.2 and 3.4, we obtain

$$||T^A(f)||_{\dot{F}_q^{\beta,\infty}} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \| f \|_{L^p}. \tag{3.14}$$

This completes the proof of (a).

(b) By same argument as in proof of (a), we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)|\,dx \\ &\quad \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)), \end{aligned} \tag{3.15}$$

thus,

$$(T^A(f))^\# \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)). \tag{3.16}$$

Now, using Lemma 3.4, we gain

$$\begin{aligned}||T^A(f)||_{L^q} &\le C||(T^A(f))^{\#}||_{L^q}\\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}(||M_{\delta+\beta,r}(f)||_{L^q}+||M_{\delta+\beta,1}(f)||_{L^q})\le C\|f\|_{L^p}.\end{aligned}\tag{3.17}$$

This completes the proof of (b) and the theorem. □

To prove Theorems 2.4 and 2.5, since g_δ and μ_δ are all bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$ (see [11, 12]), it suffices to verify that g_δ^A and μ_δ^A satisfy the size condition in *Theorem 3.1 (main theorem)*.

Suppose $\operatorname{supp} f \subset (2Q)^c$ and $x \in Q = Q(x_0,l)$. Note that $|x_0 - y| \approx |x - y|$ for $y \in (2Q)^c$.

For g_δ^A, we write

$$\begin{aligned}&F_t^{\tilde A}(f)(x) - F_t^{\tilde A}(f)(x_0)\\ &\quad= \int_{R^n\setminus\tilde Q}\left[\frac{\psi_t(x-y)}{|x-y|^m}-\frac{\psi_t(x_0-y)}{|x_0-y|^m}\right]R_m(\tilde A;x,y)f(y)dy\\ &\qquad+\int_{R^n\setminus\tilde Q}\frac{\psi_t(x_0-y)f(y)}{|x_0-y|^m}[R_m(\tilde A;x,y)-R_m(\tilde A;x_0,y)]dy\\ &\qquad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n\setminus\tilde Q}\left[\frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m}\right]D^\alpha\tilde A(y)f(y)dy\\ &\quad= I_1+I_2+I_3.\end{aligned}\tag{3.18}$$

By the condition on ψ, we obtain

$$\begin{aligned}||I_1|| &\le C\int_{R^n\setminus\tilde Q}\frac{|x-x_0|}{|x_0-y|^{m+1}}|R_m(\tilde A;x,y)||f(y)|\left(\int_0^\infty\frac{tdt}{(t+|x_0-y|)^{2(n+1-\delta)}}\right)^{1/2}dy\\ &\quad+C\int_{R^n\setminus\tilde Q}\frac{|x-x_0|^\varepsilon}{|x_0-y|^m}|R_m(\tilde A;x,y)||f(y)|\left(\int_0^\infty\frac{tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}dy\\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}|Q|^{\beta/n}\sum_{k=0}^\infty\int_{2^{k+1}\tilde Q\setminus 2^{k+1}\tilde Q}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f(y)|dy\\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}|Q|^{\beta/n}\sum_{k=1}^\infty(2^{-k}+2^{-k\varepsilon})\left(\frac{1}{|2^k\tilde Q|^{1-\delta/n}}\int_{2^k\tilde Q}|f(y)|dy\right)\\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x).\end{aligned}\tag{3.19}$$

For I_2, by the formula (see [5]):

$$R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \widetilde{A};x,x_0)(x-y)^\eta \tag{3.20}$$

and Lemma 3.5, we get

$$|R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y)| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} |x-x_0| |x_0-y|^{m-1}, \tag{3.21}$$

thus, similar to the proof of I_1,

$$\begin{aligned} ||I_2|| &\le C \int_{R^n\setminus\widetilde{Q}} \frac{|R_m(\widetilde{A};x,y) - R_m(\widetilde{A};x_0,y)|}{|x_0-y|^{m+n-\delta}} |f(y)|\,dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=0}^{\infty} \int_{2^{k+1}\widetilde{Q}\setminus 2^k\widetilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} |f(y)|\,dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \end{aligned} \tag{3.22}$$

