APPLICATIONS OF THE POINCARÉ INEQUALITY TO EXTENDED KANTOROVICH METHOD

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We apply the Poincaré inequality to study the extended Kantorovich method that was used to construct a closed-form solution for two coupled partial differential equations with mixed boundary conditions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain in $\mathbb{R}^n$. Consider the Dirichlet space $H_0^1(\Omega)$ which is the collection of all functions in the Sobolev space $L_1^2(\Omega)$ such that

$$H_0^1(\Omega) = \left\{ u \in L^2(\Omega) : u|_{\partial\Omega} = 0,\ \|u\|_{L^2} + \sum_{k=1}^{n} \left\| \frac{\partial u}{\partial x_k} \right\|_{L^2} < \infty \right\}. \tag{1.1}$$

The famous Poincaré inequality can be stated as follows: for $u \in H_0^1(\Omega)$, then there exists a universal constant C such that

$$\int_\Omega u^2(\mathbf{x})d\mathbf{x} \le C \sum_{k=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_k} \right|^2 d\mathbf{x}. \tag{1.2}$$

One of the applications of this inequality is to solve the modified version of the Dirichlet problem (see, John [5, page 97]): find a $v \in H_0^1(\Omega)$ such that

$$(u,v) = \int_\Omega \left[\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \right] d\mathbf{x} = \int_\Omega u(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}, \tag{1.3}$$

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where $\mathbf{x} = (x_1,\dots,x_n)$ with a fixed $f \in C(\bar{\Omega})$. Then the function v in (1.3) satisfied the boundary value problem

$$\begin{aligned} \Delta v &= -f, \quad \text{in } \Omega \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

In this paper, we will use the Poincaré inequality to study the extended Kantorovich method, see [6]. This method has been used extensively in many engineering problems, for example, readers can consult papers [4, 7, 8, 11, 12], and the references therein. Let us start with a model problem, see [8]. For a clamped rectangular box $\Omega = \prod_{k=1}^{n}[-a_k, a_k]$, subjected to a lateral distributed load, $\mathcal{P}(\mathbf{x}) = \mathcal{P}(x_1,\dots,x_n)$, the principle of virtual displacements yields

$$\prod_{\ell=1}^{n} \int_{-a_\ell}^{a_\ell} [\eta \nabla^4 \Phi - \mathcal{P}] \delta\Phi \, D\mathbf{x} = 0, \tag{1.5}$$

where Φ is the lateral deflection which satisfies the boundary conditions, η is the flexural rigidity of the box, and

$$\nabla^4 = \sum_{k=1}^{n} \frac{\partial^4}{\partial x_k^4} + \sum_{j\neq k} 2 \frac{\partial^4}{\partial x_j^2 \partial x_k^2}. \tag{1.6}$$

Since the domain Ω is a rectangular box, it is natural to assume the deflection in the form

$$\Phi(\mathbf{x}) = \Phi_{k_1\cdots k_n}(\mathbf{x}) = \prod_{\ell=1}^{n} f_{k_\ell}(x_\ell), \tag{1.7}$$

it follows that when $f_{k_2}(x_2)\cdots f_{k_n}(x_n)$ is prescribed a priori, (1.5) can be rewritten as

$$\int_{-a_1}^{a_1} \left[\prod_{\ell=2}^{n} \int_{-a_\ell}^{a_\ell} \left(\eta \nabla^4 \Phi_{k_1\cdots k_n} - \mathcal{P}\right) f_{k_\ell}(x_\ell)\, dx_\ell \right] \delta f_{k_1}(x_1) dx_1 = 0. \tag{1.8}$$

Equation (1.8) is satisfied when

$$\prod_{\ell=2}^{n} \int_{-a_\ell}^{a_\ell} \left(\eta \nabla^4 \Phi_{k_1\cdots k_n} - \mathcal{P}\right) f_{k_\ell}(x_\ell)\, dx_\ell = 0. \tag{1.9}$$

Similarly, when $\prod_{\ell=1,\ell\neq m}^{n} f_{k_\ell}(x_\ell)$ is prescribed a priori, (1.5) can be rewritten as

$$\int_{-a_m}^{a_m} \left[\prod_{\ell=1,\ell\neq m}^{n} \int_{-a_\ell}^{a_\ell} \left(\eta \nabla^4 \Phi_{k_1\cdots k_n} - \mathcal{P}\right) f_{k_\ell}(x_\ell)\, dx_\ell \right] \delta f_{k_m}(x_m) dx_m = 0. \tag{1.10}$$

It is satisfied when

$$\prod_{\ell=1,\ell\neq m}^{n} \int_{-a_\ell}^{a_\ell} \left(\eta \nabla^4 \Phi_{k_1\cdots k_n} - \mathcal{P}\right) f_{k_\ell}(x_\ell)\, dx_\ell = 0. \tag{1.11}$$

It is known that (1.9) and (1.11) are called the Galerkin equations of the extended Kantorovich method. Now we may first choose

$$f_{20}(x_2)\cdots f_{n0}(x_n)=\prod_{\ell=2}^{n}c_\ell\left(\frac{x_\ell^2}{a_\ell^2}-1\right)^2. \tag{1.12}$$

Then $\Phi_{10\cdots0}(\mathbf{x})=f_{11}(x_1)f_{20}(x_2)\cdots f_{n0}(x_n)$ satisfies the boundary conditions

$$\Phi_{10\cdots0}=0,\qquad \frac{\partial\Phi_{10\cdots0}}{\partial x_\ell}=0\quad \text{at } x_\ell=\pm a_\ell,\ x_1\in[-a_1,a_1], \tag{1.13}$$

for $\ell=2,\dots,n$. Now (1.9) becomes

$$\prod_{\ell=2}^{n}c_\ell\int_{-a_\ell}^{a_\ell}\left(\nabla^4\Phi_{10\cdots0}-\frac{\mathcal{P}}{\eta}\right)\left(\frac{x_\ell^2}{a_\ell^2}-1\right)^2dx_\ell=0, \tag{1.14}$$

which yields

$$C_4\frac{d^4f_{11}}{dx^4}+C_2\frac{d^2f_{11}}{dx^2}+C_0f_{11}=B. \tag{1.15}$$

After solving the above ODE, we can use $f_{11}(x_1)\prod_{\ell=3}^{n}f_{\ell0}(x_\ell)$ as a priori data and plug it into (1.10) to find $f_{21}(x_2)$. Then we obtain the function

$$\Phi_{110\cdots0}(\mathbf{x})=f_{11}(x_1)f_{21}(x_2)f_{30}(x_3)\cdots f_{n0}(x_n). \tag{1.16}$$

Continue this process until we obtain $\Phi_{1\cdots1}(\mathbf{x})=f_{11}(x_1)f_{21}(x_2)\cdots f_{n1}(x_n)$ and therefore completes the first cycle. Next, we use $f_{21}(x_2)\cdots f_{n1}(x_n)$ as our priori data and find $f_{12}(x_1)$. We continue this process and expect to find a sequence of "approximate solutions." The problem reduces to investigate the convergence of this sequence. Therefore, it is crucial to analyze (1.15). Moreover, from numerical point of view, we know that this sequence converges rapidly (see [1, 2]). Hence, it is necessary to give a rigorous mathematical proof of this method.

2. A convex linear functional on $H_0^2(\Omega)$

Denote

$$I[\phi]=\int_\Omega\{|\Delta\phi|^2-2\mathcal{P}(\mathbf{x})\phi(\mathbf{x})\}d\mathbf{x} \tag{2.1}$$

for $\Omega\subset\mathbb{R}^n$ a bounded Lipschitz domain. Here $\mathbf{x}=(x_1,\dots,x_n)$. As usual, denote

$$D^2\phi=\begin{bmatrix}\dfrac{\partial^2\phi}{\partial x^2} & \dfrac{\partial^2\phi}{\partial x\partial y}\\ \dfrac{\partial^2\phi}{\partial y\partial x} & \dfrac{\partial^2\phi}{\partial y^2}\end{bmatrix}. \tag{2.2}$$

For $\Omega \subset \mathbb{R}^2$, we define the Lagrangian function L associated to $I[\phi]$ as follows:

$$
\begin{gathered}
L : \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4 \longrightarrow \mathbb{R}, \\
(x, y; z; X, Y; U, V, S, W) \longmapsto (U+V)^2 - 2\mathcal{P}(x,y)z,
\end{gathered}
\tag{2.3}
$$

where $\mathcal{P}(x,y)$ is a fixed function on Ω which shows up in the integrand of $I[\phi]$. With the above definitions, we have

$$
L(x, y; \phi; \nabla\phi; D^2\phi) = |\Delta\phi|^2 - 2\mathcal{P}(x,y)\phi(x,y), \tag{2.4}
$$

where we have identified

$$
\begin{gathered}
z \longleftrightarrow \phi(x,y), \qquad X \longleftrightarrow \frac{\partial\phi}{\partial x}, \qquad Y \longleftrightarrow \frac{\partial\phi}{\partial y}, \\
U \longleftrightarrow \frac{\partial^2\phi}{\partial x^2}, \qquad V \longleftrightarrow \frac{\partial^2\phi}{\partial y^2}, \qquad S \longleftrightarrow \frac{\partial^2\phi}{\partial y\,\partial x}, \qquad W \longleftrightarrow \frac{\partial^2\phi}{\partial x\,\partial y}.
\end{gathered}
\tag{2.5}
$$

