A NEW SYSTEM OF GENERALIZED NONLINEAR RELAXED COCOERCIVE VARIATIONAL INEQUALITIES

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We introduce and study a new system of generalized nonlinear relaxed cocoercive inequality problems and construct an iterative algorithm for approximating the solutions of the system of generalized relaxed cocoercive variational inequalities in Hilbert spaces. We prove the existence of the solutions for the system of generalized relaxed cocoercive variational inequality problems and the convergence of iterative sequences generated by the algorithm. We also study the convergence and stability of a new perturbed iterative algorithm for approximating the solution.

1. Introduction

Variational inequality problems have various applications in mechanics and physics, optimization and control, linear and nonlinear programming, economics and finance, transportation equilibrium and engineering science, and so forth. Consequently considerable attention has been devoted to the study of the theory and efficient numerical methods for variational inequality problems (see, e.g., [2–17] and the references therein). In [15], Verma introduced a new system of nonlinear strongly monotone variational inequalities and studied the approximate of this system based on the projection method, and in [16], Verma discussed the approximate solvability of a system of nonlinear relaxed cocoercive variational inequalities in Hilbert spaces. Recently, Kim and Kim [14] introduced and studied a system of nonlinear mixed variational inequalities in Hilbert spaces, and obtained some approximate solvability results. In the recent paper [6], Cho et al. introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities. They also constructed an iterative algorithm for approximating the solution of the system of nonlinear variational inequalities. Some related works, we refer to [2, 3, 5, 7–10, 12, 13]. Motivated and inspired by these works, in this paper, we introduce and study a new system of generalized nonlinear relaxed cocoercive variational

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inequality problems and construct an iterative algorithm for approximating the solutions of the system of generalized relaxed cocoercive variational inequalities in Hilbert spaces. We prove the existence of the solutions for the system of generalized relaxed cocoercive variational inequality problems and the convergence of iterative sequences generated by the algorithm. We also study the convergence and stability of a new perturbed iterative algorithm for approximating the solution. The results presented in this paper improve and extend the previously known results in this area.

2. Preliminaries

Let H be a Hilbert space endowed with a norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, respectively. Let $\mathrm{CB}(H)$ be the family of all nonempty subsets of H and K_1, K_2 be two convex and closed subsets of H. Let $g_1, g_2, m_1, m_2 : H \to H$ and $F, G : H \times H \to H$ be mappings. We consider the following system of generalized nonlinear variational inequality problems: find $x, y \in H$ such that $g_i(x) \in K_i(x)$ for $i = 1, 2$, and

$$\begin{aligned} \big(F(x,y), z - g_1(x)\big) &\geq 0, \quad \forall z \in K_1(x),\\ \big(G(x,y), z - g_2(y)\big) &\geq 0, \quad \forall z \in K_2(y), \end{aligned} \tag{2.1}$$

where $K_i(x) = m_i(x) + K_i$ for $i = 1, 2$.

When K_1 and K_2 are both convex cones of H, it is easy to see that problem (2.1) is equivalent to the following system of generalized nonlinear co-complementarity problems: find $x, y \in H$ such that $g_i(x) \in K_i(x)$ for $i = 1, 2$, and

$$\begin{aligned} F(x,y) &\in \big(K_1(x) - g_1(x)\big)^*,\\ G(x,y) &\in \big(K_2(y) - g_2(y)\big)^*, \end{aligned} \tag{2.2}$$

where $K_i(x) = m_i(x) + K_i$ and $(K_i(x) - g_i(x))^*$ is the dual of $K_i(x) - g_i(x)$ for $i = 1, 2$, that is,

$$\big(K_i(x) - g_i(x)\big)^* = \big\{u \in H \mid (u,v) \geq 0,\ \forall v \in K_i(x) - g_i(x)\big\}. \tag{2.3}$$

Some examples of problems (2.1) and (2.2) are as follows.

(I) If $G = 0$ and $F(x,y) = Tx + Ax$ for all $x, y \in X$, where $T, A : H \to H$ are two mappings, then problem (2.2) reduces to finding $x \in H$ such that

$$Tx + Ax \in \big(K_1(x) - g_1(x)\big)^*, \tag{2.4}$$

which is called the generalized complementarity problem. The problem (2.4) was extended and studied by Jou and Yao [11] in Hilbert spaces, and by Chen et al. [5] in the setting of Banach spaces.

