# THE OPTIMIZATION FOR THE INEQUALITIES OF POWER MEANS

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Let $M_n^{[t]}(a)$ be the $t$th power mean of a sequence $a$ of positive real numbers, where $a = (a_1, a_2, \dots, a_n), n \geq 2$, and $\alpha, \lambda \in \mathbb{R}^m_{++}, m \geq 2, \sum_{j=1}^m \lambda_j = 1, \min\{\alpha\} \leq \theta \leq \max\{\alpha\}$. In this paper, we will state the important background and meaning of the inequality $\prod_{j=1}^m \{M_n^{[\alpha_j]}(a)\}^{\lambda_j} \leq (\geq) M_n^{[\theta]}(a)$; a necessary and sufficient condition and another interesting sufficient condition that the foregoing inequality holds are obtained; an open problem posed by Wang et al. in 2004 is solved and generalized; a rulable criterion of the semipositivity of homogeneous symmetrical polynomial is also obtained. Our methods used are the procedure of descending dimension and theory of majorization; and apply techniques of mathematical analysis and permanents in algebra.



## 1. Symbols and introduction

We will use some symbols in the well-known monographs [1, 5, 13]:

$A_n = a = (a_1, \dots, a_n)$, $a^\theta = (a_1^\theta, \dots, a_n^\theta)$, $I_n = (1, \dots, 1)$, $O_n = (0, \dots, 0)$,
$\alpha = (\alpha_1, \dots, \alpha_m)$; $\min\{\alpha\} = \min\{\alpha_1, \dots, \alpha_m\}$; $\max\{\alpha\} = \max\{\alpha_1, \dots, \alpha_m\}$,
$\lambda = (\lambda_1, \dots, \lambda_m)$; $\mathbb{R}^n = \{a : a_i \in \mathbb{R}, 1 \leq i \leq n\}$; $\mathbb{R}^n_+ = \{a : a_i \geq 0,\ 1 \leq i \leq n\}$,
$\mathbb{R}^n_{++} = \{a : a_i > 0,\ 1 \leq i \leq n\}$, $\mathbb{Z}^n_+ = \{a \mid a_i \geq 0,\ a_i$ is a integer, $i = 1, 2, \dots, n\}$, $(0,1]^n = \{a : 0 < a_i \leq 1,\ 1 \leq i \leq n\}$, $d \in \mathbb{R}$, $B_d \subset \{\alpha : \alpha \in \mathbb{R}^m,\ \alpha_1 + \cdots + \alpha_m = d\}$, $B_d$ is a finite set, and it is not empty.

Recall that the definitions of the $t$th power mean and Hardy mean of order $r$ for a sequence $a = (a_1, \dots, a_n)$ $(n \geq 2)$ are, respectively,

$$M_n^{[t]}(a) = \left(\frac{1}{n} \cdot \sum_{i=1}^n a_i^t\right)^{1/t}, \quad \text{if } 0 < |t| < +\infty,$$

$$M_n^{[t]}(a) = \sqrt[n]{a_1 a_2 \cdots a_n}, \quad \text{if } t = 0,$$

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$$H_n(a;r) = \left[\frac{1}{n!}\cdot\sum_{i_1,\dots,i_n}\prod_{j=1}^{n}(a_{i_j})^{r_j}\right]^{1/(r_1+\cdots+r_n)}, \quad \text{if } r_1+\cdots+r_n>0,$$
$$H_n(a;r) = \sqrt[n]{a_1a_2\cdots a_n}, \quad \text{if } r_i=0,\ i=1,\dots,n, \tag{1.1}$$

where $t\in\mathbb{R}$, $r\in\mathbb{R}^n_+$, $a\in\mathbb{R}^n_{++}$. And

$$h_n(a;r) = \frac{1}{n!}\cdot\sum_{i_1,\dots,i_n}\prod_{j=1}^{n}a_{i_j}^{r_j}, \quad a\in\mathbb{R}^n_{++},\ r\in\mathbb{R}^n, \tag{1.2}$$

is called Hardy function, where $i_1,\dots,i_n$ is the total permutation of $1,\dots,n$.

*Definition 1.1.* Let $\alpha\in\mathbb{R}^n$, let $\lambda_\alpha$ be a function of $\alpha$, $\lambda_\alpha\in\mathbb{R}$, $x\in\mathbb{R}^n_{++}$. Then the function $f(x)=\sum_{\alpha\in B_d}\lambda_\alpha h_n(x;\alpha)$ is called the generalized homogeneous symmetrical polynomial of $n$ variables and degree $d$. When $B_d\subset Z^n_+$, $f(x)$ is called the homogeneous symmetrical polynomial of $n$ variables and degree $d$, simply, homogeneous symmetrical polynomial (see [24, page 431]).

*Definition 1.2.* Let $a_{ij}$ be the complex numbers, $i,j=1,2,\dots,n$, and let the matrix $A=(a_{ij})_{n\times n}$ be an $n\times n$ matrix. Then the permanent (of order $n$) of $A$ is a function of matrix, written $\operatorname{per}A$, it is defined by

$$\operatorname{per}A = \sum_{\sigma}a_{1\sigma_1}a_{2\sigma_2}\cdots a_{n\sigma_n}, \tag{1.3}$$

where the summation extends over all one-to-one functions from $1,\dots,n$ to $1,\dots,n$. (See [12].) It is often convenient in the proof of Lemma 2.2 and Corollary 2.6 that we will also apply a symbol similar to determinant as follows:

$$\operatorname{per}A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}^{+}_{n}. \tag{1.4}$$

It should be noted that the permanent remains some properties of the common determinant, but both of them are different. For example, for the common determinant, we have "the determinant changes sign if two adjacent rows are interchanged." But the affirmative proposition and its corollaries do not hold for permanents.

As pointed out in [1], the theory of inequalities plays an important role in all the fields of mathematics. And the power mean is the most important one in all the means. Many mathematicians wrote a great number of papers, and established the inequalities involving the power means and the related problems (see, e.g., [1, 4, 5, 8, 9, 13, 14, 17–19, 21]). Recently, the authors studied the optimal real number $\lambda$ such that the following inequality:

$$\left\{M_n^{[\alpha]}(a)\right\}^{1-\lambda}\left\{M_n^{[\beta]}(a)\right\}^{\lambda} \le M_n^{[\theta]}(a) \tag{1.5}$$

or its converse holds, where $a \in \mathbb{R}^n_{++}$, $0 < \alpha < \theta < \beta$, $\lambda \in \mathbb{R}$.

The optimal concepts are multifarious and versatile in mathematics (e.g., see [8, 19, 20]). Although it is so, the true worth for inequalities is as follows: if an inequality includes some parameters, we study that these parameters should satisfy some necessary and sufficient conditions such that this inequality holds, then we call that the inequality is optimized. In this paper, we want to discuss the following optimal problems that are more general inequalities than inequality (1.5).

Let $a \in \mathbb{R}^n_{++}$, $n \ge 2$, $\alpha,\lambda \in \mathbb{R}^m_{++}$, $m \ge 2$, $\sum_{j=1}^m \lambda_j = 1$, $\min\{\alpha\} \le \theta \le \max\{\alpha\}$. A natural problem is the following: what are the necessary and sufficient conditions such that the inequalities

$$\prod_{j=1}^m \left\{M_n^{[\alpha_j]}(a)\right\}^{\lambda_j} \le M_n^{[\theta]}(a), \tag{1.6}$$

$$\prod_{j=1}^m \left\{M_n^{[\alpha_j]}(a)\right\}^{\lambda_j} \ge M_n^{[\theta]}(a) \tag{1.7}$$

hold, respectively?

Assume that the components of $a$ are complex numbers. Then inequality (1.6) (or (1.7)) can be expressed as

$$\prod_{j=1}^m \left(\|a\|_{\alpha_j}\right)^{\lambda_j} \le (\ge) C \cdot \|a\|_\theta, \tag{1.8}$$

where $\|a\|_p = \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}$, $p > 1$ is the norm of $a$, and $C = n^{\sum_{j=1}^m (\lambda_j/\alpha_j) - 1/\theta}$.

Let $1/p_j + 1/\alpha_j = 1$, $1/p + 1/\theta = 1$, $p > 1$, $p_j > 1$, $1 \le j \le m$. If $f$ is a bounded linear functional on $\bigcap_{j=1}^m L^{p_j}[a,b] \bigcap Ł^p[a,b]$. From the well-known theorem (see, e.g., [26]), there exists a unique function $y(t) \in \bigcap_{j=1}^m L^{p_j}[a,b] \bigcap Ł^p[a,b]$, such that $f(x) = \int_a^b x(t)y(t)dt$. Thus we have

$$\|f\|_{\alpha_j} = \|y\|_{\alpha_j} = \left(\int_a^b |y(t)|^{\alpha_j} dt\right)^{1/\alpha_j}, \qquad \|f\|_\theta = \|y\|_\theta = \left(\int_a^b |y(t)|^\theta dt\right)^{1/\theta}. \tag{1.9}$$

By the above facts, inequality (1.8) can also be expressed as

$$\prod_{j=1}^m \left(\|f\|_{\alpha_j}\right)^{\lambda_j} \le (\ge) C \cdot \|f\|_\theta, \tag{1.10}$$

where $C = (b-a)^{\sum_{j=1}^m (\lambda_j/\alpha_j) - 1/\theta}$.

Based on the above-mentioned definitions and the related depictions, an open problem posed in [19] and others, which will be solved in this paper, are significative. We obtain not only a necessary and sufficient condition, but also an interesting sufficient condition such that inequality (1.6) holds. Note that the inequalities (1.6), (1.8), and (1.10) play some roles in the geometry of convex body (see, e.g., [3, 7]). Our methods are, of late years, the approach of descending dimension and theory of majorization; and apply some techniques of mathematical analysis and permanents [12] in algebra. Note that the way of descending dimension used in this paper is different from [15, 23, 25]; and the majorization is an effective theory that "it can state the inwardness and the relation between the quantities" (see [4, 11, 16]). It is very interesting that the mathematical analysis and permanent can skillfully be combined.

## 2. The background of inequality (1.6)

The following theorem can display the background and meaning of inequality (1.6).