We also set $H_0^2(\Omega)$ to be the class of all square integrable functions such that

$$
H_0^2(\Omega) = \left\{ \psi \in L^2(\Omega) : \sum_{|\mathbf{k}| \le 2} \left\| \frac{\partial^{\mathbf{k}}\psi}{\partial \mathbf{x}^{\mathbf{k}}} \right\|_{L^2} < \infty,\ \psi|_{\partial\Omega} = 0,\ \nabla\psi|_{\partial\Omega} = \mathbf{0} \right\}. \tag{2.6}
$$

Fix $(x, y) \in \Omega$. We know that

$$
\nabla L(x, y; z; X, Y; U, V, S, W) = \begin{bmatrix} -2\mathcal{P}(x,y) & 0 & 0 & 2(U+V) & 2(U+V) & 0 & 0 \end{bmatrix}^T. \tag{2.7}
$$

Because the convexity of the function L in the remaining variables, then for all $(\tilde{z}; \tilde{X}, \tilde{Y}; \tilde{U}, \tilde{V}, \tilde{S}, \widetilde{W}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4$, one has

$$
\begin{aligned}
L\left(x, y; \tilde{z}; \tilde{X}, \tilde{Y}; \tilde{U}, \tilde{V}, \tilde{S}, \widetilde{W}\right) \ge{}& L(x, y; z; X, Y; U, V, S, W) - 2\mathcal{P}(x,y)(\tilde{z} - z) \\
&+ 2(U+V)\left[(\tilde{U} - U) + (\tilde{V} - V)\right].
\end{aligned}
\tag{2.8}
$$

In particular, one has, with $\tilde{z} = \tilde{\phi}(x,y)$,

$$
\begin{aligned}
L\left(x, y; \tilde{\phi}; \nabla\tilde{\phi}; D^2\tilde{\phi}\right) \ge{}& L(x, y; \phi; \nabla\phi; D^2\phi) + 2\Delta\phi[\nabla\tilde{\phi} - \nabla\phi] \\
&- 2\mathcal{P}(x,y)(\tilde{\phi} - \phi).
\end{aligned}
\tag{2.9}
$$

This implies that

$$
|\Delta\tilde{\phi}|^2 - 2\mathcal{P}(x,y)\tilde{\phi} \ge |\Delta\phi|^2 - 2\mathcal{P}(x,y)\phi + 2\Delta\phi[\Delta\tilde{\phi} - \Delta\phi] - 2\mathcal{P}(x,y)[\tilde{\phi} - \phi]. \tag{2.10}
$$

If instead we fix $(x, y; z) \in \Omega \times \mathbb{R}$, then

$$
\begin{aligned}
L\left(x, y; \tilde{z}; \tilde{X}, \tilde{Y}; \tilde{U}, \tilde{V}, \tilde{S}, \widetilde{W}\right) \ge{}& L(x, y; \tilde{z}; X, Y; U, V, S, W) \\
&+ 2(U+V)[(\tilde{U} - U) + (\tilde{V} - V)].
\end{aligned}
\tag{2.11}
$$

This implies that

$$L(x,y;\widetilde{\phi};\nabla\widetilde{\phi};D^2\widetilde{\phi}) \geq L(x,y;\widetilde{\phi};\nabla\phi;D^2\phi) + 2\Delta\phi[\nabla\widetilde{\phi} - \nabla\phi] \tag{2.12}$$

Therefore,

$$|\Delta\widetilde{\phi}|^2 - 2\mathcal{P}(x,y)\widetilde{\phi} \geq |\Delta\phi|^2 - 2\mathcal{P}(x,y)\widetilde{\phi} + 2\Delta\phi[\Delta\widetilde{\phi} - \Delta\phi]. \tag{2.13}$$

Lemma 2.1. *Suppose either*
(1) $\phi \in H_0^2(\Omega) \cap C^4(\Omega)$ *and* $\eta \in C_c^1(\Omega)$; *or*
(2) $\phi \in H_0^2(\Omega) \cap C^3(\bar{\Omega}) \cap C^4(\Omega)$ *and* $\eta \in H_0^2(\Omega)$.
Let $\delta I[\phi;\eta]$ *denote the first variation of* I *at* ϕ *in the direction* η, *that is,*

$$\delta I[\phi;\eta] = \lim_{\varepsilon\to 0} \frac{I[\phi+\varepsilon\eta] - I[\phi]}{\varepsilon}. \tag{2.14}$$

Then

$$\delta I[\phi;\eta] = 2\int_\Omega (\Delta^2\phi - \mathcal{P}(x,y))\eta\, dx\, dy. \tag{2.15}$$

Proof. We know that

$$I[\phi+\varepsilon\eta] - I[\phi] = 2\varepsilon\int_\Omega [\Delta\phi\Delta\eta - \mathcal{P}\eta]dx\,dy + \varepsilon^2\int_\Omega (\Delta\eta)^2 dx\,dy. \tag{2.16}$$

Hence,

$$\varepsilon I[\phi;\eta] = 2\int_\Omega [\Delta\phi\Delta\eta - \mathcal{P}\eta]dx\,dy. \tag{2.17}$$

If either assumption (1) or (2) holds, we can apply Green's formula to a Lipschitz domain Ω to obtain

$$\int_\Omega (\Delta\phi\Delta\eta)dx\,dy = \int_\Omega \eta(\Delta^2\phi)dx\,dy + \int_{\partial\Omega}\left[\frac{\partial\eta}{\partial\vec{n}}\Delta\phi - \eta\frac{\partial}{\partial\vec{n}}\Delta\phi\right]dx\,dy, \tag{2.18}$$

where $\partial/\partial\vec{n}$ is the derivative in the direction normal to $\partial\Omega$. Since either $\eta \in C_c^1(\Omega)$ or $\eta \in H_0^2(\Omega)$, the boundary term vanishes, which proves the lemma. □

Lemma 2.2. *Let* $\phi \in H_0^2(\Omega)$. *Then*

$$\|\phi\|_{H_0^2(\Omega)} \approx \|\Delta\phi\|_{L^2(\Omega)}. \tag{2.19}$$

Proof. The function $\phi \in H_0^2(\Omega)$ implies that there exists a sequence $\{\phi_k\} \subset C_c^\infty(\Omega)$ such that $\lim_{k\to\infty}\phi_k = \phi$ in H_0^2-norm. From a well-known result for the Calderón-Zygmund operator (see, Stein [10, page 77]), one has

$$\left\|\frac{\partial^2 f}{\partial x_j\partial x_\ell}\right\|_{L^p} \leq C\|\Delta f\|_{L^p}, \quad j,\ell = 1,\dots,n \tag{2.20}$$

for all $f \in C_c^2(\mathbb{R}^n)$ and $1 < p < \infty$. Here C is a constant that depends on n only. Applying this result to each ϕ_k, we obtain

$$\left\|\frac{\partial^2 \phi_k}{\partial x^2}\right\|_{L^2(\Omega)}, \left\|\frac{\partial^2 \phi_k}{\partial x \partial y}\right\|_{L^2(\Omega)}, \left\|\frac{\partial^2 \phi_k}{\partial y^2}\right\|_{L^2(\Omega)} \leq C\|\Delta \phi_k\|_{L^2(\Omega)}. \tag{2.21}$$

Taking the limit, we conclude that

$$\left\|\frac{\partial^2 \phi}{\partial x^2}\right\|_{L^2(\Omega)}, \left\|\frac{\partial^2 \phi}{\partial x \partial y}\right\|_{L^2(\Omega)}, \left\|\frac{\partial^2 \phi}{\partial y^2}\right\|_{L^2(\Omega)} \leq C\|\Delta \phi\|_{L^2(\Omega)}. \tag{2.22}$$