(II) Let $T : H \times H \to H$ be a mapping. If $F(x,y) = \rho T(y,x) + x - y$, $G(x,y) = \eta T(x,y) + y - x$ for all $x, y \in H$, $m_1 = m_2 = 0$, $K_1 = K_2 = K$, and $g_1 = g_2 = I$, where I is an identity mapping and $\rho > 0$, $\eta > 0$, then problem (2.1) reduces to finding $x, y \in K$ such that

$$\begin{aligned} &(\rho T(y,x) + x - y, z - x) \geq 0, \quad \forall z \in K,\\ &(\eta T(x,y) + y - x, z - y) \geq 0, \quad \forall z \in K, \end{aligned} \tag{2.5}$$

which is called the system of nonlinear variational inequality problems considered by Verma [16]. The special case of problem (2.5) was studied by Verma [15]. The problem (2.5) was extended and studied by Agarwal et al. [1], Kim and Kim [14], and Cho et al. [6].

(III) If $m_1 = m_2 = 0$, and $g_1 = g_2 = I$, then problem (2.1) reduces to finding $x \in K_1$ and $y \in K_2$ such that

$$\begin{aligned} &(F(x,y), z - x) \geq 0, \quad \forall z \in K_1,\\ &(G(x,y), z - y) \geq 0, \quad \forall z \in K_2, \end{aligned} \tag{2.6}$$

which is just the problem considered in [12] with F, G being single-valued mappings.

Definition 2.1. A mapping $N : H \times H \to H$ is said to be

(i) α-strongly monotone with respect to first argument if there exists some $\alpha > 0$ such that

$$(N(x,\cdot) - N(y,\cdot), x - y) \geq \alpha \|x - y\|^2, \quad \forall (x,y) \in H \times H; \tag{2.7}$$

(ii) ξ-Lipschitz continuous with respect to the first argument, if there exists a constant $\xi > 0$ such that

$$||N(x,\cdot) - N(y,\cdot)|| \leq \xi ||x - y||, \quad \forall (x,y) \in H \times H. \tag{2.8}$$

Similarly, we can define the strong monotonicity and Lipschitzian continuity with respect to the second argument of N.

Definition 2.2. A Mapping $N : H \times H \to H$ is said to be relaxed (a,b)-cocoercive with respect to the first argument if there exists constants $a > 0$ and $b > 0$ such that

$$(N(x,\cdot) - N(y,\cdot), x - y) \geq (-a)\|x - y\|^2 + b\|x - y\|^2, \quad \forall (x,y) \in H \times H. \tag{2.9}$$

If $a = 0$, then N is b-strongly monotone. Similarly, we can define the relaxed (a,b)-cocoercivity with respect to the second argument of N.

Lemma 2.3 [4]. *If $K \subset H$ is a closed convex subset and $z \in H$ is a given point, then there exists $x \in K$ such that*

$$(x - z, y - x) \geq 0, \quad \forall y \in K \tag{2.10}$$

if and only if $x = P_K z$, where P_K is the projection of H onto K.

Lemma 2.4 [4]. *The projection P_K is nonexpansive, that is,*

$$||P_K u - P_K v|| \leq \|u - v\|, \quad \forall u, v \in H. \tag{2.11}$$

Lemma 2.5 [18]. *Let $\{K_n\}$ be a sequence of closed convex subsets of H such that $H(K_n,K) \to 0$ as $n \to \infty$, where $H(\cdot,\cdot)$ is the Hausdorff metric, that is, for any $A,B \in \mathrm{CB}(H)$,*

$$H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}. \tag{2.12}$$

Then

$$\|P_{K_n}v - P_K v\| \longrightarrow 0 \quad (n \longrightarrow \infty),\ \forall v \in H. \tag{2.13}$$

Lemma 2.6 [4]. *If $K(u) = m(u) + K$ for all $u \in H$, then*

$$P_{K(u)}v = m(u) + P_K(v - m(u)). \tag{2.14}$$

From Lemmas 2.3 and 2.6, we have the following lemma.

Lemma 2.7. *If $K_1,K_2 \subset H$ are two closed convex cones, and $K_i(\cdot) = m(\cdot) + K_i$ $(i = 1,2)$, then $x, y \in H$ solve problem (2.1) if and only if $x, y \in H$ such that*

$$\begin{aligned} x &= x - g_1(x) + m_1(x) + P_{K_1}(g_1(x) - \rho F(x,y) - m_1(x)),\\ y &= y - g_2(y) + m_2(y) + P_{K_2}(g_2(y) - \rho G(x,y) - m_2(y)), \end{aligned} \tag{2.15}$$

where $\rho > 0$ is a constant.

Lemma 2.8 [17]. *Let $\{\mu_n\}$ be a real sequence of nonnegative numbers and $\{\nu_n\}$ be a real sequence of numbers in $[0,1]$ with $\sum_{n=0}^{\infty}\nu_n = \infty$. If there exists a constant n_1 such that*

$$\mu_{n+1} \le (1-\nu_n)\mu_n + \nu_n\delta_n, \quad \forall n \ge n_1, \tag{2.16}$$

where $\delta_n \ge 0$ for all $n \ge 0$, and $\delta_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty}\mu_n = 0$.