Theorem 2.1. *Let* $f(x)=\sum_{\alpha\in B_d}\lambda_\alpha h_n(x;\alpha)$ $(x\in\mathbb{R}^n_{++})$ *be a generalized homogeneous symmetrical polynomial of n variables and degree d, where* $d>0$, $B_d\subset\mathbb{R}^n_+$, $\lambda_\alpha>0$ *(for all* $\alpha\in B_d$*),* $\lambda=d^{-1}\cdot\alpha$, $\min\{\alpha\}\le\theta\le\max\{\alpha\}$ *(for all* $\alpha\in B_d$*). If, for arbitrary* $\alpha\in B_d$, $x\in\mathbb{R}^n_{++}$*, the inequality*

$$\prod_{j=1}^{n}\{M_n^{[\alpha_j]}(x)\}^{\lambda_j}\le M_n^{[\theta]}(x) \tag{2.1}$$

*holds, then, for arbitrary* $x\in\mathbb{R}^n_{++}$,

$$\left[\frac{f(x)}{f(I_n)}\right]^{1/d}\le M_n^{[\theta]}(x). \tag{2.2}$$

*In particular, if* $0<\theta\le 1$*, and the measurable set* $G\subset\Omega_n:=\{x\mid\sum_{i=1}^{n}x_i\le n,\ x\in\mathbb{R}^n_+\}$*, then, for arbitrary real number* $\delta>0$,

$$\int_G\left[\frac{f(x)}{f(I_n)}\right]^{\delta}dx\le\frac{n^n}{n!}, \tag{2.3}$$

*where* $dx=dx_1dx_2\cdots dx_n$.

Lemma 2.2. *If* $0\le a_{i1}\le a_{i2}\le\cdots\le a_{in}$, $i=1,2,\dots,n$*, then*

$$\frac{1}{n!}\cdot\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}_n^{+}\le\prod_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n}a_{ij}. \tag{2.4}$$

*Proof.* We will prove the general case by the induction for $m$,

$$\frac{1}{n!}\cdot\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}_n^+ \leq \prod_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n}a_{ij}. \tag{2.5}$$

All the elements of $n-m$ rows in the above permanent are 1.

When $m=1$, then the sign of equality is valid in (2.5). Assume that $m=2$ below. We delete the element at $i$th row and $j$th column from the permanent per$A$, then we construct a permanent of order $n-1$, and it is called cofactor of $a_{ij}$ and is denoted by $M_{ij}$. Note the following identities and inequalities:

$$\begin{gathered}\frac{1}{(n-1)!}M_{1j}=\frac{1}{n-1}\sum_{1\leq k\leq n,k\neq j}a_{2k}=\frac{1}{n-1}\left[\left(\sum_{k=1}^{n}a_{2k}\right)-a_{2j}\right],\\ \frac{1}{(n-1)!}M_{11}\geq\frac{1}{(n-1)!}M_{12}\geq\cdots\geq\frac{1}{(n-1)!}M_{1n},\qquad a_{11}\leq a_{12}\leq\cdots\leq a_{1n}.\end{gathered}\tag{2.6}$$

Therefore, the expansion of the permanent of the left-hand side of (2.5) in terms of elements of the first row is given by

$$\begin{aligned}\text{the left-hand side of (2.5)} &= \frac{1}{n}\cdot\sum_{j=1}^{n}a_{1j}\cdot\frac{1}{(n-1)!}M_{1j}\\ &\leq\left[\frac{1}{n}\cdot\sum_{j=1}^{n}a_{1j}\right]\left[\frac{1}{n}\cdot\sum_{j=1}^{n}\frac{1}{(n-1)!}M_{1j}\right]=\prod_{i=1}^{2}\frac{1}{n}\sum_{j=1}^{n}a_{ij},\end{aligned}\tag{2.7}$$

where we used Čebyšev's inequality.

Assume that the elements in the left-hand side of (2.5) are not all 1, and the count of these rows is equal to $m-1$ $(m\geq 3)$, inequality (2.5) holds. We will prove that inequality (2.5) holds as follows.

First we prove that inequalities (2.6) hold still.

Note that the expansion of permanent $M_{1j}$ in terms of elements of the first column is given by

$$M_{11}=\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}_{n-1}^+ =\sum_{i=2}^{n}a_{i2}M_{i2}^{*},\quad a_{ij}=1,\quad (m+1\leq i\leq n,\ 1\leq j\leq n). \tag{2.8}$$

Similarly,

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}_{n-1}^{+} = \sum_{i=2}^{n} a_{i1} M_{i1}^*, \quad a_{ij} = 1, \quad m+1 \le i \le n,\ 1 \le j \le n. \tag{2.9}$$

Since $M_{i1}^* = M_{i2}^* > 0$, $(i = 2,3,\ldots,n)$, therefore, $M_{11} - M_{12} = \sum_{i=2}^{n}(a_{i2} - a_{i1})M_{i1}^* \ge 0$, namely,

$$\frac{1}{(n-1)!}M_{11} \ge \frac{1}{(n-1)!}M_{12}. \tag{2.10}$$

Similarly,

$$\frac{1}{(n-1)!}M_{12} \ge \frac{1}{(n-1)!}M_{13} \ge \cdots \ge \frac{1}{(n-1)!}M_{1n}. \tag{2.11}$$

Thus, the first chain in (2.6) is proven; and the second chain of (2.6) is given.

By inequality (2.6) and Čebyšev's inequality, we obtain that

$$\begin{aligned} \text{the left-hand side of (2.5)} &= \frac{1}{n} \cdot \sum_{j=1}^{n} a_{1j} \cdot \frac{1}{(n-1)!}M_{1j} \\ &\le \left[\frac{1}{n} \cdot \sum_{j=1}^{n} a_{1j}\right]\left[\frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{(n-1)!}M_{1j}\right]. \end{aligned} \tag{2.12}$$

It is noteworthy that the sign of equality of (2.12) is valid when $a_{11} = a_{12} = \cdots = a_{1n} = 1$. If we change two rows (columns) in permanent, then permanent keeps invariable, then, from the assumption of the induction, we get

$$\frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{(n-1)!}M_{1j} = \frac{1}{n!} \cdot \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{mn} \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}_{n}^{+} \le \prod_{i=2}^{m} \frac{1}{n} \cdot \sum_{j=1}^{n} a_{ij}. \tag{2.13}$$

From inequalities (2.12) and (2.13), we obtain inequality (2.5).

Letting $m = n$ in (2.5), we get inequality (2.4). So the proof is complete. □

Lemma 2.3. *If $x \in \mathbb{R}_{++}^n$, $\alpha \in B_d \subset \mathbb{R}_+^n$, $d > 0$, $\lambda = d^{-1}\alpha$, $n \ge 2$, then*

$$H_n(x;\alpha) \le \prod_{j=1}^{n} \{M_n^{[\alpha_j]}(x)\}^{\lambda_j}. \tag{2.14}$$

*Proof.* Just as well assume that $0 < x_1 \le x_2 \le \cdots \le x_n$, then

$$0 < x_1^{\alpha_i} \le x_2^{\alpha_i} \le \cdots \le x_n^{\alpha_i}, \quad i = 1,2,\dots,n. \tag{2.15}$$

Thus, by the definition of permanent and by Lemma 2.2, we obtain that

$$H_n(x;\alpha) \le \left[\prod_{i=1}^{n} \frac{1}{n} \cdot \sum_{j=1}^{n} x_j^{\alpha_i}\right]^{1/d} = \prod_{j=1}^{n} \{M_n^{[\alpha_j]}(x)\}^{\lambda_j}. \tag{2.16}$$

□

*Proof of Theorem 2.1.* By Lemma 2.3, we observe that

$$\begin{aligned}\left[\frac{f(x)}{f(I_n)}\right]^{1/d} &= \left[\frac{\sum_{\alpha\in B_d} \lambda_\alpha (H_n(x;\alpha))^d}{\sum_{\alpha\in B_d}\lambda_\alpha}\right]^{1/d} \le \left[\frac{\sum_{\alpha\in B_d} \lambda_\alpha \prod_{j=1}^{n} (M_n^{[\alpha_j]}(x;\alpha))^{d\lambda_j}}{\sum_{\alpha\in B_d}\lambda_\alpha}\right]^{1/d} \\ &\le \left[\frac{\sum_{\alpha\in B_d} \lambda_\alpha (M_n^{[\theta]}(x;\alpha))^d}{\sum_{\alpha\in B_d}\lambda_\alpha}\right]^{1/d} = M_n^{[\theta]}(x;\alpha).\end{aligned} \tag{2.17}$$

Clearly, $[f(x)/f(I_n)]^{\delta}$ is integrable on $G$. Therefore, by inequality (2.2), we obtain that

$$\begin{aligned}\int_G \left[\frac{f(x)}{f(I_n)}\right]^{\delta} dx &\le \int_{\Omega_n} \left[\frac{f(x)}{f(I_n)}\right]^{\delta} dx \le \int_{\Omega_n} \{M_n^{[\theta]}(x)\}^{\delta d} dx \\ &\le \int_{\Omega_n} \{M_n^{[1]}(x)\}^{\delta d} d\mu \le \int_{\Omega_n} 1^{\delta d} dx = \int_{\Omega_n} dx = \frac{n^n}{n!}.\end{aligned} \tag{2.18}$$

□

*Remark 2.4.* The literature [6] generalizes the well-known Hardy inequality

$$\alpha \prec \beta \Longrightarrow h_n(x;\alpha) \le h_n(x;\beta) \tag{2.19}$$

to the convex functions, where $x \in \mathbb{R}_{++}^n$, $\alpha, \beta \in \mathbb{R}^n$; [24] generalizes the well-known Čebyšev inequality to the generalized homogeneous symmetrical polynomial; [22] studied a necessary and sufficient condition such that

$$H_n(x;\alpha) \le H_n(x;\beta) \tag{2.20}$$

holds.

*Remark 2.5.* Lemma 2.2 is an important theorem. We can deduce an interesting conclusion from this fact as follows.

Corollary 2.6. *Let $f(x) = \sum_{\alpha\in B_d} \lambda_\alpha h_n(x;\alpha)$ ($x \in \mathbb{R}_{++}^n$) be a generalized homogeneous symmetrical polynomial of $n$ variables and degree $d$. If $d > 0$, $B_d \subset \mathbb{R}_+^n$, $\lambda_\alpha > 0$ (for all $\alpha \in B_d$), $[1/f(x)]^{\delta}$ is integrable on measurable set $G$, $(0,1]^n \subset G \subset \mathbb{R}_{++}^n$, then, for arbitrary* *real number* $\delta > 0$,

$$\int_G \left[\frac{f(I_n)}{f(x)}\right]^\delta dx \geq \left\{\int_0^1 \left[\frac{n}{t^d+n-1}\right]^\delta dt\right\}^n, \tag{2.21}$$

*where* $dx = dx_1 dx_2 \cdots dx_n$.