Applying Poincaré inequality twice to the function $\phi \in H_0^2(\Omega)$, we have

$$\begin{aligned} \|\phi\|_{L^2(\Omega)} &\leq C_1\|\nabla \phi\|_{L^2(\Omega)} \\ &\leq C_2\left(\left\|\frac{\partial^2 \phi}{\partial x^2}\right\|_{L^2(\Omega)} + \left\|\frac{\partial^2 \phi}{\partial x \partial y}\right\|_{L^2(\Omega)} + \left\|\frac{\partial^2 \phi}{\partial y^2}\right\|_{L^2(\Omega)}\right) \\ &\leq C\|\Delta \phi\|_{L^2(\Omega)}. \end{aligned} \tag{2.23}$$

Hence, $\|\phi\|_{L^2(\Omega)} \leq C\|\Delta \phi\|_{L^2(\Omega)}$. The reverse inequality is trivial. The proof of this lemma is therefore complete. □

Lemma 2.3. *Let $\{\phi_k\}$ be a bounded sequence in $H_0^2(\Omega)$. Then there exist $\phi \in H_0^2(\Omega)$ and a subsequence $\{\phi_{k_j}\}$ such that*

$$I[\phi] \leq \liminf I[\phi_{k_j}]. \tag{2.24}$$

Proof. By a weak compactness theorem for reflexive Banach spaces, and hence for Hilbert spaces, there exist a subsequence $\{\phi_{k_j}\}$ of $\{\phi_k\}$ and ϕ in $H_0^2(\Omega)$ such that $\phi_{k_j} \to \phi$ weakly in $H_0^2(\Omega)$. Since

$$H_0^2(\Omega) \subset H_0^1(\Omega) \subset\subset L^2(\Omega), \tag{2.25}$$

by the Sobolev embedding theorem, we have

$$\phi_{k_j} \longrightarrow \phi \quad \text{in } L^2(\Omega) \tag{2.26}$$

after passing to yet another subsequence if necessary.

Now fix $(x, y, \phi_{k_j}(x,y)) \in \mathbb{R}^2 \times \mathbb{R}$ and apply inequality (2.13), we have

$$|\Delta \phi_{k_j}|^2 - 2\mathcal{P}(x,y)\phi_{k_j}(x,y) \geq |\Delta \phi|^2 - 2\mathcal{P}(x,y)\phi_{k_j}(x,y) + 2\Delta \phi[\Delta \phi_{k_j} - \Delta \phi]. \tag{2.27}$$

This implies that

$$I[\phi_{k_j}] \geq \int_\Omega [|\Delta \phi|^2 - 2\mathcal{P}(x,y)\phi_{k_j}]\,dx\,dy + 2\int_\Omega \Delta \phi \cdot [\Delta \phi_{k_j} - \Delta \phi]\,dx\,dy. \tag{2.28}$$

But $\phi_{k_j} \to \phi$ in $L^2(\Omega)$, hence

$$\int_\Omega [|\Delta \phi|^2 - 2\mathcal{P}(x,y)\phi_{k_j}]\,dx\,dy \longrightarrow \int_\Omega [|\Delta \phi|^2 - 2\mathcal{P}(x,y)\phi]\,dx\,dy = I[\phi]. \tag{2.29}$$

Besides $\phi_{k_j} \to \phi$ weakly in $H_0^2(\Omega)$ implies that

$$\int_{\Omega} \Delta\phi \cdot [\Delta\phi_{k_j} - \Delta\phi]\,dx\,dy \longrightarrow 0. \tag{2.30}$$

It follows that when taking limit

$$I[\phi] \le \liminf_{j} I[\phi_{k_j}]. \tag{2.31}$$

This completes the proof of the lemma. □

Remark 2.4. The above proof uses the convexity of $L(x,y;z;X,Y;U,V,S,W)$ when $(x,y;z)$ is fixed. We already remarked at the beginning of this section that when (x,y) is fixed, $L(x,y;z;X,Y;U,V,S,W)$ is convex in the remaining variables, including the z-variable. That is, we are not required to utilize the full strength of the convexity of L here.

3. The extended Kantorovich method

Now, we shift our focus to the extended Kantorovich method for finding an approximate solution to the minimization problem

$$\min_{\phi \in H_0^2(\Omega)} I[\phi] \tag{3.1}$$

when $\Omega = [-a,a] \times [-b,b]$ is a rectangular region in $\mathbb{R}^2$. In the sequel, we will write $\phi(x,y)$ (resp., $\phi_k(x,y)$) as $f(x)g(y)$ (resp., $f_k(x)g_k(y)$) interchangeably as notated in Kerr and Alexander [8]. More specifically, we will study the extended Kantorovich method for the case $n = 2$, which has been used extensively in the analysis of stress on rectangular plates. Equivalently, we will seek for an approximate solution of the above minimization problem in the form $\phi(x,y) = f(x)g(y)$ where $f \in H_0^2([-a,a])$ and $g \in H_0^2([-b,b])$.

To phrase this differently, we will search for an approximate solution in the tensor product Hilbert spaces $H_0^2([-a,a]) \hat{\otimes} H_0^2([-b,b])$, and all sequences $\{\phi_k\}$, $\{\phi_{k_j}\}$ involved hereinafter reside in this Hilbert space. Without loss of generality, we may assume that $\Omega = [-1,1] \times [-1,1]$ for all subsequent results remain valid for the general case where $\Omega = [-a,a] \times [-b,b]$ by approximate scalings/normalizing of the x and y variables. As in [8], we will treat the special case $\mathcal{P}(x,y) = \gamma$, that is, we assume that the load $\mathcal{P}(x,y)$ is distributed equally on a given rectangular plate.

To start the extended Kantorovich scheme, we first choose $g_0(y) \in H_0^2([-1,1]) \cap C_c^\infty(-1,1)$, and find the minimizer $f_1(x) \in H_0^2([-1,1])$ of the functional:

$$\begin{aligned}
I[fg_0] &= \int_{\Omega} \left[|\Delta(fg_0)|^2 - 2\gamma f(x) g_0(y) \right] dx\,dy \\
&= \int_{\Omega} \left[g_0^2 (f'')^2 + 2 f f'' g_0 g_0'' + f^2 (g_0'')^2 - 2\gamma f g_0 \right] dx\,dy \\
&= \int_{-1}^{1} (f'')^2 dx \int_{-1}^{1} g_0^2\, dy + 2 \int_{-1}^{1} (g_0')^2 dy \int_{-1}^{1} (f')^2 dx \\
&\quad + \int_{-1}^{1} (g_0'')^2 dy \int_{-1}^{1} f^2 dx - 2\gamma \int_{-1}^{1} g_0\, dy \int_{-1}^{1} f\, dx,
\end{aligned} \tag{3.2}$$

where the last equality was obtained via the integration by parts of ff'' and $g_0 g_0''$. Since g_0 has been chosen a priori; we can rewrite the functional I as

$$\begin{aligned} J[f] = \|g_0\|_{L^2}^2 \int_{-1}^{1} (f'')^2 dx + 2\|g_0'\|_{L^2}^2 \int_{-1}^{1} (f')^2 dx \\ + \|g_0''\|_{L^2}^2 \int_{-1}^{1} f^2 dx - 2\gamma \int_{-1}^{1} g_0(y)dy \int_{-1}^{1} f\, dx \end{aligned} \tag{3.3}$$

for all $f \in H_0^2([-1,1])$. Now we may rewrite (3.3) in the following form:

$$\begin{aligned} J[f] &= \int_{-1}^{1} [C_1(f'')^2 + C_2(f')^2 + C_3 f^2 + C_4 f]dx \\ &\equiv \int_{-1}^{1} K(x, f, f', f'')dx \end{aligned} \tag{3.4}$$

with $K : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$(x; z; V; W) \longmapsto C_1 W^2 + C_2 V^2 + C_3 z^2 + C_4 z, \tag{3.5}$$

where

$$C_1 = \|g_0\|_{L^2}^2, \qquad C_2 = \|g_0'\|_{L^2}^2, \qquad C_3 = \|g_0''\|_{L^2}^2, \qquad C_4 = -2\gamma \int_{-1}^{1} g_0(y)dy. \tag{3.6}$$

As long as $g_0 \not\equiv 0$, as we have implicitly assumed, the Poincaré inequality implies that

$$0 < C_1 \le \alpha C_2 \le \beta C_3 \tag{3.7}$$

for some positive constants α and β, independent of g_0. Consequently, $K(x; z; V; W)$ is a strictly convex function in variable z, V, W when x is fixed. In other words, K satisfies

$$\begin{aligned} &K(x; \tilde{z}; \tilde{V}; \widetilde{W}) - K(x; z; V; W) \\ &\quad \ge \frac{\partial K}{\partial z}(x; z; V; W)(\tilde{z} - z) + \frac{\partial K}{\partial V}(x; z; V; W)(\tilde{V} - V) + \frac{\partial K}{\partial W}(x; z; V; W)(\widetilde{W} - W) \end{aligned} \tag{3.8}$$

for all $(x; z; V; W)$ and $(x; \tilde{z}; \tilde{V}; \widetilde{W})$ in $\mathbb{R}^4$, and the inequality becomes equality at $(x; z; V; W)$ only if $\tilde{z} = z$, or $\tilde{V} = V$, or $\widetilde{W} = W$.