3. Existence and convergence

In this section, we construct an iterative algorithm to approximate the solution of problem (2.1) and study the convergence of the sequence generated by the algorithm.

Algorithm 3.1. For any given $x_0, y_0 \in H$, we compute

$$\begin{aligned} x_{n+1} &= x_n - g_1(x_n) + m_1(x_n) + P_{K_1}(g_1(x_n) - \rho F(x_n,y_n) - m_1(x_n)),\\ y_{n+1} &= y_n - g_2(y_n) + m_2(y_n) + P_{K_2}(g_2(y_n) - \rho G(x_n,y_n) - m_2(y_n)). \end{aligned} \tag{3.1}$$

Theorem 3.2. *Let $g_i : H \to H$ be η_i-strongly monotone and ζ_i-Lipschitz continuous and $m_i : H \to H$ be γ_i-Lipschitz continuous $(i = 1,2)$. Let $F : H \times H \to H$ be l_1, l_2-Lipschitz continuous with respect to the first, second arguments, respectively, and relaxed (a,b)-cocoercive with respect to the first argument. Let $G : H \times H \to H$ be n_1, n_2-Lipschitz continuous with respect to the first, second arguments, respectively, and relaxed (c,d)-cocoercive with respect* *to the second argument. If*

$$
\begin{aligned}
&2\sqrt{1+\zeta_1^2-2\eta_1}+2\gamma_1+\sqrt{1+\rho^2 l_1^2+2\rho a l_1^2-2\rho b}+\rho n_1<1,\\
&2\sqrt{1+\zeta_2^2-2\eta_2}+2\gamma_2+\sqrt{1+\rho^2 n_2^2+2\rho c n_2^2-2\rho d}+\rho l_2<1.
\end{aligned}
\tag{3.2}
$$

then there exist $x^*, y^* \in H$, which solve problem (2.1). Moreover, the iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 converge to x^* and y^*, respectively.

Proof. From (3.1) and Lemma 2.6, we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| = {} & \|x_n-g_1(x_n)+m_1(x_n)+P_{K_1}(g_1(x_n)-\rho F(x_n,y_n)-m_1(x_n))\\
& -[x_{n-1}-g_1(x_{n-1})+m_1(x_{n-1})\\
& +P_{K_1}(g_1(x_{n-1})-\rho F(x_{n-1},y_{n-1})-m_1(x_{n-1}))]\|\\
\le {} & \|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|+\|m_1(x_n)-m_1(x_{n-1})\|\\
& +\|P_{K_1}(g_1(x_n)-\rho F(x_n,y_n)-m_1(x_n))\\
& -P_{K_1}(g_1(x_{n-1})-\rho F(x_{n-1},y_{n-1})-m_1(x_{n-1}))\|.
\end{aligned}
\tag{3.3}
$$

Since g_1 is ζ_1-Lipschitz continuous and η_1-strongly monotone,

$$
\|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|^2 \le (1+\zeta_1^2-2\eta_1)\|x_n-x_{n-1}\|^2. \tag{3.4}
$$

From the γ_1-Lipschitzian continuity of m_1, we have

$$
\|m_1(x_n)-m_1(x_{n-1})\| \le \gamma_1\|x_n-x_{n-1}\|. \tag{3.5}
$$

Lemma 2.4 implies that P_{K_1} is nonexpansive and it follows from the strong monotonicity of g_1 that

$$
\begin{aligned}
&\|P_{K_1}(g_1(x_n)-\rho F(x_n,y_n)-m_1(x_n))-P_{K_1}(g_1(x_{n-1})-\rho F(x_{n-1},y_{n-1})-m_1(x_{n-1}))\|\\
&\quad\le \|(g_1(x_n)-\rho F(x_n,y_n)-m_1(x_n))-(g_1(x_{n-1})-\rho F(x_{n-1},y_{n-1})-m_1(x_{n-1}))\|\\
&\quad\le \|x_n-x_{n-1}-(g_1(x_n)-g_1(x_{n-1}))\|+\|m_1(x_n)-m_1(x_{n-1})\|\\
&\qquad+\|x_n-x_{n-1}-\rho(F(x_n,y_n)-F(x_{n-1},y_n))\|+\rho\|F(x_{n-1},y_n)-F(x_{n-1},y_{n-1})\|.
\end{aligned}
\tag{3.6}
$$

