THE JAMES CONSTANT OF NORMALIZED NORMS ON $\mathbb{R}^2$

WEERAYUTH NILSRAKOO AND SATIT SAEJUNG

Received 28 June 2005; Accepted 13 September 2005

We introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms. We also give a criterion for deciding whether a given norm in this class is uniformly nonsquare. Moreover, an estimate for the James constant is presented and the exact value of some certain norms is computed. This gives a partial answer to the question raised by Kato et al.

Copyright © 2006 W. Nilsrakoo and S. Saejung. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$) is said to be *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z,w \in \mathbb{C}$ (resp., $\mathbb{R}$), and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(z,w)\|_p = \begin{cases} \left(|z|^p + |w|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|z|,|w|\} & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$), and Ψ_2 the family of all continuous convex functions ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t,\ t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. According to Bonsall and Duncan [1], AN_2 and Ψ_2 are in a one-to-one correspondence under the equation

$$\psi(t) = \|(1-t,t)\| \quad (0 \le t \le 1). \tag{1.2}$$

Indeed, for all $\psi \in \Psi_2$, let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\,\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \ne (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{1.3}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 26265, Pages 1–12
DOI 10.1155/JIA/2006/26265

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1.2). From this result, we can consider many non-ℓ_p-type norms easily. Now let

$$\psi_p(t) = \begin{cases} \left((1-t)^p + t^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t,\ t\} & \text{if } p = \infty. \end{cases} \tag{1.4}$$

Then $\psi_p(t) \in \Psi_2$ and, as is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p.

If X is a Banach space, then X is *uniformly nonsquare* if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$,

$$\text{either } \|x+y\| \le 2(1-\delta) \qquad \text{or} \qquad \|x-y\| \le 2(1-\delta), \tag{1.5}$$

where $S_X = \{x \in X : \|x\| = 1\}$. The *James constant* $J(X)$ is defined by

$$J(X) = \sup\{\min\{\|x+y\|,\ \|x-y\|\} : x,\ y \in S_X\}. \tag{1.6}$$

The *modulus of convexity of* X, $\delta_X : [0,2] \to [0,1]$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x,\ y \in S_X,\ \|x-y\| \ge \varepsilon\right\}. \tag{1.7}$$

The preceding parameters have been recently studied by several authors (cf. [4–6, 8, 9]). We collect together some known results.

Proposition 1.1. *Let X be a nontrivial Banach space, then*

(i) $\sqrt{2} \le J(X) \le 2$ *(Gao and Lau [5]),*

(ii) *if X is a Hilbert space, then $J(X) = \sqrt{2}$; the converse is not true (Gao and Lau [5]),*

(iii) *X is uniformly nonsquare if and only if $J(X) < 2$ (Gao and Lau [5]),*

(iv) $2J(X) - 2 \le J(X^*) \le J(X)/2 + 1$, $J(X^{**}) = J(X)$, *and there exists a Banach space X such that $J(X^*) \ne J(X)$ (Kato et al. [8]),*

(v) *if $2 \le p \le \infty$, then $\delta_{\ell_p}(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$ (Hanner [6]),*

(vi) $J(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) \le 1 - \varepsilon/2\}$ *(Gao and Lau [5]).*

The paper is organized as follows. In Section 2 we introduce a new class of normalized norms on $\mathbb{R}^2$. This class properly contains all absolute normalized norms of Bonsall and Duncan [1]. The so-called generalized Day-James space, ℓ_ψ-ℓ_φ, where $\psi, \varphi \in \Psi_2$, is introduced and studied. More precisely, we prove that $(\ell_\psi\text{-}\ell_\varphi)^* = \ell_{\psi^*}\text{-}\ell_{\varphi^*}$ where ψ^* and φ^* are the dual functions of ψ and φ, respectively. In Section 3, the upper bound of the James constant of the generalized Day-James space is given. Furthermore, we compute $J(\ell_\psi\text{-}\ell_\infty)$ and deduce that every generalized Day-James space except ℓ_1-ℓ_1 and ℓ_∞-ℓ_∞ is uniformly nonsquare. This result strengthens Corollary 3 of Saito et al. [10].

2. Generalized Day-James spaces

In this section, we introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms of Bonsall and Duncan [1]. Moreover, we introduce a two-dimensional normed space which is a generalization of Day-James ℓ_p-ℓ_q spaces.

LEMMA 2.1. *Let $\psi \in \Psi_2$ and let $\|\cdot\|_{\psi,\psi_\infty}$ be a function on $\mathbb{R}^2$ defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$\begin{aligned} \|(z,w)\|_{\psi,\psi_\infty} &:= \max\{\|(z^+,w^+)\|_\psi, \|(z^-,w^-)\|_\psi\}, \\ &= \begin{cases} \|(z,w)\|_\psi & \text{if } zw \geq 0, \\ \|(z,w)\|_\infty & \text{if } zw \leq 0, \end{cases} \end{aligned} \tag{2.1}$$

where x^+ and x^- are positive and negative parts of $x \in \mathbb{R}$, that is, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$. Then $\|\cdot\|_{\psi,\psi_\infty}$ is a norm on $\mathbb{R}^2$.