*Proof.* For all $\alpha \in B_d$, just as well assume that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0. \tag{2.22}$$

Then, when $x \in (0,1]^n$, we have

$$0 < x_j^{\alpha_1} \leq x_j^{\alpha_2} \leq \cdots \leq x_j^{\alpha_n}, \quad j = 1,2,\ldots,n. \tag{2.23}$$

From Lemma 2.2, we get

$$h_n(x;\alpha) = \frac{1}{n!} \cdot \begin{vmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{vmatrix}_n^+ \leq \prod_{i=1}^n \frac{1}{n} \cdot \sum_{j=1}^n x_i^{\alpha_j}. \tag{2.24}$$

Since the exponential function $c^t$ $(c > 0)$ is a convex function on $\mathbb{R}$, therefore, by [16, page 59], we observe that $1/n \cdot \sum_{j=1}^n x_i^{\alpha_j}$ is a Schur-convex function of $\alpha$ on $\mathbb{R}_+^n$. For all $\alpha \in \mathbb{R}_+^n$, $\alpha \prec (\sum_{j=1}^n \alpha_j, O_{n-1}) = (d, O_{n-1})$ from which [16, page 54], we conclude that

$$g(\alpha) := \frac{1}{n} \cdot \sum_{j=1}^n x_i^{\alpha_j} \leq g(d, O_{n-1}) = \frac{x_i^d + n - 1}{n}, \tag{2.25}$$

$$\begin{aligned} \frac{f(I_n)}{f(x)} &= \frac{\sum_{\alpha\in B_d} \lambda_\alpha}{\sum_{\alpha\in B_d} \lambda_\alpha h_n(x;\alpha)} \\ &\geq \frac{\sum_{\alpha\in B_d} \lambda_\alpha}{\sum_{\alpha\in B_d} \lambda_\alpha \prod_{i=1}^n (x_i^d + n - 1)/n} = \prod_{i=1}^n \frac{n}{x_i^d + n - 1}, \end{aligned}$$

$$\begin{aligned} \int_G \left(\frac{f(I_n)}{f(x)}\right)^\delta dx &\geq \int_{(0,1]^n} \left(\frac{f(I_n)}{f(x)}\right)^\delta dx \geq \int_{(0,1]^n} \left(\prod_{i=1}^n \frac{n}{x_i^d + n - 1}\right)^\delta dx \\ &= \int_{(0,1]^n} \prod_{i=1}^n \left(\frac{n}{x_i^d + n - 1}\right)^\delta dx = \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{i=1}^n \left(\frac{n}{x_i^d + n - 1}\right)^\delta dx_1 dx_2 \cdots dx_n \\ &= \prod_{i=1}^n \int_0^1 \left(\frac{n}{x_i^d + n - 1}\right)^\delta dx_i = \left[\int_0^1 \left(\frac{n}{t^d + n - 1}\right)^\delta dt\right]^n. \end{aligned} \tag{2.26}$$

In Section 1 through Section 2, these pioneer studies that the authors attempted would demonstrate that these results of this paper occupy some important positions in the theory of inequalities, as well as they are often used in several function spaces. □

## 3. A necessary and sufficient condition that inequality (1.6) holds

We have known from Section 2 that investigation that inequalities (1.6) and (1.7) hold has considerable meaning. In this section, we will discuss how to transform inequality (1.6) into an inequality involving fewer variables so that there is a possibility that inequality (1.6) can be proven by means of mathematical software.

Theorem 3.1. *Let* $a \in \mathbb{R}^n_{++}$, $\alpha,\lambda \in \mathbb{R}^m_{++}$, $n \geq m \geq 2$, $\sum_{j=1}^m \lambda_j = 1$, $\min\{\alpha\} \leq \theta \leq \max\{\alpha\}$. *Then, a necessary and sufficient condition such that inequality (1.6) holds is that inequality*

$$\prod_{j=1}^{m} \{M_n^{[\alpha_j]}(A_{m-1},I_k,O_{n-m-k+1})\}^{\lambda_j} \leq M_n^{[\theta]}(A_{m-1},I_k,O_{n-m-k+1}) \tag{3.1}$$

*holds for all the* $A_{m-1} = (a_1,a_2,\dots,a_{m-1}) \in \mathbb{R}^{m-1}_{++}$, $k = 0,1,2,\dots,n-m+1$.

Lemma 3.2. *Let*

$$u(t) = \sum_{j=0}^{m} a_j t^{r_j}, \quad a_j \in \mathbb{R} - \{0\},\ r_j \in \mathbb{R},\ j = 1,2,\dots,m,\ m \geq 1,\ r_0 = 0,\ a_0 \in \mathbb{R},\ t \in \mathbb{R}^1_{++}, \tag{3.2}$$

*be a common polynomial of one variable. Then* $u(t)$ *has at most* $m$ *zeroes on* $\mathbb{R}^1_{++}$, *that is, the count of elements of the set* $U_m = \{t \mid u(t) = 0,\ t > 0\}$ *is* $|U_m|$, *where* $|U_m| \leq m$.

*Proof.* We will prove by means of the induction for $m$.

When $m = 1$, the conclusion is clear. Assume that when $1 \leq k \leq m-1$ $(m \geq 2)$, the inequality $|U_k| \leq k$ holds. We will prove that $|U_m| \leq m$ holds as follows. We can assume $r_m > r_{m-1} > \cdots > r_1$, $r_j \neq 0$, $j = 1,2,\dots,m$, then

$$\begin{gathered} u'(t) = \sum_{j=0}^{m} r_j a_j t^{r_j - 1} = t^{r_1 - 1} \cdot \sum_{j=1}^{m} r_j a_j t^{r_j - r_1}, \\ (r_j a_j \in \mathbb{R} - \{0\},\ j = 1,2,\dots,m,\ m \geq 2,\ t \in \mathbb{R}^1_{++}). \end{gathered} \tag{3.3}$$

Based on the assumption of induction, the common polynomial $\sum_{j=1}^m r_j a_j t^{r_j - r_1}$ has at most $m-1$ zeroes on $\mathbb{R}^1_{++}$. Since $t^{r_1 - 1} > 0$, therefore $u'(t)$ has at most $m-1$ zeroes on $\mathbb{R}^1_{++}$, $u(t)$ has at most $m-1$ extreme points on $\mathbb{R}^1_{++}$. Let all the extreme points of $u(t)$ on $\mathbb{R}^1_{++}$ be

$$t_1,t_2,\dots,t_p, \quad t_1 < t_2 < \cdots < t_p,\ 0 \leq p \leq m-1. \tag{3.4}$$

If $p = 0$, then $u(t)$ is a monotonic function on $\mathbb{R}^1_{++}$. We may assume that $u(t)$ is a increasing function on $\mathbb{R}^1_{++}$. We will prove that $u(t)$ is a strictly increasing function on $\mathbb{R}^1_{++}$ as follows.

Let $0 < t_1' < t_2'$, then $u(t_1') \le u(t_2')$. If $u(t_1') = u(t_2')$, then for all $t \in [t_1', t_2']$, $u(t_1') \le u(t) \le u(t_2') = u(t_1')$, $u(t) \equiv u(t_1')$, $u'(t) \equiv 0$. Thus, for all $t \in [t_1', t_2']$, $t$ is the zero of $u'(t)$ on $\mathbb{R}^1_{++}$. This contradicts with $u'(t)$ which has $m-1$ zeroes on $\mathbb{R}^1_{++}$. Therefore, $u(t_1') < u(t_2')$, $u(t)$ is a strictly increasing function on $\mathbb{R}^1_{++}$. Based on these facts, the count of the zeroes of $u(t)$ on $\mathbb{R}^1_{++}$ is $|U_m| \le 1 \le m$.

If $p \ge 1$, we can assert by using the above method that $u(t)$ is a strictly monotonic function on each of the following $p+1$ intervals: $(0,t_1],[t_1,t_2],\dots,[t_p,+\infty)$. And the number of zeroes of $u(t)$ is at most 1 on each of these intervals, then the amount of zeroes of $u(t)$ on $\mathbb{R}^1_{++}$ is $|U_m| \le p+1 \le m$. This ends the proof of Lemma 3.2. □

Lemma 3.3. *Let $A_n = a \in \mathbb{R}^n_+$, $\alpha,\lambda \in \mathbb{R}^m_{++}$, $n \ge m \ge 2$. $F(a)$ denotes $\prod_{j=1}^m \{M_n^{[\alpha_j]}(a)\}^{\lambda_j}$. If $A_q$ is a critical point of $F(A_q,O_{n-q})$ (for all $q : m \le q \le n$) on $D_q := \{A_q \mid \sum_{r=1}^q a_r = q,\ A_q \in \mathbb{R}^q_+\}$, then $a_1,a_2,\dots,a_q$ satisfying $r \le m$, where $m$ denotes largest number of the pair $(a_i,a_j)$ with $a_i \ne a_j$, $i<j$, for $i,j = 1,2,\dots,q$, that is, the amount of the elements in the set $\{a_1,a_2,\dots,a_q\}$ is $|\{a_1,a_2,\dots,a_q\}|$, and $|\{a_1,a_2,\dots,a_q\}| \le m$.*

*Proof.* Make the Lagrange function $L = F(A_q,O_{n-q}) + \mu(\sum_{r=1}^q a_r - q)$. Then $A_q$ is a critical point of $F(A_q,O_{n-q})$ on the domain $D_q$ if and only if $\partial L/\partial a_k = \partial F(A_q,O_{n-q})/(\partial a_k) + \mu = 0$, $k = 1,2,\dots,q$, and $A_q \in D_q$. Since $\ln F(A_q,O_{n-q}) = \sum_{j=1}^m \ln\{M_n^{[\alpha_j]}(A_q,O_{n-q})\}^{\lambda_j} = \sum_{j=1}^m (\lambda_j/\alpha_j)\ln(\sum_{r=1}^q a_r^{\alpha_j}/n)$, therefore,

$$\frac{\partial F(A_q,O_{n-q})}{\partial a_k}\left[F(A_q,O_{n-q})\right]^{-1} = \sum_{j=1}^m \frac{\lambda_j}{\alpha_j}\cdot\frac{\alpha_j\cdot a_k^{\alpha_j-1}}{\sum_{r=1}^q a_r^{\alpha_j}} = \sum_{j=1}^m \frac{\lambda_j}{\sum_{r=1}^q a_r^{\alpha_j}}\cdot a_k^{\alpha_j-1}. \tag{3.5}$$

Let $A_q$ be a critical point of $F(A_q,O_{n-q})$ on the domain $D_q$. We take the auxiliary function as follows:

$$\begin{gathered} u(t) = \sum_{j=0}^m b_j t^{r_j} : b_j = \frac{\lambda_j\cdot F(A_q,O_{n-q})}{\sum_{r=1}^q a_r^{\alpha_j}} \ne 0, \\ r_j = \alpha_j - 1 \in \mathbb{R}, \quad j = 1,2,\dots,m, \qquad b_0 = \mu. \end{gathered} \tag{3.6}$$

Then

$$\begin{gathered} \frac{\partial L}{\partial a_k} = 0, \quad k = 1,2,\dots,q \Longleftrightarrow u(a_k) = 0, \\ a_k \in U_m = \{t \mid u(t) = 0,\ t > 0\}, \quad k = 1,2,\dots,q \Longleftrightarrow \{a_1,a_2,\dots,a_q\} \subset U_m. \end{gathered} \tag{3.7}$$

By Lemma 3.2, we obtain that $|\{a_1,a_2,\dots,a_q\}| \le |U_m| \le m$. Lemma 3.3 is thus proved. □

*Proof of Theorem 3.1. Necessity.* If inequality (1.6) holds, in (1.6), we put that $a = (A_{m-1},I_k,O_{n-m-k+1})$, (for all $k : 0 \le k \le n-m+1$), then, (1.6) reduces to (3.1), thus (3.1) holds.