Since F is relaxed (a,b)-cocoercive and l_1-Lipschitz continuous with respect to the first argument,

$$\begin{aligned}
&\|x_n - x_{n-1} - \rho(F(x_n,y_n) - F(x_{n-1},y_n))\|^2 \\
&\quad = \|x_n - x_{n-1}\|^2 + \rho^2\|F(x_n,y_n) - F(x_{n-1},y_n)\|^2 \\
&\qquad - 2(x_n - x_{n-1}, \rho(F(x_n,y_n) - F(x_{n-1},y_n))) \\
&\quad \le \|x_n - x_{n-1}\|^2 + \rho^2\|F(x_n,y_n) - F(x_{n-1},y_n)\|^2 \\
&\qquad + 2\rho a\|F(x_n,y_n) - F(x_{n-1},y_n)\|^2 - 2\rho b\|x_n - x_{n-1}\|^2 \\
&\quad = (1 + l_1^2\rho^2 + 2\rho a l_1^2 - 2\rho b)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.7}$$

Since F is l_2-Lipschitz continuous with respect to the second argument,

$$\|F(x_{n-1},y_n) - F(x_{n-1},y_{n-1})\| \le l_2\|y_n - y_{n-1}\|. \tag{3.8}$$

It follows from (3.3)–(3.8) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \left(2\sqrt{1 + \zeta_1^2 - 2\eta_1} + 2\gamma_1 + \sqrt{1 + \rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b}\right)\|x_n - x_{n-1}\| + \rho l_2\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad \le \left(2\sqrt{1 + \zeta_2^2 - 2\eta_2} + 2\gamma_2 + \sqrt{1 + \rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d}\right)\|y_n - y_{n-1}\| + \rho n_1\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.10}$$

Now (3.9) and (3.10) imply

$$\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\quad \le \left(2\sqrt{1 + \zeta_1^2 - 2\eta_1} + 2\gamma_1 + \sqrt{1 + \rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + \rho n_1\right)\|x_n - x_{n-1}\| \\
&\qquad + \left(2\sqrt{1 + \zeta_2^2 - 2\eta_2} + 2\gamma_2 + \sqrt{1 + \rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + \rho l_2\right), \\
&\qquad\qquad \|y_n - y_{n-1}\| \le \omega(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
\omega = \max\Big\{&2\sqrt{1 + \zeta_1^2 - 2\eta_1} + 2\gamma_1 + \sqrt{1 + \rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + \rho n_1, \\
&2\sqrt{1 + \zeta_2^2 - 2\eta_2} + 2\gamma_2 + \sqrt{1 + \rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + \rho l_2\Big\}.
\end{aligned} \tag{3.12}$$

It follows from (3.2) that $\omega < 1$. Thus (3.11) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in H, and $\{x_n\}$ converges to $x^* \in H$, $\{y_n\}$ converges to $y^* \in H$. Since m_1, m_2, $g_1, g_2, P_{K_1}, P_{K_2}, F, G$ are all continuous, we have

$$\begin{aligned} x^* &= x^* - g_1(x^*) + m_1(x^*) + P_K\big(g_1(x^*) - \rho F(x^*, y^*) - m_1(x^*)\big), \\ y^* &= y^* - g_2(y^*) + m_2(y^*) + P_K\big(g_2(y^*) - \rho G(x^*, y^*) - m_2(y*)\big), \end{aligned} \tag{3.13}$$

The result follows then from Lemma 2.7. This completes the proof. □

Remark 3.3. Let $\rho > 0$ be a number satisfying the conditions.

$$\begin{aligned} &\left|\rho - \frac{b - al_1^2 - (1-e_1)n_1}{l_1^2 - n_1^2}\right| \\ &\qquad < \frac{(1-e_1)^2 - 1 + \left((b - al_1^2 - (1-e_1)n_1)^2\right)/(l_1^2 - n_1^2)}{l_1^2 - n_1^2}, \quad \rho n_1 < 1 - e_1,\ n_1 < l_1, \\ &\left|\rho - \frac{d - cn_2^2 - (1-e_2)l_2}{n_2^2 - l_2^2}\right| \\ &\qquad < \frac{(1-e_2)^2 - 1 + \left((d - cn_2^2 - (1-e_2)l_2)^2\right)/(n_2^2 - l_2^2)}{n_2^2 - l_2^2}, \quad \rho l_2 < 1 - e_2,\ l_2 < n_2, \end{aligned} \tag{3.14}$$

where $e_1 = 2\sqrt{1 + \zeta_1^2 - 2\eta_1} + 2\gamma_1$ and $e_2 = 2\sqrt{1 + \zeta_2^2 - 2\eta_2} + 2\gamma_2$. Then (3.2) holds.