Proposition 3.1. *Let $\mathscr{L} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying the following convexity condition:*

$$\begin{aligned} &\mathscr{L}(x; z + z'; V + V'; W + W') - \mathscr{L}(x; z; V; W) \\ &\quad \ge \frac{\partial \mathscr{L}}{\partial z}(x; z; V; W)z' + \frac{\partial \mathscr{L}}{\partial V}(x; z; V; W)V' + \frac{\partial \mathscr{L}}{\partial W}(x; z; V; W)W' \end{aligned} \tag{3.9}$$

for all $(x;z;V;W)$ *and* $(x;z+z';V+V';W+W')\in\mathbb{R}^4$, *with equality at* $(x;z;V;W)$ *only if* $z'=0$, *or* $V'=0$, *or* $W'=0$. *Also, let*

$$J[f]=\int_\alpha^\beta \mathcal{L}(x,f(x),f'(x),f''(x))\,dx,\quad \forall f\in H_0^2(\alpha,\beta). \tag{3.10}$$

Then

$$J[f+\eta]-J[f]\geq \delta J[f,\eta],\quad \forall \eta\in C_c^\infty(\alpha,\beta) \tag{3.11}$$

and equality holds only if $\eta\equiv 0$. *Here* $\delta J[f,\eta]$ *is the first variation of* J *at* f *in the direction* η.

Proof. Condition (3.9) means that at each x,

$$\begin{aligned}&\mathcal{L}(x;f+\eta;f'+\eta';f''+\eta'')-\mathcal{L}(x;f;f';f'')\\&\quad\geq \frac{\partial\mathcal{L}}{\partial z}(x;f;f';f'')\eta(x)+\frac{\partial\mathcal{L}}{\partial V}(x;f;f';f'')\eta'(x)+\frac{\partial\mathcal{L}}{\partial W}(x;f;f';f'')\eta''(x)\end{aligned} \tag{3.12}$$

for all $\eta\in C_c^\infty(\alpha,\beta)$ with equality only if $\eta(x)=0$, or $\eta'(x)=0$, or $\eta''(x)=0$. Equivalently, the equality holds in (3.12) at x only if $\eta(x)\eta'(x)=0$ or $\eta''(x)=0$. In other words,

$$\eta''(x)\frac{d}{dx}(\eta^2(x))=0. \tag{3.13}$$

Integrating (3.12) gives

$$J[f+\eta]-J[f]\geq\int_\alpha^\beta\left[\frac{\partial\mathcal{L}}{\partial z}\eta+\frac{\partial\mathcal{L}}{\partial V}\eta'+\frac{\partial\mathcal{L}}{\partial W}\eta''\right]dx=\delta J[f,\eta]. \tag{3.14}$$

Now suppose there exists $\eta\in C_c^\infty(\alpha,\beta)$ such that (3.14) is an equality. Since $\mathcal{L}$ is a smooth function, this equality forces (3.12) to be a pointwise equality, which implies, in view of (3.13), that

$$\eta''(x)\frac{d}{dx}(\eta^2(x))=0,\quad \forall x. \tag{3.15}$$

If $\eta''(x)\equiv 0$, then $\eta'(x)=$ constant which implies that $\eta'(x)\equiv 0$ (since $\eta\in C_c^\infty(\alpha,\beta)$). This tells us that $\eta\equiv$ constant and conclude that $\eta\equiv 0$ on the interval (α,β).

If $\eta''(x)\not\equiv 0$, set $U=\{x\in(\alpha,\beta):\eta''(x)\neq 0\}$. Then U is a non-empty open set which implies that there exist $x_0\in U$ and some open set $\mathbb{O}_{x_0}$ of x_0 contained in U. Then $\eta''(\xi)\neq 0$ for all $\xi\in\mathbb{O}_{x_0}\subset U$. Thus

$$\frac{d}{dx}(\eta^2)=0\quad\text{on }\mathbb{O}_{x_0}. \tag{3.16}$$

Hence, $\eta(\xi)\equiv$ constant on $\mathbb{O}_{x_0}$. But this creates a contradiction because $\eta''(\xi)\equiv 0$ on $\mathbb{O}_{x_0}$. Therefore,

$$J[f+\eta]-J[f]=\delta J[f,\eta] \tag{3.17}$$

only if $\eta(x)\equiv 0$, as desired. This completes the proof of the proposition. □

Corollary 3.2. *Let $J[f]$ be as in (3.4). Then $f_1 \in H_0^2([-1,1])$ is the unique minimizer for $J[f]$ if and only if f_1 solves the following ODE:*

$$||g_0||_{L^2}^2 \frac{d^4 f}{dx^4} - 2||g_0'||_{L^2}^2 \frac{d^2 f}{dx^2} + ||g_0''||_{L^2}^2 f = \gamma \int_{-1}^{1} g_0\, dy. \tag{3.18}$$

Proof. Suppose f_1 is the unique minimizer. Then f_1 is a local extremum of $J[f]$. This implies that $\delta J[f,\eta] = 0$ for all $\eta \in H_0^2([-1,1])$. Using the notations in (3.4), we have

$$\begin{aligned} 0 &= \delta J[f,\eta] \\ &= \int_{-1}^{1} \left[\frac{\partial K}{\partial z}\eta + \frac{\partial K}{\partial V}\eta' + \frac{\partial K}{\partial W}\eta'' \right] dx \\ &= \int_{-1}^{1} \left[\frac{\partial K}{\partial z} - \frac{d}{dx}\left(\frac{\partial K}{\partial V}\right) + \frac{d^2}{dx^2}\left(\frac{\partial K}{\partial W}\right) \right] \eta(x)\, dx \end{aligned} \tag{3.19}$$

for all $\eta \in H_0^2([-1,1])$. This implies that

$$\frac{\partial K}{\partial z} - \frac{d}{dx}\left(\frac{\partial K}{\partial V}\right) + \frac{d^2}{dx^2}\left(\frac{\partial K}{\partial W}\right) = 0, \tag{3.20}$$

which is the Euler-Lagrange equation (3.18). This also follows from Lemma 2.1 directly.

Conversely, assume f_1 solves (3.18). Then the above argument shows that $\delta J[f,\eta] = 0$ for all $\eta \in H_0^2([-1,1])$. Since K satisfies condition (3.9) in Proposition 3.1, we conclude that

$$J[f_1 + \eta] - J[f_1] \geq \delta J[f_1, \eta], \quad \forall \eta \in C_c^\infty([-1,1]). \tag{3.21}$$

This tells us that $J[f_1 + \eta] \geq J[f_1]$ for all $\eta \in C_c^\infty([-1,1])$ and $J[f_1+\eta] > J[f_1]$ if $\eta \not\equiv 0$. Observe that $J : H_0^1([-1,1]) \to \mathbb{R}$ as given in (3.4) is a continuous linear functional in the H_0^2-norm. This fact, combined with the density of $C_c^\infty([-1,1])$ in $H_0^2([-1,1])$ (in the H_0^2-norm), implies that

$$J[f_1 + \eta] \geq J[f_1], \quad \forall \eta \in C_c^\infty([-1,1]). \tag{3.22}$$

This means that for all $\varphi \in H_0^2([-1,1])$, we have $J[\varphi] \geq J[f_1]$ and if $\varphi \not\equiv f_1$ (almost everywhere), then $\varphi - f_1 \not\equiv 0$ and hence, $J[\varphi] > J[f_1]$. Thus f_1 is the unique minimum for J. □

Reversing the roles of f and g, that is, fixing f_0 and finding $g_1 \in H_0^2$ to minimize $I[f_0 g]$ over $g \in H_0^2([-1,1])$, we obtain the same conclusion by using the same arguments.