*Sufficiency.* Assume that (3.1) holds. We will prove that inequality (1.6) holds.

Note that we will prove a more general conclusion, that is,

$$\prod_{j=1}^{m}\left\{M_n^{[\alpha_j]}(A_q,O_{n-q})\right\}^{\lambda_j} \le M_n^{[\theta]}(A_q,O_{n-1}), \quad \forall q: m\le q\le n,\ \forall A_q\in\mathbb{R}_+^q. \tag{3.8}$$

First we prove a special case $\theta=1$. Since both sides of (3.8) are a linear homogeneous function of $A_q$, therefore we may assume that $A_q\in D_q:=\{A_q\mid \sum_{r=1}^{q}a_r=q,\ A_q\in\mathbb{R}_+^q\}$. Thus, inequality (3.8) is equivalent to

$$F(A_q,O_{n-q})\le\frac{q}{n}, \quad \forall q: m\le q\le n, \forall A_q\in D_q, \tag{3.9}$$

where the definitions of $F(a)$ and $D_q$ are in Lemma 3.3. We can prove that (3.9) holds for $q$ by the induction.

First we prove that (3.9) holds for the case $q=m$. If $a_m=0$, from (3.1), we get

$$F(A_q,O_{n-q})=F(A_{m-1},I_0,O_{n-m+1})\le M_n^{[1]}(A_{m-1},I_0,O_{n-m+1})=\frac{q}{n}, \tag{3.10}$$

therefore (3.9) holds. Let $a_m>0$ below. Taking $k=1$ in (3.1), we have

$$\prod_{j=1}^{m}\{M_n^{[\alpha_j]}(A_{m-1},1,O_{n-m})\}^{\lambda_j}\le M_n^{[1]}(A_{m-1},1,O_{n-m}). \tag{3.11}$$

Replacing $A_{m-1}$ by $A_{m-1}/a_m$ in (3.11), we obtain that

$$\prod_{j=1}^{m}\left\{M_n^{[\alpha_j]}\left(\frac{A_{m-1}}{a_m},1,O_{n-m}\right)\right\}^{\lambda_j}\le M_n^{[1]}\left(\frac{A_{m-1}}{a_m},1,O_{n-m}\right). \tag{3.12}$$

Multiplying both sides of (3.12) by $a_m$, then (3.12) reduces to (3.8), thus (3.9) holds.

Assume that we replace $q$ by $q-1(m+1\le q\le n)$ in (3.9), we have (3.9). We will prove that (3.9) holds as follows. From the continuity and differentiability of $F(A_q,O_{n-q})$ on $D_q$, we just have to prove that for the critical point $A_q$ of $F(A_q,O_{n-q})$ on $D_q$, for the point $A_q$ on the boundary of $D_q$, (3.9) holds still.

*Case 1.* If $A_q$ is a critical point of $F(A_q,O_{n-q})$ on $D_q$, from Lemma 3.3, we know that the amount of unequal terms of $a_1,a_2,\dots,a_q$ is at most $m$.

By the symmetry, we may assume that $a_m=a_{m+1}=\cdots=a_q>0$. Thus, taking $k=q-m+1$ in (3.1), we obtain that

$$\begin{aligned}F(A_q,O_{n-q})&=F(A_{m-1},a_m\cdot I_{q-m+1},O_{n-q})=a_m\cdot F\left(\frac{A_{m-1}}{a_m},I_{q-m+1},O_{n-q}\right)\\&\le a_m\cdot M_n^{[1]}\left(\frac{A_{m-1}}{a_m},I_{q-m+1},O_{n-q}\right)=M_n^{[1]}(A_q,O_{n-q})=\frac{q}{n}.\end{aligned} \tag{3.13}$$

In other words, (3.9) holds.

*Case 2.* Let $A_q$ be a point on the boundary of $D_q$. Then there exists a term in $a_1, a_2, \ldots, a_q$, this term must be zero. We may assume that $a_q = 0$. From $A_q \in D_q$, $a_1 + a_2 + \cdots + a_{q-1} = q$, $((q-1)/q) \cdot a_1 + (q-1)/q \cdot a_2 + \cdots + ((q-1)/q) \cdot a_{q-1} = q-1$, therefore, if we take $X_{q-1} = ((q-1)/q) \cdot A_{q-1}$, then $A_{q-1} = q/(q-1) \cdot X_{q-1}$, $X_{q-1} \in D_{q-1}$. Thus, by the assumption of induction, we obtain that

$$\begin{aligned} F(A_q, O_{n-q}) &= F(A_{q-1}, O_{n-q+1}) = F\left(\frac{q}{q-1} \cdot X_{q-1}, O_{n-q+1}\right) \\ &= \frac{q}{q-1} \cdot F(X_{q-1}, O_{n-q+1}) \\ &\le \frac{q}{q-1} \cdot M_n^{[1]}(X_{q-1}, O_{n-q+1}) \\ &= \frac{q}{q-1} \cdot \frac{q-1}{n} = \frac{q}{n}. \end{aligned} \tag{3.14}$$

Based on the principle of induction, (3.9) has been proven.

Second, we will prove the general case $\theta \neq 1$. Letting

$$\begin{aligned} a^\theta &= (a_1^\theta, a_2^\theta, \ldots, a_n^\theta) = y = (y_1, y_2, \ldots, y_n) \\ &\in \mathbb{R}_+^n \Longleftrightarrow a = (y_1^{1/\theta}, y_2^{1/\theta}, \ldots, y_n^{1/\theta}) \in \mathbb{R}_+^n, \end{aligned} \tag{3.15}$$

then inequality (3.8) is equivalent to

$$\prod_{j=1}^{m} \{M_n^{[\alpha_j/\theta]}(Y_q, O_{n-q})\}^{\lambda_j} \le M_n^{[1]}(Y_q, O_{n-q}), \quad \forall q : m \le q \le n, \forall A_q \in \mathbb{R}_+^q. \tag{3.16}$$

Since $\alpha/\theta \in \mathbb{R}_{++}^n$, $\min\{\alpha/\theta\} \le 1 \le \max\{\alpha/\theta\}$, inequality (3.16) reduces to the case $\theta = 1$, therefore inequality (3.16) holds.

Summarizing the above mentioned, inequality (3.8) has been proven. Taking $q = n$ in inequality (3.8), we obtain inequality (1.6). Theorem 3.1 is thus proved. □

Corollary 3.4. *Let $a \in \mathbb{R}_{++}^n$, $0 < \alpha < \theta < \beta$, $\lambda \in \mathbb{R}$. Then the maximal value of $\lambda$ such that inequality (1.5) holds is*

$$\lambda^* := \inf_{t>0, 0 \le k \le n-1} \left\{ \frac{\ln M_n^{[\theta]}([a]_{t,n,k}) - \ln M_n^{[\alpha]}([a]_{t,n,k})}{\ln M_n^{[\beta]}([a]_{t,n,k}) - \ln M_n^{[\alpha]}([a]_{t,n,k})} \right\}, \tag{3.17}$$

*where $[a]_{t,n,k} = (t, I_k, O_{n-k-1})$. Inequality (1.5) holds if and only if $\lambda \le \lambda^*$.*

*Proof.* By Theorem 3.1, inequality (1.5) holds if and only if

$$\begin{aligned}
&\{M_n^{[\alpha]}([a]_{t,n,k})\}^{1-\lambda}\{M_n^{[\beta]}([a]_{t,n,k})\}^{\lambda} \\
&\quad\le M_n^{[\theta]}([a]_{t,n,k}), \quad \forall t>0, \forall k: 0\le k\le n-1, \\
&\quad\Longleftrightarrow \lambda \le \frac{\ln M_n^{[\theta]}([a]_{t,n,k})-\ln M_n^{[\alpha]}([a]_{t,n,k})}{\ln M_n^{[\beta]}([a]_{t,n,k})-\ln M_n^{[\alpha]}([a]_{t,n,k})}, \quad \forall t>0, \forall k: 0\le k\le n-1 \Longleftrightarrow \lambda\le\lambda^*.
\end{aligned} \tag{3.18}$$

□

*Example 3.5.* Let $\alpha = 1/2, \theta = 1, \beta = 3/2, n = 15$. By using (3.17) in Corollary 3.4, we have

$$\begin{aligned}
\lambda^* &:= \inf_{t>0,0\le k\le n-1}\left\{\frac{\ln M_n^{[\theta]}([a]_{t,n,k})-\ln M_n^{[\alpha]}([a]_{t,n,k})}{\ln M_n^{[\beta]}([a]_{t,n,k})-\ln M_n^{[\alpha]}([a]_{t,n,k})}\right\} \\
&= \inf_{t>0,0\le k\le 14}\left\{\frac{\ln(t+k)/15-2\ln(\sqrt{t}+k)/15}{(2/3)\ln(t^{3/2}+k)/15-2\ln(\sqrt{t}+k)/15}\right\}.
\end{aligned} \tag{3.19}$$

In fact, by means of Mathematica software, we can sketch the graphs of the functions of two variables $g(t,k) := (\ln((t+k)/15) - 2\ln((\sqrt{t}+k)/15))/((2/3)\ln((t^{3/2}+k)/15) - 2\ln(\sqrt{t}+k)/15)$ and $-g(t,k)$. Thus our problem can be explained from the graphs, our result is the following: if $a \in \mathbb{R}_+^{15}$, then $\lambda \le 0.4160944179212302\ldots$ if and only if

$$\{M_{15}^{[1/2]}(a)\}^{1-\lambda}\{M_{15}^{[3/2]}(a)\}^{\lambda} \le M_{15}^{[1]}(a). \tag{3.20}$$

*Remark 3.6.* Corollary 3.4 is an open problem posed in [19].