4. Perturbed algorithm and stability

In this section, we construct a new perturbed iterative algorithm for solving problem (2.1) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Definition 4.1. Let T be a self-map of H, $x_0 \in H$ and let $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in H. Suppose that $\{x \in H : Tx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converge to a fixed point x^* of T. Let $\{u_n\} \subset H$ and let $\epsilon_n = \|u_{n+1} - f(T, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $\lim u_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T. Some results for the stability of various iterative processes, we refer to [1, 10] and the references therein.

Let $\{K_n^1\}$ and $\{K_n^2\}$ be two sequences of closed convex subsets of H such that $H(K_n^1, K) \to 0$, $H(K_n^2, K) \to 0$, when $n \to \infty$. Now we consider the following perturbed algorithm for solving problem (2.1).

Algorithm 4.2. For any given $x_0, y_0 \in H$, we compute

$$x_{n+1} = (1-t_n)x_n + t_n\left(x_n - g_1(x_n) + m_1(x_n) + P_{K_n^1}\left(g_1(x_n) - \rho F(x_n, y_n) - m_1(x_n)\right)\right) + t_n e_n,$$
$$y_{n+1} = (1-t_n)y_n + t_n\left(y_n - g_2(y_n) + m_2(y_n) + P_{K_n^2}\left(g_2(y_n) - \rho G(x_n, y_n) - m_2(y_n)\right)\right) + t_n j_n, \tag{4.1}$$

for all $n = 0, 1, 2, \ldots$, where $\{e_n\}$ and $\{j_n\}$ are two sequences of the elements of H, and the sequence $\{t_n\}$ satisfies the following conditions

$$0 \le t_n \le 1, \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} t_n = \infty. \tag{4.2}$$

Let $\{u_n\}$ and $\{v_n\}$ be any sequences in H and define $\epsilon_n = \epsilon_n^1 + \epsilon_n^2$ by

$$\begin{aligned}\epsilon_n^1 &= \| u_{n+1} - \{(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) \\ &\qquad + P_{K_1}(g_1(u_n) - \rho F(u_n, v_n) + m_1(u_n))] + t_n e_n\}\| \\ \epsilon_n^2 &= \| v_{n+1} - \{(1-t_n)v_n + t_n[v_n - g_2(v_n) + m_2(v_n) \\ &\qquad + P_{K_2}(g_2(v_n) - \rho G(u_n, v_n) + m_2(v_n))] + t_n j_n\}\|.\end{aligned} \tag{4.3}$$

THEOREM 4.3. *Let $g_i : X \to X$ be η_i-strongly monotone and ζ_i-Lipschitz continuous, and $m_i : X \to X$ be τ_i-Lipschitz continuous for $i = 1, 2$. Let $F : X \times X \to X$ be l_1, l_2-Lipschitz continuous with respect to the first and second arguments, respectively, and relaxed (a, b)-cocoercive with respect to the first argument. Let $G : X \times X \to X$ be n_1, n_2-Lipschitz continuous with respect to the first and second arguments, respectively, and relaxed (c, d)-cocoercive with respect to the second argument. Suppose $H(K_n, K) \to 0$ $(n \to \infty)$ and*

$$\begin{aligned}&\left|\rho - \frac{b - al_1^2 - (1-e_1)n_1}{l_1^2 - n_1^2}\right| \\ &\qquad < \frac{(1-e_1)^2 - 1 + \left((b - al_1^2 - (1-e_1)n_1)^2\right)/(l_1^2 - n_1^2)}{l_1^2 - n_1^2}, \quad \rho n_1 < 1 - e_1,\ n_1 < l_1, \\ &\left|\rho - \frac{d - cn_2^2 - (1-e_2)l_2}{n_2^2 - l_2^2}\right| \\ &\qquad < \frac{(1-e_2)^2 - 1 + \left((d - cn_2^2 - (1-e_2)l_2)^2\right)/(n_2^2 - l_2^2)}{n_2^2 - l_2^2}, \quad \rho l_2 < 1 - e_2,\ l_2 < n_2,\end{aligned} \tag{4.4}$$

where $e_i = 2\sqrt{1 + \zeta_i^2 - 2\eta_i} + 2\gamma_i$ for $i = 1, 2$. If $\lim_{n\to\infty} \|e_n\| = 0$ and $\lim_{n\to\infty} \|j_n\| = 0$, then we have the following conclusions.