For convenience, we put $\mathcal{B}_{\psi_1,\psi_2} := \{(z,w) \in \mathbb{R}^2 : \|(z,w)\|_{\psi_1,\psi_2} \leq 1\}$.

THEOREM 2.2. *Let $\psi, \varphi \in \Psi_2$ and*

$$\|(z,w)\|_{\psi,\varphi} := \begin{cases} \|(z,w)\|_\psi & \text{if } zw \geq 0, \\ \|(z,w)\|_\varphi & \text{if } zw \leq 0 \end{cases} \tag{2.2}$$

for all $(z,w) \in \mathbb{R}^2$. Then $\|\cdot\|_{\psi,\varphi}$ is a norm on $\mathbb{R}^2$. Denote by N_2 the family of all such preceding norms.

Proof. Let $\psi, \varphi \in \Psi_2$, we only show $\|\cdot\|_{\psi,\varphi}$ satisfies the triangle inequality. To this end, it suffices to prove that $\mathcal{B}_{\psi,\varphi}$ is convex. By Lemma 2.1, we have that $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are closed unit balls of $\|\cdot\|_{\psi,\psi_\infty}$ and $\|\cdot\|_{\varphi,\psi_\infty}$, respectively, and so $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are convex sets. We define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T((z,w)) = (-z,w) \quad \forall (z,w) \in \mathbb{R}^2. \tag{2.3}$$

Then T is a linear operator and $T(\mathcal{B}_{\varphi,\psi_\infty}) = \mathcal{B}_{\psi_\infty,\varphi}$, which implies that $\mathcal{B}_{\psi_\infty,\varphi}$ is convex and so $\mathcal{B}_{\psi,\varphi} = \mathcal{B}_{\psi_\infty,\varphi} \cap \mathcal{B}_{\psi,\psi_\infty}$ is convex. □

Taking $\psi = \psi_p$ and $\varphi = \psi_q$ $(1 \leq p,q \leq \infty)$ in Theorem 2.2, we obtain the following.

COROLLARY 2.3 (Day-James ℓ_p-ℓ_q spaces). *For $1 \leq p$, $q \leq \infty$, denote by ℓ_p-ℓ_q the Day-James space, that is, $\mathbb{R}^2$ with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$\|(z,w)\|_{p,q} = \begin{cases} \|(z,w)\|_p & \text{if } zw \geq 0, \\ \|(z,w)\|_q & \text{if } zw \leq 0. \end{cases} \tag{2.4}$$

James [7] considered the ℓ_p-$\ell_{p'}$ space as an example of a Banach space which is isometric to its dual but which is not given by a Hilbert norm when $p \neq 2$. Day [2] considered even more general spaces, namely, if $(X, \|\cdot\|)$ is a two-dimensional Banach space and $(X^*, \|\cdot\|^*)$ its dual, then the X-X^* space is the space X with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,

$$\|(z,w)\|_{X,X^*} = \begin{cases} \|(z,w)\| & \text{if } zw \geq 0, \\ \|(z,w)\|^* & \text{if } zw \leq 0. \end{cases} \tag{2.5}$$

For $\psi,\varphi \in \Psi_2$, denote by ℓ_ψ-ℓ_φ the *generalized Day-James space*, that is, $\mathbb{R}^2$ with the norm $\|\cdot\|_{\psi,\varphi}$ defined by (2.2). For ψ_p defined by (1.4), we write ℓ_ψ-ℓ_p for ℓ_ψ-ℓ_{ψ_p}. For example, if $1 \le p,q \le \infty$, ℓ_p-ℓ_q means ℓ_{ψ_p}-ℓ_{ψ_q}.

It is worthwhile to mention that there is a normalized norm which is not absolute.

Proposition 2.4. *There is $\psi \in \Psi_2$ such that ℓ_ψ-ℓ_∞ is not isometrically isomorphic to ℓ_φ-ℓ_φ for all $\varphi \in \Psi_2$.*

Proof. Let

$$\psi(t) := \begin{cases} 1-t & \text{if } 0 \le t \le \dfrac{1}{8}, \\ \dfrac{11-4t}{12} & \text{if } \dfrac{1}{8} \le t \le \dfrac{1}{2}, \\ \dfrac{1+t}{2} & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases} \tag{2.6}$$