For I_3, similar to the proof of I_1, we obtain

$$\begin{aligned} ||I_3|| &\le C \sum_{|\alpha|=m} \int_{R^n\setminus\widetilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f(y)| |D^\alpha \widetilde{A}(y)|\,dy \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} (2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)}) M_{\delta,1}(f)(x) \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x) \end{aligned} \tag{3.23}$$

so that

$$||F_t^{\widetilde{A}}(f)(x) - F_t^{\widetilde{A}}(f)(x_0)|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \tag{3.24}$$

For μ_δ^A, we write

$$\begin{aligned}
&\|F_t^{\widetilde{A}}(f)(x)-F_t^{\widetilde{A}}(f)(x_0)\| \\
&\quad\le\left(\int_0^\infty\left[\int_{|x-y|\le t,\ |x_0-y|>t}\frac{|\Omega(x-y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}}|f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+\left(\int_0^\infty\left[\int_{|x-y|>t,\ |x_0-y|\le t}\frac{|\Omega(x_0-y)|\,|R_m(\widetilde{A};x_0,y)|}{|x_0-y|^{m+n-1-\delta}}|f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{\Omega(x-y)R_m(\widetilde{A};x,y)}{|x-y|^{m+n-1-\delta}}\right.\right.\right. \\
&\qquad\qquad\left.\left.\left.-\frac{\Omega(x_0-y)R_m(\widetilde{A};x_0,y)}{|x_0-y|^{m+n-1-\delta}}\right||f(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2} \\
&\qquad+C\sum_{|\alpha|=m}\left(\int_0^\infty\left|\int_{|x-y|\le t}\left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1-\delta}}-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1-\delta}}\right)\right.\right. \\
&\qquad\qquad\left.\left.\times D^\alpha\widetilde{A}(y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}:=J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.25}$$

Then

$$\begin{aligned}
J_1&\le C\int_{R^n\setminus\widetilde{Q}}\frac{|f(y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy \\
&\le C\int_{R^n\setminus\widetilde{Q}}\frac{|f(y)|\,|R_m(\widetilde{A};x,y)|}{|x-y|^{m+n-1-\delta}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^k\widetilde{Q}|^{1-\delta/n}}\int_{2^k\widetilde{Q}}|f(y)|dy \\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x),
\end{aligned}\tag{3.26}$$

similarly, we have $J_2\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x)$.

For J_3, by the following inequality (see [12]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{m+n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{m+n-1-\delta}}\right|\le C\left(\frac{|x-x_0|}{|x_0-y|^{m+n-\delta}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{m+n-1-\delta+\gamma}}\right),\tag{3.27}$$

we gain

$$J_3 \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot\Lambda_\beta} |Q|^{\beta/n} \int_{R^n\setminus\tilde Q} \left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1-\delta+\gamma}} \right) \times \left(\int_{|x_0-y|\le t,\ |x-y|\le t} \frac{dt}{t^3} \right)^{1/2} |f(y)|\,dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot\Lambda_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} (2^{-k}+2^{-\gamma k}) M_{\delta,1}(f)(x)$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot\Lambda_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \tag{3.28}$$

For J_4, similar to the proof of J_1, J_2, and J_3, we obtain

$$J_4 \le C \sum_{|\alpha|=m} \int_{R^n\setminus\tilde Q} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta}} \right) \times |D^\alpha \tilde A(y)||f(y)|\,dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot\Lambda_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} \left(2^{k(\beta-1)} + 2^{k(\beta-1/2)} + 2^{k(\beta-\gamma)}\right) \frac{1}{|2^k\tilde Q|} \int_{2^k\tilde Q} |f(y)|\,dy$$
$$\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot\Lambda_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x). \tag{3.29}$$

These yield the desired results.

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