(I) *The iterative sequences generated by Algorithm 4.2 converge to the unique solution of (2.1).*

(II) *Moreover, if $0 < t \le t_n$, then $\lim u_n = x^*$, $\lim v_n = y^*$ if and only if $\lim(\epsilon_n^1 + \epsilon_n^2) = 0$, where ϵ_n^1 and ϵ_n^2 are defined by (4.3).*

Proof. By Theorem 3.2, problem (2.1) admits a solution (x^*,y^*). It is easy to prove that (x^*,y^*) is the unique solution of (4.1). From Lemma 2.7, we have

$$\begin{aligned} x^* &= (1-t_n)x^* + t_n(x^* - g_1(x^*) + m_1(x^*) + P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))), \\ y^* &= (1-t_n)y^* + t_n(y^* - g_2(y^*) + m_2(y^*) + P_{K_2}(g_2(y^*) - \rho G(x^*,y^*) - m_2(y*))), \end{aligned} \tag{4.5}$$

Since P_K is nonexpansive and it follows from (4.1) and (4.5) that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad = \|(1-t_n)x_n + t_n[x_n - g_1(x_n) + m_1(x_n) + P_{K_n^1}(g_1(x_n) - \rho F(x_n,y_n) - m_1(x_n))] + t_n e_n \\ &\qquad - (1-t_n)x^* - t_n[x^* - g_1(x^*) + m_1(x^*) + P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))]\| \\ &\quad \le (1-t_n)\|(x_n - x^*)\| + t_n\|(x_n - x^*) + g_1(x_n) - g_1(x^*)\| + t_n\|m_1(x) - m_1(x^*)\| + t_n\|e_n\| \\ &\qquad + t_n\|P_{K_n^1}(g_1(x_n) - \rho F(x_n,y_n) - m_1(x_n)) - P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))\| \\ &\quad \le (1-t_n)\|(x_n - x^*)\| + t_n\|(x_n - x^*) + g_1(x_n) - g_1(x^*)\| + t_n\|m_1(x) - m_1(x^*)\| + t_n\|e_n\| \\ &\qquad + t_n\|P_{K_n^1}(g_1(x_n) - \rho F(x_n,y_n) - m_1(x_n)) - P_{K_n^1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))\| \\ &\qquad + t_n\|P_{K_n^1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*)) - P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))\| \\ &\quad \le (1-t_n)\|(x_n - x^*)\| + t_n\|(x_n - x^*) + g_1(x_n) - g_1(x^*)\| + t_n\|m_1(x) - m_1(x^*)\| + t_n\|e_n\| \\ &\qquad + t_n\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| + t_n\|m_1(x_n) - m_1(x^*)\| \\ &\qquad + t_n\|x_n - x^* - \rho(F(x_n,y_n) - F(x^*,y_n))\| + \rho t_n\|F(x^*,y_n) - F(x^*,y^*)\| \\ &\qquad + t_n\|P_{K_n^1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*)) - P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))\|. \end{aligned} \tag{4.6}$$

Since F is l_2-Lipschitz continuous with respect to the second argument,

$$\|F(x^*,y_n) - F(x^*,y^*)\| \le l_2\|y_n - y^*\|. \tag{4.7}$$

From the strong monotonicity and Lipschitzian continuity of g_1, we obtain

$$\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|^2 \le (1 + \zeta_1^2 - 2\eta_1)\|x_n - x^*\|^2. \tag{4.8}$$

The Lipschitzian continuity of m_1 implies

$$\|m_1(x_n) - m_1(x^*)\| \le \gamma_1\|x_n - x^*\|. \tag{4.9}$$

Since F is relaxed (a,b)-cocoercive and l_1-Lipschitz continuous with respect to the first argument,

$$\|x_n - x^* - \rho(F(x_n,y_n) - F(x^*,y_n))\| \le \sqrt{1+\rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b}\,\|x_n - x^*\|. \tag{4.10}$$

It follows from (4.6)–(4.10) that

$$\begin{aligned}\|x_{n+1} - x^*\| \le{}& \Big(2t_n\sqrt{1+\zeta_1^2 - 2\eta_1} + 2t_n\gamma_1 + t_n\sqrt{1+\rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + 1 - t_n\Big)\|x_n - x^*\| \\ &+ t_n\rho l_2\|y_n - y^*\| + t_n b_n + t_n\|e_n\|,\end{aligned} \tag{4.11}$$

where

$$b_n = \|P_{K_n^1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*)) - P_{K_1}(g_1(x^*) - \rho F(x^*,y^*) - m_1(x^*))\|. \tag{4.12}$$

From the fact of $H(K_n^1,K_1) \to 0$ and Lemma 2.5, we know that $b_n \to 0$.