Corollary 3.3. *Fix $f_0 \in H_0^2([-1,1])$. Then $g_1 \in H_0^2([-1,1])$ is the unique minimizer for*

$$\begin{aligned} J[g] = I[f_0 g] = ||f_0||_{L^2}^2 \int_{-1}^{1} (g'')^2 dy + 2||f_0'||_{L^2}^2 \int_{-1}^{1} (g')^2 dy \\ + ||f_0''||_{L^2}^2 \int_{-1}^{1} g^2 dy - 2\gamma ||f_0||_{L^1} \int_{-1}^{1} g\, dy \end{aligned} \tag{3.23}$$

if and only if g_1 solves the Euler-Lagrange equation

$$||f_0||_{L^2}^2 \frac{d^4 g}{dy^4} - 2||f_0'||_{L^2}^2 \frac{d^2 g}{dy^2} + ||f_0''||_{L^2}^2 g = 2\gamma \int_{-1}^{1} f_0(x)dx. \tag{3.24}$$

Now we search for the solution $f_1 \in H_0^2([-1,1])$ in (3.18), that is,

$$||g_0||_{L^2}^2 \frac{d^4 f}{dx^4} - 2||g_0'||_{L^2}^2 \frac{d^2 f}{dx^2} + ||g_0''||_{L^2}^2 f = 2\gamma \int_{-1}^{1} g_0(y)dy. \tag{3.25}$$

Rewrite the above ODE in the following form:

$$||g_0||_{L^2}^2 \left[\left(D - \frac{||g'||_{L^2}^2}{||g_0||_{L^2}^2} \right)^2 + \frac{||g''||_{L^2}^2}{||g_0||_{L^2}^2} - \frac{||g'||_{L^2}^4}{||g_0||_{L^2}^4} \right] f = 2\gamma \int_{-1}^{1} g_0(y)dy, \tag{3.26}$$

where $D = d^2/dx^2$.

Remark 3.4. In general when $g \in H^2$, that is, g needs not satisfy the zero boundary conditions for function in H_0^2, then the quantity

$$\left(\frac{||g_0''||_{L^2}^2}{||g_0||_{L^2}^2} - \frac{||g_0'||_{L^2}^4}{||g_0||_{L^2}^4} \right) \tag{3.27}$$

can take on any values. However, if $g \in H_0^2$ and $g_0 \not\equiv 0$, as proved below, this quantity is always positive.

Lemma 3.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 1$. Let $g \in H_0^2(\Omega)$ be arbitrary. Then*

$$\|\nabla g\|_{L^2}^2 \leq \|g\|_{L^2} \cdot \|\Delta g\|_{L^2}, \tag{3.28}$$

and equality holds if and only if $g \equiv 0$.

Proof. Integration by parts yields

$$\|\nabla g\|_{L^2}^2 = \int_\Omega \nabla g \cdot \nabla g \, d\mathbf{x} = -\int_\Omega g \Delta g \, d\mathbf{x} + \int_{\partial\Omega} g \frac{\partial g}{\partial \vec{n}} d\sigma = -\int_\Omega g \Delta g \, d\mathbf{x}. \tag{3.29}$$

By the Cauchy-Schwartz inequality, we have

$$\|\nabla g\|_{L^2}^2 \leq \|g\|_{L^2} \cdot \|\Delta g\|_{L^2}, \tag{3.30}$$

and the equality holds if and only if (see Lieb-Loss [9])

(i) $|g(\mathbf{x})| = \lambda |\Delta g(\mathbf{x})|$ almost everywhere for some $\lambda > 0$,

(ii) $g(\mathbf{x})\Delta g(\mathbf{x}) = e^{i\theta} |g(\mathbf{x})| \cdot |\Delta g(\mathbf{x})|$.

Since g is real-valued, (i) and (ii) imply

$$g(\mathbf{x})\Delta g(\mathbf{x}) = \lambda (\Delta g(\mathbf{x}))^2. \tag{3.31}$$

So, g must satisfy the following PDE:

$$\Delta g - \frac{1}{\lambda} g = 0, \tag{3.32}$$

where $g \in H_0^2(\Omega)$. But the only solution to this PDE is $g \equiv 0$ (see, Evans [3, pages 300–302]). This completes the proof of the lemma. □

Remark 3.6. If $n = 1$, one can solve $g'' - \lambda^{-1} g = 0$ directly without having to appeal to the theory of elliptic PDEs.

PROPOSITION 3.7. *The solutions of (3.18) and (3.24) have the same form.*

Proof. Using either Lemma 3.5 in case $n = 1$ to the above remark, we see that

$$\frac{\|g''\|_{L^2}^2}{\|g_0\|_{L^2}^2} - \frac{\|g'\|_{L^2}^4}{\|g_0\|_{L^2}^4} > 0 \quad \text{if } g_0 \not\equiv 0. \tag{3.33}$$

Hence the characteristic polynomial associated to (3.26) has two pairs of complex conjugate roots as long as $g_0 \not\equiv 0$. Apply the same arguments to the ODE in (3.24) and the proposition is proved. □

Remark 3.8. The statement in Proposition 3.7 was claimed in [8] without verification. Indeed the authors stated therein that the solutions of (3.18) and (3.24) are of the same form because of the positivity of the coefficients on the left-hand side of (3.18) and (3.24). As observed in Remark 3.4 and proved in Proposition 3.7, the positivity requirement is not sufficient. The fact that $f_0, g_0 \in H_0^2$ must be used to conclude this assumption.

4. Explicit solution for (3.26)

We now find the explicit solution for (3.26), and hence for (3.18). Let

$$\begin{aligned} r &= \frac{\|g'\|_{L^2}}{\|g_0\|_{L^2}}, & t &= \frac{\|g''\|_{L^2}}{\|g_0\|_{L^2}}, \\ \rho &= \sqrt{\frac{t + r^2}{2}}, & \kappa &= \sqrt{\frac{t - r^2}{2}}. \end{aligned} \tag{4.1}$$

Then from Proposition 3.7 and its proof, the 4 roots of the characteristic polynomial associated to ODE (3.26) are

$$\rho + i\kappa, \qquad \rho - i\kappa, \qquad -\rho - i\kappa, \qquad -\rho + i\kappa. \tag{4.2}$$

Thus the homogeneous solution of (3.26) is

$$\begin{aligned} f_h(x) = {} & c_1 \cosh(\rho x)\cos(\kappa x) + c_2 \sinh(\rho x)\cos(\kappa x) \\ & + c_3 \cosh(\rho x)\sin(\kappa x) + c_4 \sinh(\rho x)\sin(\kappa x). \end{aligned} \tag{4.3}$$

It follows that a particular solution of (3.26) is

$$f_p(x) = \frac{2\gamma \int_{-1}^{1} g_0(y)dy}{\|g_0''\|_{L^2}^2}. \tag{4.4}$$

Thus the solution of (3.18) is

$$\begin{aligned} f(x) = {} & c_1 \cosh(\rho x)\cos(\kappa x) + c_2 \sinh(\rho x)\cos(\kappa x) \\ & + c_3 \cosh(\rho x)\sin(\kappa x) + c_4 \sinh(\rho x)\sin(\kappa x) + c_p, \end{aligned} \tag{4.5}$$

where $c_p = 2\gamma \int_{-1}^{1} g_0(y)dy/\|g_0''\|_{L^2}^2$ is a known constant. This implies that

$$\begin{aligned} f'(x) = {} & \rho c_1 \sinh(\rho x)\cos(\kappa x) - \kappa c_1 \cosh(\rho x)\sin(\kappa x) \\ & + \rho c_2 \cosh(\rho x)\cos(\kappa x) - \kappa c_2 \sinh(\rho x)\sin(\kappa x) \\ & + \rho c_3 \sinh(\rho x)\sin(\kappa x) + \kappa c_3 \cosh(\rho x)\cos(\kappa x) \\ & + \rho c_4 \cosh(\rho x)\sin(\kappa x) + \kappa c_4 \sinh(\rho x)\cos(\kappa x). \end{aligned} \tag{4.6}$$

Apply the boundary conditions $f(1) = f(-1) = f'(1) = f'(-1) = 0$, we get

$$\begin{aligned} & c_1 \cosh(\rho)\cos(\kappa) + c_2 \sinh(\rho)\cos(\kappa) + c_3 \cosh(\rho)\sin(\kappa) + c_4 \sinh(\rho)\sin(\kappa) = -c_p, \\ & c_1 \cosh(\rho)\cos(\kappa) - c_2 \sinh(\rho)\cos(\kappa) - c_3 \cosh(\rho)\sin(\kappa) + c_4 \sinh(\rho)\sin(\kappa) = -c_p, \\ & c_1\big[\rho \sinh(\rho)\cos(\kappa) - \kappa \cosh(\rho)\sin(\kappa)\big] + c_2\big[\rho \cosh(\rho)\cos(\kappa) - \kappa \sinh(\rho)\sin(\kappa)\big] \\ & \quad + c_3\big[\rho \sinh(\rho)\sin(\kappa) + \kappa \cosh(\rho)\cos(\kappa)\big] + c_4\big[\rho \cosh(\rho)\sin(\kappa) + \kappa \sinh(\rho)\cos(\kappa)\big] = 0, \\ & c_1\big[-\rho \sinh(\rho)\cos(\kappa) + \kappa \cosh(\rho)\sin(\kappa)\big] + c_2\big[\rho \cosh(\rho)\cos(\kappa) - \kappa \sinh(\rho)\sin(\kappa)\big] \\ & \quad + c_3\big[\rho \sinh(\rho)\sin(\kappa) + \kappa \cosh(\rho)\cos(\kappa)\big] - c_4\big[\rho \cosh(\rho)\sin(\kappa) + \kappa \sinh(\rho)\cos(\kappa)\big] = 0. \end{aligned} \tag{4.7}$$