In this section, we merely discuss the optimal problem of inequality (1.6) under the condition $n \ge m \ge 2$. When $m$ is sufficiently large, it is impossible that we apply Theorem 3.1 artificially. Owing to this reason, we will discuss the general case of inequality (1.6) in Section 4. In other words, we will search for the necessary and sufficient condition such that $m \ge 2$, $n \ge 2$ hold. Our aim is to work artificially.

## 4. The sufficient condition that inequality (1.6) holds

THEOREM 4.1. *Let* $\alpha \in \mathbb{R}_{++}^m$, $m \ge 2$, $0 < \alpha_1 \le \cdots \le \alpha_{p-1} \le \theta \le \alpha_p \le \cdots \le \alpha_m$, $2 \le p \le m$, $\lambda \in \mathbb{R}_{++}^m$, $\sum_{j=1}^m \lambda_j = 1$. *If* $\alpha_m \le 2(\alpha_p + \theta)$, $\inf_{t>0}\sum_{j=1}^m(\lambda_j(\theta-\alpha_j)/(2+(n-2)t^{\alpha_j})) \ge 0$, *then, for all* $a \in \mathbb{R}_{++}^n$, $n \ge 2$, *inequality (1.6) holds.*

Recall the definition (see, e.g., [5, pages 41–42] and [9, 19]) of generalized logarithmic means $E(r,s;x,y)$,

$$E(r,s;x,y)=\begin{cases}\left[\dfrac{r}{s}\cdot\dfrac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, & rs(r-s)(x-y)\neq 0,\\ \left[\dfrac{1}{r}\cdot\dfrac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, & s=0,\ r(x-y)\neq 0,\\ e^{-1/r}\left[\dfrac{x^{x^r}}{y^{y^r}}\right]^{1/(x^r-y^r)}, & r=s,\ r(x-y)\neq 0,\\ \sqrt{xy}, & r=s=0,\ x\neq y,\\ x, & x=y.\end{cases} \tag{4.1}$$

Lemma 4.2 [9, 19]. *Let $a_1$ and $a_2$ be two positive real numbers, and let $r$, $s$, $u$, $v$ be real numbers, where $r\neq s$, $u\neq v$. Then, for all the $a_1,a_2>0$, a necessary and sufficient condition such that the inequality*

$$E(r,s;a_1,a_2)\leq E(u,v;a_1,a_2) \tag{4.2}$$

*holds is that*

$$\begin{aligned} r+s&\leq u+v,\\ e(r,s)&\leq e(u,v),\end{aligned} \tag{4.3}$$

*where* $0\leq \min\{r,s,u,v\}$ *or* $\max\{r,s,u,v\}\leq 0$,

$$e(x,y)=\begin{cases}\dfrac{x-y}{\ln x-\ln y}, & xy>0,\ x\neq y,\\ 0, & xy=0,\end{cases} \tag{4.4}$$

*when* $\min\{r,s,u,v\}<0<\max\{r,s,u,v\}$,

$$e(x,y)=\frac{|x|-|y|}{x-y},\quad x,y\in\mathbb{R},\ x\neq y. \tag{4.5}$$

Lemma 4.3. *Let $\alpha\in\mathbb{R}^m_{++}$, $m\geq 2$, $0<\alpha_1\leq\cdots\leq\alpha_{p-1}\leq 1\leq\alpha_p\leq\cdots\leq\alpha_m$, $2\leq p\leq m$, $\lambda\in\mathbb{R}^m_{++}$, $\alpha_m\leq 2(\alpha_p+1)$. Define the function $\Phi:\mathbb{R}^n_{++}\to\mathbb{R}$, $\Phi(a):=-\ln\prod_{j=1}^m\{M_n^{[\alpha_j]}(a)\}^{\lambda_j}$, then a necessary and sufficient condition such that the function $\Phi$ is a Schur-convex function is that*

$$\inf_{t>0}\left\{\sum_{j=1}^m\frac{\lambda_j(1-\alpha_j)}{2+(n-2)t^{\alpha_j}}\right\}\geq 0. \tag{4.6}$$

*Proof.* From the literature [11, 16], we only have to prove that a necessary and sufficient condition such that the inequality

$$(a_1-a_2)\left(\frac{\partial\Phi}{\partial a_1}-\frac{\partial\Phi}{\partial a_2}\right)\geq 0,\quad \forall a\in\mathbb{R}^n_{++}, \tag{4.7}$$

holds is that inequality (4.6) holds.

Without loss of generality, we may assume that $a_1 > a_2$, $0 < \alpha_{p-1} < 1 < \alpha_p$. Note that

$$\Phi(a) = -\sum_{j=1}^{m} \frac{\lambda_j}{\alpha_j} \ln \frac{\sum_{i=1}^{n} a_i^{\alpha_j}}{n},$$
$$\frac{\partial\Phi}{\partial a_1} - \frac{\partial\Phi}{\partial a_2} = -\sum_{j=1}^{m} \frac{\lambda_j\left(a_1^{\alpha_j-1} - a_2^{\alpha_j-1}\right)}{\sum_{i=1}^{n} a_i^{\alpha_j}}, \qquad a_1^{\alpha_p-1} - a_2^{\alpha_p-1} > 0. \tag{4.8}$$

Thus, inequality (4.7) is equivalent to

$$\sum_{j=1}^{m} \frac{\lambda_j}{\sum_{i=1}^{n} a_i^{\alpha_j}} \cdot \frac{a_1^{\alpha_j-1} - a_2^{\alpha_j-1}}{a_1^{\alpha_p-1} - a_2^{\alpha_p-1}} \le 0. \tag{4.9}$$

Since

$$\begin{aligned} \frac{a_1^{\alpha_j-1} - a_2^{\alpha_j-1}}{a_1^{\alpha_p-1} - a_2^{\alpha_p-1}} &= \frac{a_1^{\alpha_p\cdot((\alpha_j-1)/\alpha_p)} - a_2^{\alpha_p\cdot((\alpha_j-1)/\alpha_p)}}{a_1^{\alpha_p\cdot((\alpha_p-1)/\alpha_p)} - a_2^{\alpha_p\cdot((\alpha_p-1)/\alpha_p)}} \\ &= \frac{\alpha_j-1}{\alpha_p-1} \cdot \left[E\left(\frac{\alpha_j-1}{\alpha_p}, \frac{\alpha_p-1}{\alpha_p}; a_1^{\alpha_p}, a_2^{\alpha_p}\right)\right]^{(\alpha_j-\alpha_p)/\alpha_p}, \end{aligned} \tag{4.10}$$

so, inequality (4.9) is equivalent to

$$\sum_{j=1}^{m} \frac{\lambda_j(1-\alpha_j)}{\sum_{i=1}^{n} a_i^{\alpha_j}} \left[E\left(\frac{\alpha_j-1}{\alpha_p}, \frac{\alpha_p-1}{\alpha_p}; a_1^{\alpha_p}, a_2^{\alpha_p}\right)\right]^{(\alpha_j-\alpha_p)/\alpha_p} \ge 0. \tag{4.11}$$

*Sufficiency.* Assume that (4.6) holds, we will prove that inequality (4.11) holds. In fact, we will follow every step in the following.

*Step 1.* We will prove that

$$\frac{\lambda_j(1-\alpha_j)}{\sum_{i=1}^{n} a_i^{\alpha_j}} \ge \frac{\lambda_j(1-\alpha_j)}{2u^{\alpha_j} + (n-2)v^{\alpha_j}}, \quad j = 1,2,\dots,m, \tag{4.12}$$

where $u = ((a_1^{\alpha_p} + a_2^{\alpha_p})/2)^{1/\alpha_p}$, when $n > 2$, we have $v = (\sum_{i=3}^{n} a_i^{\alpha_p}/(n-2))^{1/\alpha_p}$, when $n = 2$, we may define an arbitrary value of $v$. Now we define that $v = u$. When $1 \le j \le p-1$, we have $\lambda_j(1-\alpha_j) > 0$, $0 < \alpha_j < \alpha_p$. Therefore, by the inequality with power means, we obtain that

$$\sum_{i=1}^{n} a_i^{\alpha_j} = 2 \cdot \left(\frac{a_1^{\alpha_j} + a_2^{\alpha_j}}{2}\right)^{\alpha_j/\alpha_j} + (n-2)\left(\frac{\sum_{i=3}^{n} a_i^{\alpha_j}}{n-2}\right)^{\alpha_j/\alpha_j} \le 2u^{\alpha_j} + (n-2)v^{\alpha_j}. \tag{4.13}$$

Thus, inequality (4.12) holds. When $j \ge p$, $\lambda_j(1-\alpha_j) < 0$, $\alpha_j \ge \alpha_p > 1$, the reverse inequality of (4.13) holds, therefore inequality (4.12) holds still.

*Step 2.* We will prove that

$$E\left(\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p};a_1^{\alpha_p},a_2^{\alpha_p}\right)\le E(1,2;a_1^{\alpha_p},a_2^{\alpha_p})=\frac{a_1^{\alpha_p}+a_2^{\alpha_p}}{2},\quad 1\le j\le m,\ j\ne p. \tag{4.14}$$

When $1\le j\le p-1$,

$$\begin{gathered}\min\left\{\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p},1,2\right\}=\frac{\alpha_j-1}{\alpha_p}<0<2=\max\left\{\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p},1,2\right\},\\ \frac{\alpha_i-1}{\alpha_p}+\frac{\alpha_p-1}{\alpha_p}<0+1<1+2,\\ \frac{|(\alpha_j-1)/\alpha_p|-|(\alpha_p-1)/\alpha_p|}{((\alpha_j-1)/\alpha_p)-((\alpha_p-1)/\alpha_p)}=\frac{2-\alpha_p-\alpha_j}{\alpha_j-\alpha_p}=\frac{2(1-\alpha_j)}{\alpha_j-\alpha_p}+1<1=\frac{|1|-|2|}{1-2},\end{gathered} \tag{4.15}$$

therefore, by Lemma 4.2, inequality (4.13) holds.