Similarly, we have

$$\begin{aligned}\|y_{n+1} - y^*\| \le{}& \Big(2t_n\sqrt{1+\zeta_2^2 - 2\eta_2} + 2t_n\gamma_2 + t_n\sqrt{1+\rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + 1 - t_n\Big)\|y_n - y^*\| \\ &+ t_n\rho n_1\|x_n - x^*\| + t_n c_n + t_n\|j_n\|,\end{aligned} \tag{4.13}$$

where

$$c_n = \|P_{K_n^2}(g_2(y^*) - \rho G(x^*,y^*) - m_2(y^*)) - P_{K_2}(g_2(y^*) - \rho F(x^*,y^*) - m_2(y^*))\|, \tag{4.14}$$

and $c_n \to 0$. Now (4.11) and (4.13) imply

$$\begin{aligned}&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\ &\quad\le \Big(2t_n\sqrt{1+\zeta_1^2 - 2\eta_1} + 2t_n\gamma_1 + 1 - t_n + t_n\sqrt{1+\rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + t_n\rho n_1\Big)\|x_n - x^*\| \\ &\qquad + \Big(2t_n\sqrt{1+\zeta_2^2 - 2\eta_2} + 2t_n\gamma_2 + 1 - t_n + t_n\sqrt{1+\rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + t_n\rho l_2\Big)\|y_n - y^*\| \\ &\qquad + t_n c_n + t_n b_n + t_n\|e_n\| + t_n\|j_n\|.\end{aligned} \tag{4.15}$$

Let

$$h_1 = 2\sqrt{1+\zeta_1^2 - 2\eta_1} + 2\gamma_1 + \sqrt{1+\rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + \rho n_1,$$
$$h_2 = 2\sqrt{1+\zeta_2^2 - 2\eta_2} + 2\gamma_2 + \sqrt{1+\rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + \rho l_3. \tag{4.16}$$

From (4.4), it is easy to see that $0 \le h_1 < 1$ and $0 \le h_2 < 1$. Let $h = \max\{h_1, h_2\}$. Then $h < 1$ and so (4.15) reduces to

$$\begin{aligned}
&||x_{n+1} - x^*|| + ||y_{n+1} - y^*|| \\
&\quad\le (1-(1-h)t_n)(||x_n - x^*|| + ||y_n - y^*||) + t_n(b_n + c_n + ||e_n|| + ||j_n||) \\
&\quad= (1-(1-h)t_n)(||x_n - x^*|| + ||y_n - y^*||) + (1-h)t_n\delta_n,
\end{aligned} \tag{4.17}$$

where

$$\delta_n = \frac{b_n + c_n + ||e_n|| + ||j_n||}{1-h}. \tag{4.18}$$

From (4.12), (4.14) and Lemma 2.5, we have

$$b_n \longrightarrow 0, \quad c_n \longrightarrow 0, \quad \delta_n = \frac{b_n + c_n + ||e_n|| + ||j_n||}{1-h} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.19}$$

It follows from (4.2), (4.17), (4.19) and Lemma 2.8 that

$$x_n \longrightarrow x^*, \quad y_n \longrightarrow y^* \quad (n \longrightarrow \infty). \tag{4.20}$$

This completes the proof of Conclusion I.

Next we prove Conclusion II. By using (4.1), we obtain

$$\begin{aligned}
&||u_{n+1} - x^*|| \\
&\le ||u_{n+1} - \{(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) + P_{K_1}(g_1(u_n) - \rho F(u_n, v_n) + m_1(u_n))] + t_n e_n\}|| \\
&\quad + ||\{(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) + P_{K_1}(g_1(u_n) - \rho F(u_n, v_n) + m_1(u_n))] + t_n e_n\} - x^*|| \\
&\le ||(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) + P_{K_1}(g_1(u_n) - \rho F(u_n, v_n) + m_1(u_n))] + t_n e_n - x^*|| + \epsilon_n^1.
\end{aligned} \tag{4.21}$$

As the proof of inequality (4.11), we have

$$
\begin{aligned}
&\|\{(1-t_n)u_n+t_n[u_n-g_1(u_n)+m_1(u_n)+P_{K_1}(g_1(u_n)-\rho F(u_n,v_n)+m_1(u_n))]+t_ne_n\}-x^*\|\\
&\quad\le\Big(2t_n\sqrt{1+\zeta_1^2-2\eta_1}+2t_n\gamma_1+t_n\sqrt{1+\rho^2l_1^2+2\rho a l_1^2-2\rho b}+1-t_n\Big)\|u_n-x^*\|\\
&\qquad+t_n\rho l_2\|v_n-y^*\|+t_nb_n+t_n\|e_n\|,
\end{aligned}
\tag{4.22}
$$

where b_n is defined by (4.12). From (4.21) and (4.22), we have

$$
\begin{aligned}
\|u_{n+1}-x^*\|\le{}&\Big(2t_n\sqrt{1+\zeta_1^2-2\eta_1}+2t_n\gamma_1+t_n\sqrt{1+\rho^2l_1^2+2\rho a l_1^2-2\rho b}+1-t_n\Big)\|u_n-x^*\|\\
&+t_n\rho l_2\|v_n-y^*\|+t_nb_n+t_n\|e_n\|+\epsilon_n^1.
\end{aligned}
\tag{4.23}
$$