Hence,

$$c_1 \cosh(\rho)\cos(\kappa) + c_4 \sinh(\rho)\sin(\kappa) = -c_p, \tag{4.8}$$

$$c_2 \sinh(\rho)\cos(\kappa) + c_3 \cosh(\rho)\sin(\kappa) = 0, \tag{4.9}$$

$$\begin{aligned} & c_2\big[\rho \cosh(\rho)\cos(\kappa) - \kappa \sinh(\rho)\sin(\kappa)\big] \\ & \quad + c_3\big[\rho \sinh(\rho)\sin(\kappa) + \kappa \cosh(\rho)\cos(\kappa)\big] = 0, \end{aligned} \tag{4.10}$$

$$\begin{aligned} & c_1\big[\rho \sinh(\rho)\cos(\kappa) - \kappa \cosh(\rho)\sin(\kappa)\big] \\ & \quad + c_4\big[\rho \cosh(\rho)\sin(\kappa) + \kappa \sinh(\rho)\cos(\kappa)\big] = 0. \end{aligned} \tag{4.11}$$

We know, beforehand, that there must be a unique solution. Thus (4.9) and (4.23) force

$c_2 = c_3 = 0$. We are left to solve for c_1 and c_4 from (4.8) and (4.11). But (4.11) tells us that

$$c_1 = -c_4 \frac{\rho\cosh(\rho)\sin(\kappa) + \kappa\sinh(\rho)\cos(\kappa)}{\rho\sinh(\rho)\cos(\kappa) - \kappa\cosh(\rho)\sin(\kappa)}. \tag{4.12}$$

Substituting (4.12) into (4.8), we have

$$c_4 = c_p \frac{\rho\sinh(\rho)\cos(\kappa) - \kappa\cosh(\rho)\sin(\kappa)}{\rho\sin(\kappa)\cos(\kappa) + \kappa\sinh(\rho)\cosh(\rho)}. \tag{4.13}$$

Plugging (4.13) into (4.12), we have

$$c_1 = -c_p \frac{\rho\cosh(\rho)\sin(\kappa) + \kappa\sinh(\rho)\cos(\kappa)}{\rho\sin(\kappa)\cos(\kappa) + \kappa\sinh(\rho)\cosh(\rho)}. \tag{4.14}$$

Therefore, the solution $f_1(x)$ can be written in the form

$$f_1(x) = c_p\left[\frac{K_1}{K_0}\cosh(\rho x)\cos(\kappa x) + \frac{K_2}{K_0}\sinh(\rho x)\sin(\kappa x) + 1\right], \tag{4.15}$$

where

$$\begin{gathered}
c_p = \frac{2\gamma\int_{-1}^{1} g_0(y)\,dy}{\|g_0''\|_{L^2}^2}, \\
\rho = \sqrt{\frac{t+r^2}{2}} = \sqrt{\frac{\|g_0''\|_{L^2}/\|g_0\|_{L^2} + \|g_0'\|_{L^2}^2/\|g_0\|_{L^2}^2}{2}}, \\
\kappa = \sqrt{\frac{t-r^2}{2}} = \sqrt{\frac{\|g_0''\|_{L^2}/\|g_0\|_{L^2} - \|g_0'\|_{L^2}^2/\|g_0\|_{L^2}^2}{2}}, \\
K_0 = \rho\sin(\kappa)\cos(\kappa) + \kappa\sinh(\rho)\cosh(\rho), \\
K_1 = -\rho\cosh(\rho)\sin(\kappa) - \kappa\sinh(\rho)\cos(\kappa), \\
K_2 = \rho\sinh(\rho)\cos(\kappa) - \kappa\cosh(\rho)\sin(\kappa).
\end{gathered} \tag{4.16}$$

The next step in the extended Kantorovich method is to fix $f_1(x)$ just found above and solve for $g_1(y) \in H_0^2([-1,1])$ from (3.24). Lemma 2.2 and the computation above show that

$$g_1(y) = \tilde{c}_p\left[\frac{\tilde{K}_1}{\tilde{K}_0}\cosh(\tilde{\rho} y)\cos(\tilde{\kappa} y) + \frac{\tilde{K}_2}{\tilde{K}_0}\sinh(\tilde{\rho} y)\sin(\tilde{\kappa} y) + 1\right], \tag{4.17}$$

where

$$
\begin{aligned}
\widetilde{c}_p &= \frac{2\gamma\int_{-1}^{1} f_1(x)dx}{\|f_1''\|_{L^2}^2}, \\
\widetilde{\rho} &= \sqrt{\frac{\|f_1''\|_{L^2}/\|f_1\|_{L^2} + \|f_1'\|_{L^2}^2/\|f_1\|_{L^2}^2}{2}}, \\
\widetilde{\kappa} &= \sqrt{\frac{\|f_1''\|_{L^2}/\|f_1\|_{L^2} - \|f_1'\|_{L^2}^2/\|f_1\|_{L^2}^2}{2}}, \\
\widetilde{K}_0 &= \widetilde{\rho}\sin(\widetilde{\kappa})\cos(\widetilde{\kappa}) + \widetilde{\kappa}\sinh(\widetilde{\rho})\cosh(\widetilde{\rho}), \\
\widetilde{K}_1 &= -\widetilde{\rho}\cosh(\widetilde{\rho})\sin(\widetilde{\kappa}) - \widetilde{\kappa}\sinh(\widetilde{\rho})\cos(\widetilde{\kappa}), \\
\widetilde{K}_2 &= \widetilde{\rho}\sinh(\widetilde{\rho})\cos(\widetilde{\kappa}) - \widetilde{\kappa}\cosh(\widetilde{\rho})\sin(\widetilde{\kappa}).
\end{aligned}
\tag{4.18}
$$

Now we start the next iteration by fixing $g_1(y)$ and solving for $f_2(x)$ in (3.18), and so forth. In particular, we will write

$$
f_n(x) = c_n\left[\frac{K_{1n}}{K_{0n}}\cosh(\rho_n x)\cos(\kappa_n x) + \frac{K_{2n}}{K_{0n}}\sinh(\rho_n x)\sin(\kappa_n x) + 1\right], \tag{4.19}
$$

where

$$
\begin{aligned}
c_n &= \frac{2\gamma\int_{-1}^{1} g_{n-1}(y)dy}{\|g_{n-1}''\|_{L^2}^2}, \\
\rho_n &= \sqrt{\frac{\|g_{n-1}''\|_{L^2}/\|g_{n-1}\|_{L^2} + \|g_{n-1}'\|_{L^2}^2/\|g_{n-1}\|_{L^2}^2}{2}}, \\
\kappa_n &= \sqrt{\frac{\|g_{n-1}''\|_{L^2}/\|g_{n-1}\|_{L^2} - \|g_{n-1}'\|_{L^2}^2/\|g_{n-1}\|_{L^2}^2}{2}}, \\
K_{0n} &= \rho_n\sin(\kappa_n)\cos(\kappa_n) + \kappa_n\sinh(\rho_n)\cosh(\rho_n), \\
K_{1n} &= -\rho_n\cosh(\rho_n)\sin(\kappa_n) - \kappa_n\sinh(\rho_n)\cos(\kappa_n), \\
K_{2n} &= \rho_n\sinh(\rho_n)\cos(\kappa_n) - \kappa_n\cosh(\rho_n)\sin(\kappa_n).
\end{aligned}
\tag{4.20}
$$

Similarly,

$$
g_n(y) = \widetilde{c}_n\left[\frac{\widetilde{K}_{1n}}{\widetilde{K}_{0n}}\cosh(\widetilde{\rho}_n y)\cos(\widetilde{\kappa}_n y) + \frac{\widetilde{K}_{2n}}{\widetilde{K}_{0n}}\sinh(\widetilde{\rho}_n y)\sin(\widetilde{\kappa}_n y) + 1\right], \tag{4.21}
$$

where

$$\widetilde{c}_n = \frac{2\gamma\int_{-1}^{1} f_n(x)dx}{\|f_n''\|_{L^2}^2},$$
$$\widetilde{\rho} = \sqrt{\frac{\|f_n''\|_{L^2}/\|f_n\|_{L^2} + \|f_n'\|_{L^2}^2/\|f_n\|_{L^2}^2}{2}},$$
$$\widetilde{\kappa}_n = \sqrt{\frac{\|f_n''\|_{L^2}/\|f_n\|_{L^2} - \|f_n'\|_{L^2}^2/\|f_n\|_{L^2}^2}{2}}, \tag{4.22}$$
$$\widetilde{K}_{0n} = \widetilde{\rho}_n \sin(\widetilde{\kappa}_n)\cos(\widetilde{\kappa}_n) + \widetilde{\kappa}_n \sinh(\widetilde{\rho}_n)\cosh(\widetilde{\rho}_n),$$
$$\widetilde{K}_{1n} = -\widetilde{\rho}_n \cosh(\widetilde{\rho}_n)\sin(\widetilde{\kappa}_n) - \widetilde{\kappa}_n \sinh(\widetilde{\rho}_n)\cos(\widetilde{\kappa}_n),$$
$$\widetilde{K}_{2n} = \widetilde{\rho}_n \sinh(\widetilde{\rho}_n)\cos(\widetilde{\kappa}_n) - \widetilde{\kappa}_n \cosh(\widetilde{\rho}_n)\sin(\widetilde{\kappa}_n).$$