When $p+1\le j\le m$,

$$\begin{gathered}\min\left\{\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p},1,2\right\}=\frac{\alpha_p-1}{\alpha_p}>0,\\ \frac{\alpha_j-1}{\alpha_p}+\frac{\alpha_p-1}{\alpha_p}=\frac{\alpha_j+\alpha_p-2}{\alpha_p}\le\frac{\alpha_m+\alpha_p-2}{\alpha_p}\le\frac{2(\alpha_p+1)+\alpha_p-2}{\alpha_p}=1+2.\end{gathered} \tag{4.16}$$

Based on the above discussion and Lemma 4.2, we only have to prove that

$$\left(\frac{\alpha_j-1}{\alpha_p}-\frac{\alpha_p-1}{\alpha_p}\right)\Big/\left(\ln\frac{\alpha_j-1}{\alpha_p}-\ln\frac{\alpha_p-1}{\alpha_p}\right)\le\frac{1}{\ln 2}=\frac{1-2}{\ln 1-\ln 2}. \tag{4.17}$$

Let $x=\ln((\alpha_j-1)/\alpha_p)$, $y=\ln((\alpha_p-1)/\alpha_p)$, then $y\le x\le\ln((\alpha_m-1)/\alpha_p)\le\ln((2\alpha_p+1)/\alpha_p)=x_0$, from this fact, we get

$$\begin{aligned}&\left(\frac{\alpha_j-1}{\alpha_p}-\frac{\alpha_p-1}{\alpha_p}\right)\Big/\left(\ln\frac{\alpha_j-1}{\alpha_p}-\ln\frac{\alpha_p-1}{\alpha_p}\right)\\ &\quad=\frac{e^x-e^y}{x-y}=e^y\cdot\frac{e^{x-y}-1}{x-y}=e^y\cdot\sum_{k=0}^{\infty}\frac{(x-y)^k}{(k+1)!}\le e^y\cdot\sum_{k=0}^{\infty}\frac{(x_0-y)^k}{(k+1)!}\\ &\quad=\left(\frac{2\alpha_p+1}{\alpha_p}-\frac{\alpha_p-1}{\alpha_p}\right)\Big/\left(\ln\frac{2\alpha_p+1}{\alpha_p}-\ln\frac{\alpha_p-1}{\alpha_p}\right)\\ &\quad=\frac{1+2t}{\ln(2+t)-\ln(1-t)},\quad 0<t=\frac{1}{\alpha_p}<1.\end{aligned} \tag{4.18}$$

Thus, we only have to prove that

$$\frac{1+2t}{\ln(2+t)-\ln(1-t)} \le \frac{1}{\ln 2} \Longleftrightarrow \phi(t) := \ln(2+t)-\ln(1-t)-(\ln 2)(1+2t) \ge 0, \quad \forall t \in (0,1). \tag{4.19}$$

Since $\phi'(t) = 3/(2-t-t^2) - 2\ln 2$ is increasing on $(0,1)$, we have

$$\phi'(t) > \phi'(0) = \frac{3}{2} - 2\ln 2 = 0.11371\cdots > 0, \qquad \phi(t) > \phi(0) = 0. \tag{4.20}$$

It follows that inequality (4.17) and the assertion of Step 2 have been proven.

*Step 3.* We will prove that inequality (4.11) holds.

Since $(1-\alpha_j)(\alpha_j-\alpha_p) < 0$ $(j = 1,\dots,p-1,p+1,\dots,m)$, therefore, by the inequalities (4.12) and (4.14), when $j \ne p$,

$$\frac{\lambda_j(1-\alpha_j)}{\sum_{i=1}^n a_i^{\alpha_j}}\left[E\left(\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p};a_1^{\alpha_p},a_2^{\alpha_p}\right)\right]^{(\alpha_j-\alpha_p)/\alpha_p} \ge \frac{\lambda_j(1-\alpha_j)u^{\alpha_j-\alpha_p}}{2u^{\alpha_j}+(n-2)v^{\alpha_j}}, \tag{4.21}$$

when $j = p$, inequality (4.21) reduces to the equality, so (4.21) holds still. Thus, letting $t = v/u > 0$, we have

$$\begin{aligned}
&\sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)}{\sum_{i=1}^n a_i^{\alpha_j}}\left[E\left(\frac{\alpha_j-1}{\alpha_p},\frac{\alpha_p-1}{\alpha_p};a_1^{\alpha_p},a_2^{\alpha_p}\right)\right]^{(\alpha_j-\alpha_p)/\alpha_p}\\
&\quad\ge \sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)u^{\alpha_j-\alpha_p}}{2u^{\alpha_j}+(n-2)v^{\alpha_j}} = u^{-\alpha_p}\sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)}{2+(n-2)t^{\alpha_j}}\\
&\quad\ge u^{-\alpha_p}\cdot \inf_{t>0} u^{-\alpha_p}\sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)}{2+(n-2)t^{\alpha_j}} \ge 0.
\end{aligned} \tag{4.22}$$

It follows that inequality (4.11) holds.

*Necessity.* Assume that inequality (4.11) holds. We will prove that inequality (4.6) holds as follows.

Putting $a_1 = a_2 = 1$, $a_3 = \cdots = a_n = t > 0$ in (4.11), then inequality (4.11) reduces to

$$\sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)}{2+(n-2)t^{\alpha_j}} \ge 0 \quad (\forall t > 0) \Longrightarrow \inf_{t>0}\sum_{j=1}^m \frac{\lambda_j(1-\alpha_j)}{2+(n-2)t^{\alpha_j}} \ge 0. \tag{4.23}$$

This completes the proof. □

*Proof of Theorem 4.1.* We first prove a special case $\theta = 1$. By the hypothesis of Theorem 4.1 and Lemma 4.3, the function

$$\Phi : \mathbb{R}_{++}^n \longrightarrow \mathbb{R}, \qquad \Phi(a) := -\ln\prod_{j=1}^m \left\{M_n^{[\alpha_j]}(a)\right\}^{\lambda_j} \tag{4.24}$$

is a Schur-convex function.

Let $A = (a_1 + a_2 + \cdots + a_n)/n$. Then $\overline{a} = (A,A,\dots,A) \prec a$. From the definition of Schur-convex function, we observe that $\Phi(\overline{a}) \le \Phi(a)$. By reason of $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, it is easy to see that inequality (1.6) is equivalent to the inequality $\Phi(\overline{a}) \le \Phi(a)$. Thus inequality (1.6) holds.

Second, we prove the general case $\theta \ne 1$ as follows.

By the hypothesis of Theorem 4.1, we obtain that

$$0 < \frac{\alpha_1}{\theta} \le \cdots \le \frac{\alpha_{p-1}}{\theta} \le 1 \le \frac{\alpha_p}{\theta} \le \cdots \le \frac{\alpha_m}{\theta}, \quad \frac{\alpha_m}{\theta} \le 2\left(\frac{\alpha_p}{\theta} + 1\right),$$
$$\inf_{t>0} \sum_{j=1}^{m} \frac{\lambda_j(1-\alpha_j/\theta)}{2+(n-2)t^{\alpha_j/\theta}} = \frac{1}{\theta} \inf_{t^{1/\theta}>0} \sum_{j=1}^{m} \frac{\lambda_j(\theta-\alpha_j)}{2+(n-2)(t^{1/\theta})^{\alpha_j}} \ge 0. \tag{4.25}$$

Combining the above with the conclusion of the special case $\theta = 1$, we have

$$\prod_{j=1}^{m} \{M_n^{[\alpha_j/\theta]}(a)\}^{\lambda_j} \le M_n^{[1]}(a). \tag{4.26}$$

Replacing $a$ by $a^\theta$ in (4.26), then inequality (4.26) reduces to inequality (1.6). This completes our proof. □

*Remark 4.4.* Let $m = 2$ in Theorem 4.1. Then we get [19, Theorem 1].

COROLLARY 4.5. *If $a,b,x,y \in (0,+\infty)$, then*

$$\exp \frac{\int_a^b (\ln(x^t + y^t)/2) \cdot (dt/t)}{b-a} \le \left(\frac{x^{(a+b)/2} + y^{(a+b)/2}}{2}\right)^{2/(a+b)}, \tag{4.27}$$

*with equality holds if and only if $x = y$ or $a = b$.*

*Proof.* In Theorem 4.1, letting $n = 2$, $m = 2p-1$, $p \ge 2$, $\lambda_j = 1/m$, $\alpha_j = a + (j/m)(b-a)$, $b > a > 0$, $j = 1,2,\dots,m$, $(a_1,a_2) = (x,y)$, $\theta = (1/m)\sum_{j=1}^{m}\alpha_j = a + (p/m)(b-a) = \alpha_p$, then we have $\alpha \in \mathbb{R}_{++}^m$, $m \ge 2$, $0 < \alpha_1 \le \cdots \le \alpha_{p-1} \le \theta = \alpha_p \le \cdots \le \alpha_m$, $2 \le p \le m$, $\lambda \in \mathbb{R}_{++}^m$, $\sum_{j=1}^{m}\lambda_j = 1$. Since

$$\alpha_m = b \le b + a < 2\left[a + \frac{p}{2p-1}(b-a)\right] < 2\left[a + \frac{p}{2p-1}(b-a) + \theta\right] = 2(\alpha_p + \theta),$$
$$\sum_{j=1}^{m} \frac{\lambda_j(\theta-\alpha_j)}{2+(n-2)t^{\alpha_j}} = \sum_{j=1}^{m} \frac{\lambda_j(\theta-\alpha_j)}{2} = 0, \qquad \inf_{t>0} \sum_{j=1}^{m} \frac{\lambda_j(\theta-\alpha_j)}{2+(n-2)t^{\alpha_j}} \ge 0, \tag{4.28}$$

therefore, by Theorem 4.1 we have

$$\begin{aligned}
&\prod_{j=1}^{m}\left(\frac{x^{\alpha_j}+y^{\alpha_j}}{2}\right)^{1/m\alpha_j}\le\left(\frac{x^{\theta}+y^{\theta}}{2}\right)^{1/\theta},\\
&\frac{1}{b-a}\sum_{j=1}^{m}\frac{1}{\alpha_j}\ln\left(\frac{x^{\alpha_j}+y^{\alpha_j}}{2}\right)\cdot\frac{b-a}{m}\le\ln\left(\frac{x^{\theta}+y^{\theta}}{2}\right)^{1/\theta},\\
\frac{\int_a^b(\ln(x^t+y^t)/2)\cdot dt/t}{b-a}&=\lim_{m\to\infty}\frac{1}{b-a}\sum_{j=1}^{m}\frac{1}{\alpha_j}\ln\left(\frac{x^{\alpha_j}+y^{\alpha_j}}{2}\right)\cdot\frac{b-a}{m}\\
&\le\lim_{m\to\infty}\ln\left(\frac{x^{\theta}+y^{\theta}}{2}\right)^{1/\theta}=\ln\left(\frac{x^{(a+b)/2}+y^{(a+b)/2}}{2}\right)^{2/(a+b)}.
\end{aligned}\tag{4.29}$$

In other words, inequality (4.27) has been proven. Corollary 4.5 is thus proved. □

*Example 4.6.* Consider the condition such that the inequality

$$\left\{\prod_{j=1}^{5}M_{10}^{[2j]}(a)\right\}^{(1-\lambda)/5}\left\{\prod_{j=6}^{10}M_{10}^{[2j]}(a)\right\}^{\lambda/5}\le M_{10}^{[11]}(a),\quad\forall a\in\mathbb{R}_{++}^{10},\tag{4.30}$$

holds.