Similarly, we have

$$
\begin{aligned}
\|v_{n+1}-y^*\|\le{}&\Big(2t_n\sqrt{1+\zeta_2^2-2\eta_2}+2t_n\gamma_2+t_n\sqrt{1+\rho^2n_2^2+2\rho c n_2^2-2\rho d}+1-t_n\Big)\|v_n-y^*\|\\
&+t_n\rho n_1\|u_n-x^*\|+t_nc_n+t_n\|j_n\|+\epsilon_n^2,
\end{aligned}
\tag{4.24}
$$

where c_n is defined by (4.14). As the proof of inequality (4.17), and since $0<t\le t_n$, (4.23) and (4.24) yield

$$
\begin{aligned}
&\|u_{n+1}-x^*\|+\|v_{n+1}-y^*\|\\
&\quad\le(1-(1-h)t_n)(\|u_n-x^*\|+\|v_n-y^*\|)+t_n(b_n+c_n+\|e_n\|+\|j_n\|)+\epsilon_n^1+\epsilon_n^2\\
&\quad\le(1-(1-h)t_n)(\|u_n-x^*\|+\|v_n-y^*\|)+t_n[b_n+c_n+\|e_n\|+\|j_n\|+(\epsilon_n^1+\epsilon_n^2)/t]\\
&\quad=(1-(1-h)t_n)(\|u_n-x^*\|+\|v_n-y^*\|)+(1-h)t_n\delta_n,
\end{aligned}
\tag{4.25}
$$

where

$$
\delta_n=\frac{b_n+c_n+\|e_n\|+\|j_n\|+(\epsilon_n^1+\epsilon_n^2)/t}{1-h}.
\tag{4.26}
$$

Suppose that $\lim\epsilon_n^1+\epsilon_n^2=0$, then from $b_n\to 0$, $c_n\to 0$, $\|e_n\|\to 0$ and $\|j_n\|\to 0$, we have $\delta_n\to 0$ (as $n\to\infty$). Then from the fact of $t_n\to 0$, $\sum t_n=\infty$, (4.25) and Lemma 2.8, we have $\lim u_n=x^*$ and $\lim v_n=y^*$.

Conversely, suppose that $\lim u_n = x^*$ and $\lim v_n = y^*$. Then we have

$$
\begin{aligned}
\epsilon_n^1 + \epsilon_n^2 = {} & \|u_{n+1} - \{(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) \\
& + P_{K_1}(g_1(u_n) - \rho F(u_n,v_n) + m_1(u_n))] + t_n e_n\}\| \\
& + \|v_{n+1} - \{(1-t_n)v_n + t_n[v_n - g_2(v_n) + m_2(v_n) \\
& + P_{K_2}(g_2(v_n) - \rho G(u_n,v_n) + m_2(v_n))] + t_n j_n\}\| \\
\le {} & \|u_{n+1} - x^*\| + \|(1-t_n)u_n + t_n[u_n - g_1(u_n) + m_1(u_n) \\
& + P_{K_1}(g_1(u_n) - \rho F(u_n,v_n) + m_1(u_n))] \\
& + t_n e_n - x^*\| + \|v_{n+1} - y^*\| \\
& + \|(1-t_n)v_n + t_n[v_n - g_2(v_n) + m_2(v_n) \\
& + P_{K_2}(g_2(v_n) - \rho G(u_n,v_n) + m_2(v_n))] + t_n j_n - y^*\| \\
\le {} & \|u_{n+1} - x^*\| + \|v_{n+1} - y^*\| + t_n\rho l_2\|v_n - y^*\| + t_n\rho n_1\|u_n - x^*\| \\
& + \left(2t_n\sqrt{1+\zeta_1^2 - 2\eta_1} + 2t_n\gamma_1 + t_n\sqrt{1+\rho^2 l_1^2 + 2\rho a l_1^2 - 2\rho b} + 1 - t_n\right)\|u_n - x^*\| \\
& + \left(2t_n\sqrt{1+\zeta_2^2 - 2\eta_2} + 2t_n\gamma_2 + t_n\sqrt{1+\rho^2 n_2^2 + 2\rho c n_2^2 - 2\rho d} + 1 - t_n\right)\|v_n - y^*\| \\
& + t_n b_n + t_n\|e_n\| + t_n c_n + t_n\|j_n\|
\end{aligned}
\tag{4.27}
$$

and so $\epsilon_n^1 + \epsilon_n^2 \to 0$ as $n \to \infty$. This completes the proof. □

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