In summary, a solution $\phi_n(x, y)$ in Lemma 2.3 can be written into the following form:

$$\begin{aligned}
\phi_n(x,y) &= f_n(x)g_n(y)\\
&= c_n\widetilde{c}_n\Big[\frac{K_{1n}\widetilde{K}_{1n}}{K_{0n}\widetilde{K}_{0n}}\cosh(\rho_n x)\cosh(\widetilde{\rho}_n y)\cos(\kappa_n x)\cos(\widetilde{\kappa}_n y)\\
&\quad + \frac{K_{1n}\widetilde{K}_{2n}}{K_{0n}\widetilde{K}_{0n}}\cosh(\rho_n x)\sinh(\widetilde{\rho}_n y)\cos(\kappa_n x)\sin(\widetilde{\kappa}_n y)\\
&\quad + \frac{K_{2n}\widetilde{K}_{1n}}{K_{0n}\widetilde{K}_{0n}}\sinh(\rho_n x)\cosh(\widetilde{\rho}_n y)\sin(\kappa_n x)\cos(\widetilde{\kappa}_n y)\\
&\quad + \frac{K_{2n}\widetilde{K}_{2n}}{K_{0n}\widetilde{K}_{0n}}\sinh(\rho_n x)\sinh(\widetilde{\rho}_n y)\sin(\kappa_n x)\cos(\widetilde{\kappa}_n y)\\
&\quad + \frac{K_{1n}}{K_{0n}}\cosh(\rho_n x)\cos(\kappa_n x) + \frac{K_{2n}}{K_{0n}}\sinh(\rho_n x)\sin(\kappa_n x)\\
&\quad + \frac{\widetilde{K}_{1n}}{\widetilde{K}_{0n}}\cosh(\widetilde{\rho}_n y)\sin(\widetilde{\kappa}_n y) + \frac{\widetilde{K}_{2n}}{\widetilde{K}_{0n}}\sinh(\widetilde{\rho}_n y)\sin(\widetilde{\kappa}_n y) + 1\Big].
\end{aligned} \tag{4.23}$$

5. Convergence of the solutions

In order to discuss the convergence of the extended Kantorovich method, let us start with the following auxiliary lemma.

LEMMA 5.1. *Let $\phi_n(x, y) = f_n(x)g_n(y)$ and $\psi_n(x, y) = f_{n+1}(x)g_n(y)$. Then these two sequences are bounded in $H_0^2(\Omega)$.*

Proof. We will verify the boundedness of $\{\psi_n\}$ for the arguments which is identical for the sequence $\{\phi_n\}$. Fix an integer $n \in \mathbf{Z}_+$ and assume that g_n has been determined from the extended Kantorovich scheme when $n \geq 1$ or g_n has been chosen a priori when $n = 0$.

Then f_{n+1} is determined by minimizing

$$\begin{aligned} I[fg_n] &= J[f] \\ &= \|g_n\|_{L^2}^2 \int (f'')^2 dx + 2\|g_n'\|_{L^2}^2 \int (f')^2 dx \\ &\quad + \|g_n''\|_{L^2}^2 \int f^2 dx - 2\gamma \int g_n\, dy \cdot \int f\, dx. \end{aligned} \tag{5.1}$$

By Corollary 3.2, if f_{n+1} is as in (4.19), then f_{n+1} is the unique minimum for $J[f]$ over $H_0^2(\Omega)$. Thus we must have

$$I[f_{n+1}g_n] = I[f_{n+1}] < I[0] = 0. \tag{5.2}$$

This implies that

$$\int_\Omega |\Delta\psi_n|^2 - \gamma \int_\Omega \psi_n dx\, dy < 0. \tag{5.3}$$

Lemma 2.2 then yields

$$\|\psi_n\|_{H_0^2(\Omega)}^2 < C\gamma \|\psi_n\|_{L^2(\Omega)}^2 < C\gamma \|\psi_n\|_{H_0^2(\Omega)}. \tag{5.4}$$

Therefore, $\|\psi_n\|_{H_0^2(\Omega)} < C\gamma$ as desired. □

Now we are in a position to prove the main theorem of this section.

Theorem 5.2. *There exist subsequences $\{\phi_{n_j}\}_j$ and $\{\psi_{n_j}\}_j$ of $\{\phi_n\}$ and $\{\psi_n\}$ which converge in $L^2(\Omega)$ to some functions $\phi, \psi \in H_0^2(\Omega)$. Furthermore if*

$$\mathscr{L} = \left\{ g \in H_0^2([-1,1]) : \int_{-1}^{1} g(y)dy = 0 \right\} \tag{5.5}$$

and if $g_0 \notin \mathscr{L}$, then

$$\lim_j \|\phi_{n_j}\|_{L^2} > 0, \qquad \lim_j \|\psi_{n_j}\|_{L^2} > 0, \qquad \lim_j \|\phi_{n_j}\|_{L^1} > 0, \qquad \lim_j \|\psi_{n_j}\|_{L^1} > 0. \tag{5.6}$$

Therefore, the above limits are zero if and only if $g_0 \in \mathscr{L}$.

Proof. From Lemma 5.1, $\{\phi_n\}$ and $\{\psi_n\}$ are bounded in $H_0^2(\Omega)$. As a consequence of a weak compactness theorem, there are subsequences $\{\phi_{n_j}\}$ and $\{\psi_{n_j}\}$ and functions ϕ and ψ in $H_0^2(\Omega)$ such that

$$\phi_{n_j} \longrightarrow \phi, \quad \psi_{n_j} \longrightarrow \psi, \quad \text{weakly in } H_0^2(\Omega). \tag{5.7}$$

By the Sobolev embedding theorem on the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we conclude that after passing to another subsequence if necessary,

$$\phi_{n_j} \longrightarrow \phi, \quad \psi_{n_j} \longrightarrow \psi, \quad \text{in } L^2(\Omega). \tag{5.8}$$

From (4.19), we see that $g_0 \in \mathcal{Z}$ if and only if $f_1 \equiv 0$. Hence if $g_0 \in \mathcal{Z}$, the iteration process of the extended Kantorovich method stops and we have $\psi_1(x,y) = f_1(x)g_0(y) \equiv 0$. Now suppose $g_0 \notin \mathcal{Z}$, that is, $f_1 \not\equiv 0$. As in the proof of Lemma 5.1, Corollary 3.2 implies that

$$I[f_1g_0] < I[0] = 0, \tag{5.9}$$

since f_1 is the unique minimizer of $I[fg_0]$ and $f_1 \not\equiv 0$. Applying Corollary 3.2 repeatedly, one has

$$I[f_{m+1}g_m] < \cdots < I[f_2g_1] < I[f_1g_1] < I[f_1g_0] < 0. \tag{5.10}$$

But by Lemma 2.3,

$$I[\psi] \le \liminf_j I[\psi_{n_j}] := \liminf_j I[f_{n_j+1}g_{n_j}]. \tag{5.11}$$

In view of (5.10), we must have $J[\psi] < 0$, which implies $\lim_j \|\psi_{n_j}\|_{L^2} = \|\psi\|_{L^2} > 0$; otherwise, we would have $\|\psi\|_{L^2} = 0$ which implies that $J[\psi] = 0$. Similarly, $\lim_j \|\phi_{n_j}\|_{L^2} = \|\phi\|_{L^2} > 0$. Since $\psi_{n_j} \to \psi$ and $\phi_{n_j} \to \phi$ in L^2, we also have $\psi_{n_j} \to \psi$ and $\phi_{n_j} \to \phi$ in L^1. Thus

$$\lim_j ||\psi_{n_j}||_{L^1} = \|\psi\|_{L^1} > 0, \qquad \lim_j ||\psi_{n_j}||_{L^1} = \|\psi\|_{L^1} > 0. \tag{5.12}$$