Since $0<2<4<6<8<10<\theta=11<12<14<16<18<20<2(12+11)$, therefore, from Theorem 4.1, we know that, when

$$\begin{gathered}
\inf_{t>0}\left\{\frac{1-\lambda}{5}\sum_{j=1}^{5}\frac{11-2j}{2+8t^{2j}}+\frac{\lambda}{5}\sum_{j=6}^{10}\frac{11-2j}{2+8t^{2j}}\right\}\ge0\Longleftrightarrow\lambda\le\inf_{t>0}\{g(t)\},\\
g(t)=\left(\sum_{j=1}^{5}\frac{11-2j}{1+4t^{2j}}\right)\Big/\left(\sum_{j=1}^{5}\frac{11-2j}{1+4t^{2j}}+\sum_{j=6}^{10}\frac{2j-11}{1+4t^{2j}}\right),
\end{gathered}\tag{4.31}$$

inequality (4.30) holds. By means of Mathematica software, we can work out $\inf g(t)=0.297911\ldots$. Namely, when $\lambda\le0.297911\ldots$, inequality (4.30) holds.

## 5. The necessary and sufficient condition that inequality (1.7) holds

THEOREM 5.1. *Let $a\in\mathbb{R}_{++}^{n}$, $n\ge2$, $\alpha,\lambda\in\mathbb{R}_{++}^{m}$, $m\ge2$, $\sum_{j=1}^{m}\lambda_j=1$, $\min\{\alpha\}\le\theta\le\max\{\alpha\}$. Then, a necessary and sufficient condition such that inequality (1.7) holds is that*

$$\sum_{j=1}^{m}\frac{\lambda_j}{\alpha_j}\le\frac{1}{\theta}.\tag{5.1}$$

Lemma 5.2. *Let $a_{jk} > 0$, $q_j > 0$, $\sum_{j=1}^{m} q_j \le 1$, $1 \le j \le m$, $1 \le k \le n$. Then, an analogue of Hölder's inequality is*

$$\frac{1}{n} \cdot \sum_{k=1}^{n} \prod_{j=1}^{m} a_{jk}^{q_j} \le \prod_{j=1}^{m} \left( \frac{1}{n} \cdot \sum_{k=1}^{n} a_{jk} \right)^{q_j}. \tag{5.2}$$

*Proof of Theorem 5.1. Sufficiency*. Assume that inequality (5.1) holds. We will prove that inequality (1.7) holds as follows: by (5.1), we have $\sum_{j=1}^{m}(\theta\lambda_j/\alpha_j) \le 1$, $\theta\lambda_j/\alpha_j > 0$, $j = 1,2,\dots,m$. using Lemma 5.2, we obtain that

$$\begin{aligned} \left[ \prod_{j=1}^{m} \{ M_n^{[\alpha_j]}(a) \}^{\lambda_j} \right]^{\theta} &= \prod_{j=1}^{m} \left( \frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{\alpha_j} \right)^{\theta\lambda_j/\alpha_j} \ge \frac{1}{n} \cdot \sum_{i=1}^{n} \prod_{j=1}^{m} \left( a_i^{\alpha_j} \right)^{\theta\lambda_j/\alpha_j} \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n} \prod_{j=1}^{m} a_i^{\theta\lambda_j} = \frac{1}{n} \cdot \sum_{i=1}^{n} (a_i)^{\theta \cdot \sum_{j=1}^{m} \lambda_j} = \frac{1}{n} \cdot \sum_{i=1}^{n} (a_i)^{\theta}. \end{aligned} \tag{5.3}$$

In other words, inequality (1.7) holds.

*Necessity.* Assume that inequality (1.7) holds. We will prove that inequality (5.1) holds as follows: letting $a_1 = 1$, $a_2 = a_3 = \cdots = a_n \to 0$ in inequality (1.7), (1.7) can be reduced to

$$\prod_{j=1}^{m} \left\{ \left( \frac{1}{n} \right)^{1/\alpha_j} \right\}^{\lambda_j} \ge \left( \frac{1}{n} \right)^{1/\theta} \Longleftrightarrow \prod_{j=1}^{m} n^{-\lambda_j/\alpha_j} \ge n^{-1/\theta} \Longleftrightarrow n^{\sum_{j=1}^{m}(\lambda_j/\alpha_j)} \le n^{1/\theta} \Longleftrightarrow \sum_{j=1}^{m} \frac{\lambda_j}{\alpha_j} \le \frac{1}{\theta}. \tag{5.4}$$

Up to now, Theorem 5.1 is proven. □

*Remark 5.3.* Applying the approach of [19], we can establish some results that are similar to [19, (33) and (37)]. By using the definition of Riemann' integral, we can obtain an analogue of integral of (1.7) as follows.

Corollary 5.4. *Let the measurable function on the measurable sets $E$ and $E_0$,*

$$f : E \longrightarrow \mathbb{R}_{++}^1, \quad g : E_0 \longrightarrow \mathbb{R}_{++}^1, \quad p : E_0 \longrightarrow \mathbb{R}_{++}^1, \quad E, E_0 \subset \mathbb{R}^n, \tag{5.5}$$

*satisfy that $\int_{E_0} p(t)dt = 1$, $\inf_{t\in E_0} g(t) \le \theta \le \sup_{t\in E_0} g(t)$, $\int_{E_0}(p(t)/g(t))dt \le 1/\theta$. Then*

$$\exp \frac{\int_{E_0} (p(t)/g(t)) \ln \{ \int_E [f(x)]^{g(t)} dx / |E| \} dt}{|E_0|} \ge \left\{ \frac{\int_E [f(x)]^{\theta} dx}{|E|} \right\}^{1/\theta}, \tag{5.6}$$

*where $|E|$ and $|E_0|$ denote the measures of $E$ and $E_0$.*

Inequality (5.6) has important background in the geometry of convex body (see, e.g., [3, 7]).

## 6. The criterion of the semipositivity of homogeneous symmetric polynomial

In this section, we will use the following symbols:

$\sigma_k = \sum_{1\le i_1<\cdots<i_k\le n\le n}\prod_{j=1}^{k} x_{i_j}$, $s_k = \sum_{i=1}^{n} x_i^k$, $A(x^k) = (1/n)\cdot s_k$, $k = 1,2,\ldots,d\wedge n$, $d\wedge n = \min\{d,n\}$, $\overrightarrow{t(d\wedge n)} := t(d\wedge n) := (t_1,\ldots,t_{d\wedge n}) \in Z_+^{d\wedge n}$, $\overrightarrow{d\wedge n}\cdot\overrightarrow{t(d\wedge n)} = \sum_{k=1}^{d\wedge n} kt_k$, $T_d = \{\overrightarrow{t(d\wedge n)} \mid \overrightarrow{d\wedge n}\cdot\overrightarrow{t(d\wedge n)} = d, \overrightarrow{t(d\wedge n)} \in Z_+^{d\wedge n}\}$, $\lambda = (\lambda_1,\ldots,\lambda_i,\ldots,\lambda_{d\wedge n}) = d^{-1}\cdot(t_1, 2t_2,\ldots,it_i,\ldots,d\wedge nt_{d\wedge n})$.

Lemma 6.1. *Let $f(x) = \sum_{\alpha\in B_d}\lambda_\alpha h_n(x;\alpha)$ be a homogeneous symmetrical polynomial of $n$ ($n \ge 2$) variables of degree $d$ ($d\ge 2$), and let $f$ satisfy that $f(I_n) = 0$. Then $f$ can be expressed as*

$$f(x) = \sum_{t(d\wedge n)\in T_{d,1}}\lambda_{t(d\wedge n)}\left\{\prod_{i=1}^{d\wedge n}\left[M_n^{[i]}(x)\right]^{\lambda_i}\right\}^d - \sum_{t(d\wedge n)\in T_{d,2}}\lambda_{t(d\wedge n)}\left\{\prod_{i=1}^{d\wedge n}\left[M_n^{[i]}(x)\right]^{\lambda_i}\right\}^d, \tag{6.1}$$

*where $T_{d,1}, T_{d,2} \subset T_d$, $T_{d,1}\cap T_{d,2} = \Phi$, $T_{d,1}\cup T_{d,2} = T_d$, $\lambda_{t(d\wedge n)} \ge 0$ (for all $t(d\wedge n)\in T_d$), and $\sum_{t(d\wedge n)\in T_{d,1}}\lambda_{t(d\wedge n)} = \sum_{t(d\wedge n)\in T_{d,2}}\lambda_{t(d\wedge n)}$.*

*Proof.* By [2, Theorem 15, page 41], $f$ can be expressed as

$$f(x) = \sum_{t(d\wedge n)\in T_d}\lambda_{t(d\wedge n)}\prod_{i=1}^{d\wedge n}\sigma_i^{t_i}. \tag{6.2}$$

Using Newton's formula (see[2, page 49] and [10, page 28]),

$$\sigma_0 = 1, \qquad \sum_{i=1}^{k}(-1)^{k-i}\sigma_{k-i}s_i + (-1)^k k\sigma_k = 0, \quad 1\le k\le n, \tag{6.3}$$

or

$$\sigma_0 = 1, \qquad k!\sigma_k = \begin{vmatrix} s_1 & 1 & 0 & \cdots & 0 \\ s_2 & s_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_{k-2} & s_{k-3} & \cdots & k-1 \\ s_k & s_{k-1} & s_{k-2} & \cdots & s_1 \end{vmatrix}, \quad 1\le k\le n, \tag{6.4}$$

$\sigma_i$ can be expressed as

$$\sigma_i = \sum_{t(i)\in T_i}\lambda_{t(i)}\prod_{j=1}^{i}s_j^{t_j} \quad (1\le i\le n), \quad T_i = \left\{t(i) \mid \sum_{j=1}^{i} jt_j = i,\ t(i)\in Z_+^i\right\}. \tag{6.5}$$