This completes the proof of the proposition. □

Corollary 5.3. *Let $g_0 \notin \mathcal{Z}$ and set*

$$r_n = \frac{||g'_{n-1}||_{L^2}}{||g_{n-1}||_{L^2}}, \qquad \tilde{r}_n = \frac{||f'_n||_{L^2}}{||f_n||_{L^2}}, \qquad t_n = \frac{||g''_{n-1}||_{L^2}}{||g_{n-1}||_{L^2}}, \qquad \tilde{t}_n = \frac{||f''_n||_{L^2}}{||f_n||_{L^2}}. \tag{5.13}$$

Then there exist subsequences $\{f_{n_j}\}$ and $\{g_{n_j}\}$ such that the following limits exist and are positive:

$$\lim_j r_{n_j}, \qquad \lim_j \tilde{r}_{n_j}, \qquad \lim_j t_{n_j}, \qquad \lim_j \tilde{t}_{n_j}. \tag{5.14}$$

Proof. In the proof of Theorem 5.2, we showed that for each n,

$$I[\phi_n] = \int_\Omega |\Delta\phi_n|^2 dx\,dy - \gamma\phi_n < 0 \tag{5.15}$$

as long as $g_0 \notin \mathcal{Z}$. Consequently,

$$||f''_n||^2_{L^2}||g_n||^2_{L^2} + ||g''_n||^2_{L^2}||f_n||^2_{L^2} \le \gamma||f_n||^2_{L^2}||g_n||^2_{L^2}. \tag{5.16}$$

This implies that

$$||f''_n||^2_{L^2}||g_n||^2_{L^2} \le \gamma||f_ng_n||_{L^2} \Longrightarrow \frac{||f''_n||^2_{L^2}}{||f_n||^2_{L^2}} \le \frac{\gamma}{||\phi_n||_{L^2}}. \tag{5.17}$$

Combining with the Poincaré inequality, it follows that

$$0 < C' \leq C\frac{||f_n''||_{L^2}^2}{||f_n||_{L^2}^2} \leq \frac{||f_n''||_{L^2}^2}{||f_n||_{L^2}^2} \leq \frac{\gamma}{||\phi_n||_{L^2}} \tag{5.18}$$

for some universal constants C and C'. With Theorem 5.2, the above string of inequalities yields

$$\begin{aligned} \widetilde{C}_1 \leq \limsup_j \widetilde{r}_{n_j} \leq \widetilde{C}_2, \qquad \widetilde{C}_1 \leq \limsup_j \widetilde{t}_{n_j} \leq \widetilde{C}_2, \\ \widetilde{C}_1 \leq \liminf_j \widetilde{r}_{n_j} \leq \widetilde{C}_2, \qquad \widetilde{C}_1 \leq \liminf_j \widetilde{t}_{n_j} \leq \widetilde{C}_2, \end{aligned} \tag{5.19}$$

for some positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$. Similar inequalities hold for r_{n_j} and t_{n_j} with some positive constants C_1 and C_2. Thus after further extracting subsequences of $\{f_{n_j}\}$ and $\{g_{n_j}\}$, we may conclude that the following limits exist and are non-zero:

$$\lim_j \frac{||f_n''||_{L^2}}{||f_n||_{L^2}}, \qquad \lim_j \frac{||f_n'||_{L^2}}{||f_n||_{L^2}}, \qquad \lim_j \frac{||g_n''||_{L^2}}{||g_n||_{L^2}}, \qquad \lim_j \frac{||g_n'||_{L^2}}{||g_n||_{L^2}}. \tag{5.20}$$

This completes the proof of the corollary. □

Corollary 5.4. *If $g_0 \notin \mathcal{L}$, then there exists a subsequence $\{f_{n_j} g_{n_j}\}_j$ that converges pointwisely to a function of the form*

$$\Theta(x,y) = \sum_{k=1}^{N} F_k(x) G_k(y) \in H_0^2(\Omega). \tag{5.21}$$

Furthermore, the derivatives of all orders of $\{f_{n_j} g_{n_j}\}_j$ also converge pointwisely to that of $F(x)G(y)$.

Proof. Let us observe the expression of $\phi_n(x,y) = f_n(x) g_n(y)$ in (4.23). Applying Corollary 5.3 to the constants on the right-hand side of (4.23), we can find convergent subsequences:

$$\{K_{0n_j}\}, \{K_{1n_j}\}, \{K_{2n_j}\}, \{\widetilde{K}_{0n_j}\}, \{\widetilde{K}_{1n_j}\}, \{\widetilde{K}_{2n_j}\}, \tag{5.22}$$

and $\{\rho_{n_j}\}$, $\{\kappa_{n_j}\}$, $\{\widetilde{\rho}_{n_j}\}$, $\{\widetilde{\kappa}_{n_j}\}$. In addition, the constants $c_n \widetilde{c}_n$ can be rewritten as

$$\begin{aligned} c_n \widetilde{c}_n &= \frac{\gamma^2 \int_{-1}^{1} g_{n-1}(y) dx \int_{-1}^{1} f_n(x) dx}{||g_{n-1}''||_{L^2}^2 ||f_n''||_{L^2}^2} \\ &= \frac{\gamma^2 \int_\Omega f_n(x) g_{n-1}(y) dx\, dy}{||f_n g_{n-1}||_{L^2}^2} \cdot \frac{||g_{n-1}||_{L^2}^2}{||g_{n-1}''||_{L^2}^2} \cdot \frac{||f_n||_{L^2}^2}{||f_n''||_{L^2}^2}; \end{aligned} \tag{5.23}$$

hence Theorem 5.2 and Corollary 5.3 guarantee the convergence of the subsequence $\{c_{n-1} \widetilde{c}_{n-j}\}$. Altogether, after replacing all sequences on the right-hand side of (4.23) with either convergent subsequences, we get

$$\begin{aligned}
\Theta(x,y) &= \lim_j f_{n_j} g_{n_j} \\
&= C\Big\{ \frac{K_{1\infty}\widetilde{K}_{1\infty}}{K_{0\infty}\widetilde{K}_{0\infty}} \cosh(\rho_\infty x)\cosh(\widetilde{\rho}_\infty y)\cos(\kappa_\infty x)\cos(\widetilde{\kappa}_\infty y) \\
&\quad + \frac{K_{1\infty}\widetilde{K}_{2\infty}}{K_{0\infty}\widetilde{K}_{0\infty}} \cosh(\rho_\infty x)\sinh(\widetilde{\rho}_\infty y)\cos(\kappa_\infty x)\sin(\widetilde{\kappa}_\infty y) \\
&\quad + \frac{K_{2\infty}\widetilde{K}_{1\infty}}{K_{0\infty}\widetilde{K}_{0\infty}} \sinh(\rho_\infty x)\cosh(\widetilde{\rho}_\infty y)\sin(\kappa_\infty x)\cos(\widetilde{\kappa}_\infty y) \\
&\quad + \frac{K_{2\infty}\widetilde{K}_{2\infty}}{K_{0\infty}\widetilde{K}_{0\infty}} \sinh(\rho_\infty x)\sinh(\widetilde{\rho}_\infty y)\sin(\kappa_\infty x)\cos(\widetilde{\kappa}_\infty y) \\
&\quad + \frac{K_{1\infty}}{K_{0\infty}} \cosh(\rho_\infty x)\cos(\kappa_\infty x) + \frac{K_{2\infty}}{K_{0\infty}} \sinh(\rho_\infty x)\sin(\kappa_\infty x) \\
&\quad + \frac{\widetilde{K}_{1\infty}}{\widetilde{K}_{0\infty}} \cosh(\widetilde{\rho}_\infty y)\sin(\widetilde{\kappa}_\infty y) + \frac{\widetilde{K}_{2\infty}}{\widetilde{K}_{0\infty}} \sinh(\widetilde{\rho}_\infty y)\sin(\widetilde{\kappa}_\infty y) + 1\Big\}.
\end{aligned} \tag{5.24}$$

Now if we differentiate $f_n g_n$ a finite number of times, then from (4.23) we have each summand scaled by integral powers of ρ_n, $\widetilde{\rho}_n$, κ_n and $\widetilde{\kappa}_n$. But we just argued above that these sequences have convergent subsequences. Hence when x, y are fixed, we conclude that all derivatives of $f_{n_j} g_{n_j}$ at (x,y) will converge to that of $\Theta(x,y)$ as $k \to \infty$. The proof of the corollary is therefore complete. □

Remark 5.5. Corollary 5.4 implies that

$$I[f_{n_j} g_{n_j}] \longrightarrow I[\Theta(x,y)], \tag{5.25}$$

by directly using the definition of $I[fg]$. Without Corollary 5.4, we can only assert that

$$I[\Theta(x,y)] \le \liminf_j I[f_{n_j} g_{n_j}]. \tag{5.26}$$

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