Equation (6.2) is substituted by (6.5); and using expansion formula for polynomial, $f$ can be expressed as

$$\begin{aligned} f(x) &= \sum_{t(d\wedge n)\in T_d} \lambda^*_{t(d\wedge n)} \prod_{i=1}^{d\wedge n} s_i^{t_i} = \sum_{t(d\wedge n)\in T_d} \lambda_{t(d\wedge n)} \prod_{i=1}^{d\wedge n} A^{t_i}(x^i) \\ &= \sum_{t(d\wedge n)\in T_d} \lambda_{t(d\wedge n)} \prod_{i=1}^{d\wedge n} \{M_n^{[i]}(x)\}^{it_i} = \sum_{t(d\wedge n)\in T_d} \lambda_{t(d\wedge n)} \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\lambda_i} \right\}^d. \end{aligned} \tag{6.6}$$

Since $f(I_n) = 0$, $\sum_{t(d\wedge n)\in T_d} \lambda_{t(d\wedge n)} = 0$, by (6.6), there exist

$$T_{d,1}, T_{d,2} \subset T_d, \quad T_{d,1} \cap T_{d,2} = \Phi, \quad T_{d,1} \cup T_{d,2} = T_d, \quad \lambda_{t(d\wedge n)} \geq 0 \quad (\forall t(d\wedge n) \in T_d), \tag{6.7}$$

such that $f$ can be expressed as

$$f(x) = \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\lambda_i} \right\}^d - \sum_{t(d\wedge n)\in T_{d,2}} \lambda_{t(d\wedge n)} \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\lambda_i} \right\}^d, \tag{6.8}$$

where $\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} = \sum_{t(d)\in T_{d,2}} \lambda_{t(d\wedge n)}$. This completes the proof. □

THEOREM 6.2. *Let $f(x)$ be a homogeneous symmetrical polynomial of $n$ $(n \geq 2)$ variables of degree $d$ $(d \geq n)$, and let $f(I_n) = 0$, and let the expression of $f(x)$ be given by (6.1). The following can be written:*

$$\begin{aligned} \mu_i &= \frac{\sum_{t(d\wedge n)\in T_{d,1}} \lambda_i \lambda_{t(d\wedge n)}}{\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)}} = \frac{\sum_{t(d\wedge n)\in T_{d,1}} it_i \lambda_{t(d\wedge n)}}{\sum_{t(d\wedge n)\in T_{d,1}} d \cdot \lambda_{t(d\wedge n)}}, \quad i = 1, 2, \ldots, d\wedge n, \\ \theta_1 &= \left( \sum_{j=1}^{d\wedge n} \frac{\mu_j}{j} \right)^{-1} = \frac{d \cdot \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right)}{\sum_{j=1}^{d\wedge n} \sum_{t(d\wedge n)\in T_{d,1}} t_j \lambda_{t(d\wedge n)}}, \\ \theta_2 &= \sup_{t>0,\, t(d\wedge n)\in T_{d,2}} \left\{ \left[ \sum_{j=1}^{d\wedge n} \frac{j^2 t_j}{2+(n-2)t^j} \right] \Big/ \left[ \sum_{j=1}^{d\wedge n} \frac{j t_j}{2+(n-2)t^j} \right] \right\}. \end{aligned} \tag{6.9}$$

*If, for arbitrary $t(d\wedge n) \in T_{d,2}$, there exists $p : 2 \leq p \leq d\wedge n$ such that*

$$1 \leq \cdots \leq p-1 \leq \theta_2 \leq p \leq \cdots \leq d\wedge n \leq 2(p+\theta_2), \tag{6.10}$$

*then, when $\theta_1 \geq \theta_2$,*

$$f(x) \geq 0, \quad x \in \mathbb{R}^n_{++}. \tag{6.11}$$

*Proof.* By the related theorem of continuous function, if $\lambda \in \mathbb{R}_+^m$ (or, $\lambda \in \mathbb{R}_+^n$), then the above theorem and lemma are valid.

By the arithmetic-geometric mean inequality and Theorem 5.1, we obtain that

$$\begin{aligned}
&\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\lambda_i} \right\}^d \\
&\geq \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right) \left\{ \prod_{t(d\wedge n)\in T_{d,1}} \left[ \prod_{i=1}^{d\wedge n} (M_n^{[i]}(x))^{\lambda_i} \right]^{d\cdot\lambda_{t(d\wedge n)}} \right\}^{1/\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)}} \\
&= \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right) \left\{ \prod_{i=1}^{d\wedge n} \prod_{t(d\wedge n)\in T_{d,1}} [(M_n^{[i]}(x))^{\lambda_i}]^{d\cdot\lambda_{t(d\wedge n)}} \right\}^{1/\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)}} \\
&= \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right) \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\sum_{t(d\wedge n)\in T_{d,1}} \lambda_i\cdot d\cdot\lambda_{t(d\wedge n)}} \right\}^{1/\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)}} \\
&= \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right) \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\mu_i} \right\}^d \geq \left( \sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} \right) \{M_n^{[\theta_1]}(x)\}^d.
\end{aligned} \tag{6.12}$$

By the definition of $\theta_2$ and $\lambda = d^{-1}\cdot(t_1, 2t_2, \ldots, jt_j, \ldots, d\wedge nt_{d\wedge n})$, $\overrightarrow{d\wedge n}\cdot\overrightarrow{t(d\wedge n)} = d$, for for all $t(d\wedge n) \in T_{d,2}$, we have $\inf_{t>0}\{\sum_{j=1}^{d\wedge n}(\lambda_j(\theta_2 - j)/(2+(n-2)t^j))\} \geq 0$, and by the hypothesis of Theorems 4.1 and 6.2, for all $t(d\wedge n) \in T_{d,2}$, we have

$$\begin{aligned}
&\sum_{t(d\wedge n)\in T_{d,2}} \lambda_{t(d\wedge n)} \left\{ \prod_{i=1}^{d\wedge n} [M_n^{[i]}(x)]^{\lambda_i} \right\}^d \\
&\quad \leq \sum_{t(d\wedge n)\in T_{d,2}} \lambda_{t(d\wedge n)} \{M_n^{[\theta_2]}(x)\}^d = \left( \sum_{t(d\wedge n)\in T_{d,2}} \lambda_{t(d\wedge n)} \right) \{M_n^{[\theta_2]}(x)\}^d.
\end{aligned} \tag{6.13}$$

By (6.1), (6.12), (6.13), $\theta_1 \geq \theta_2$, and $\sum_{t(d\wedge n)\in T_{d,1}} \lambda_{t(d\wedge n)} = \sum_{t(d\wedge n)\in T_{d,2}} \lambda_{t(d\wedge n)}$, inequality (6.11) holds. This completes the proof. □

*Example 6.3.* Consider the condition such that the following inequality holds:

$$\begin{aligned}
&(1-s)\left(\frac{1}{10}\sum_{i=1}^{10} x_i^6\right)\left(\frac{1}{10}\sum_{i=1}^{10} x_i^{10}\right)^2 + s\left(\frac{1}{10}\sum_{i=1}^{10} x_i^2\right)^3\left(\frac{1}{10}\sum_{i=1}^{10} x_i^4\right)^5 \\
&\quad - \frac{1}{2}\left[\left(\frac{1}{10}\sum_{i=1}^{10} x_i\right)^{10}\left(\frac{1}{10}\sum_{i=1}^{10} x_i^4\right)^4 + \left(\frac{1}{10}\sum_{i=1}^{10} x_i^2\right)^9\left(\frac{1}{10}\sum_{i=1}^{10} x_i^8\right)\right] \geq 0,
\end{aligned} \tag{6.14}$$

where $d = 26$, $n = 10$, $d \wedge n = 10$,

$$T_{d,1} = \{(0,0,0,0,0,1,0,0,0,2),(0,3,0,5,0,0,0,0,0,0)\},$$
$$T_{d,2} = \{(10,0,0,4,0,0,0,0,0,0),(0,9,0,0,0,0,0,1,0,0)\},$$

$$\theta_1 = \left(\sum_{j=1}^{d\wedge n}\frac{\mu_j}{j}\right)^{-1} = \frac{d\cdot\left(\sum_{t(d\wedge n)\in T_{d,1}}\lambda_{t(d\wedge n)}\right)}{\sum_{j=1}^{d\wedge n}\sum_{t(d\wedge n)\in T_{d,1}} t_j\lambda_{t(d\wedge n)}} = \frac{26\times 1}{(1-s)+2(1-s)+3s+5s} = \frac{26}{3+5s},$$

$$\begin{aligned}\theta_2 &= \sup_{t>0,\,t(d\wedge n)\in T_{d,2}}\left\{\left[\sum_{j=1}^{d\wedge n}\frac{j^2t_j}{2+(n-2)t^j}\right]\Big/\left[\sum_{j=1}^{d\wedge n}\frac{jt_j}{2+(n-2)t^j}\right]\right\}\\ &= \sup_{t>0,\,t(10)\in T_{d,2}}\left\{\left[\sum_{j=1}^{10}\frac{j^2t_j}{2+8t^j}\right]\Big/\left[\sum_{j=1}^{10}\frac{jt_j}{2+8t^j}\right]\right\}\\ &= \sup_{t>0,\,t(10)\in T_{d,2}}\left\{\left[\frac{5}{1+4t}+\frac{32}{1+4t^4}\right]\Big/\left[\frac{5}{1+4t}+\frac{8}{1+4t^4}\right],\left[\frac{18}{1+4t}+\frac{32}{1+4t^4}\right]\Big/\left[\frac{9}{1+4t}+\frac{4}{1+4t^4}\right]\right\}.\end{aligned} \tag{6.15}$$

Using Mathematica software, we obtain that $\theta_2 = 510279565184453\ldots$.

It follows that

$$\theta_1 \geq \theta_2 \Longleftrightarrow \frac{26}{3+5s} \geq \theta_2 \Longleftrightarrow s \leq \frac{1}{5}\left(\frac{26}{\theta_2}-3\right) = 0.41904923394695076\ldots. \tag{6.16}$$

Namely, for $0 < s \leq 0.41904923394695076\ldots$, inequality (6.14) holds.

*Remark 6.4.* It must be pointed out that Theorem 6.2 can be operated artificially. Theorem 6.2 is different from the result in [15], because that of [15] only has meaning for $n \geq [d/2]$ (i.e., the greatest integer function of $d/2$) and can be operated artificially. The problem in Example 6.3 is too difficult, and furthermore, it cannot be solved by all the softwares in the existing circumstances.

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