We observe that the sphere of ℓ_ψ-ℓ_∞ is the octagon whose right half consists of 4 segments of different lengths. Suppose that there are $\varphi \in \Psi_2$ and an isometric isomorphism from ℓ_ψ-ℓ_∞ onto ℓ_φ-ℓ_φ. Since the image of each segment in ℓ_ψ-ℓ_∞ is again a segment of the same length in ℓ_φ-ℓ_φ, the sphere of ℓ_φ-ℓ_φ must be the octagon whose each corresponding side has the same length (measured by $\|\cdot\|_\varphi$). We show that this cannot happen. Consider $(1,0) \in S_{\ell_\varphi\text{-}\ell_\varphi}$. If $(1,0)$ is an extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, then $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths since $\|\cdot\|_\varphi$ is absolute. On the other hand, if $(1,0)$ is an not extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, again $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths. □

Next, we prove that the dual of a generalized Day-James space is again a generalized Day-James space. Recall that, for $\psi \in \Psi_2$, the *dual function* ψ^* of ψ is defined by

$$\psi^*(s) = \max_{0\le t\le 1} \frac{(1-s)(1-t)+st}{\psi(t)} \tag{2.7}$$

for all $s \in [0,1]$. It was proved that $\psi^* \in \Psi_2$ and $(\ell_\psi$-$\ell_\psi)^* = \ell_{\psi^*}$-$\ell_{\psi^*}$ (see [3, Proposition 1 and Theorem 2]). We generalize this result to our spaces as follows.

Theorem 2.5. *For $\psi,\varphi \in \Psi_2$, there is an isometric isomorphism that identifies $(\ell_\psi$-$\ell_\varphi)^*$ with ℓ_{ψ^*}-ℓ_{φ^*} such that if $f \in (\ell_\psi$-$\ell_\varphi)^*$ is identified with the element $(z,w) \in \ell_{\psi^*}$-ℓ_{φ^*}, then*

$$f(u,v) = zu + wv \tag{2.8}$$

for all $(u,v) \in \mathbb{R}^2$.

Proof. We can prove analogous to [3, Theorem 2]. □

3. The James constant and uniform nonsquareness

The next lemmas are crucial for proving the main theorems.

Lemma 3.1. *Let $\psi,\varphi \in \Psi_2$. Then*

(i) $\|\cdot\|_\infty \le \|\cdot\|_{\psi,\varphi} \le \|\cdot\|_1$,

(ii) $(1/M_{\psi,\varphi})\|\cdot\|_\psi \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_\psi$,

(iii) $(1/M_{\varphi,\psi})\|\cdot\|_\varphi \le \|\cdot\|_{\psi,\varphi} \le M_{\psi,\varphi}\|\cdot\|_\varphi$,

where $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$ *and* $M_{\psi,\varphi} = \max_{0\le t\le 1}\psi(t)/\varphi(t)$.

LEMMA 3.2. *Let* $\psi,\varphi \in \Psi_2$ *and let* Q_i $(i = 1,\dots,4)$ *denote the ith quadrant in* $\mathbb{R}^2$. *Suppose that* $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$, *then the following statements are true.*

(i) *If* $x, y \in Q_1$, *then* $x + y \in Q_1$ *and* $x - y \in Q_2 \cup Q_4$.

(ii) *If* $x, y \in Q_2$, *then* $x + y \in Q_2$ *and* $x - y \in Q_1 \cup Q_3$.

(iii) *If* $\psi(t) \le \varphi(t)$ *for all* $t \in [0,1]$ *and* $x - y \in Q_2^\circ \cup Q_4^\circ$, *where* Q_2° *and* Q_4° *are the interiors of* Q_2 *and* Q_4, *respectively, then* $x + y \in Q_1 \cup Q_3$.

We will estimate the James constant of ℓ_ψ-ℓ_φ.

THEOREM 3.3. *Let* $\psi,\varphi\in\Psi_2$ *with* $\psi(t)\le\varphi(t)$ *for all* $t\in[0,1]$, *let* $M_{\varphi,\psi}=\max_{0\le t\le 1}\varphi(t)/\psi(t)$, *and let* $\delta_\psi(\cdot)$ *be the modulus of convexity of* ℓ_ψ-ℓ_ψ. *Then for* $\varepsilon \in [0,2]$,

$$\delta_{\psi,\varphi}(\varepsilon) \ge \min\left\{1 - M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)),\ \delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right\}, \tag{3.1}$$

where $\delta_{\psi,\varphi}(\cdot)$ *is the modulus of convexity of* ℓ_ψ-ℓ_φ. *Consequently,*

$$J(\ell_\psi\text{-}\ell_\varphi) \le \sup\left\{\varepsilon \in (0,2) : \varepsilon \le 2M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)) \text{ or } \varepsilon \le 2\left(1-\delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right)\right\}. \tag{3.2}$$

Proof. By Lemma 3.1(ii), we have

$$\|\cdot\|_\psi \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_\psi. \tag{3.3}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. We consider two cases.

Case 1. Take $\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1$ with $\|x-y\|_{\psi,\varphi} \ge \varepsilon$, where $x - y \in Q_1 \cup Q_3$. Thus $\|x\|_\psi \le 1$, $\|y\|_\psi \le 1$, and $\|x-y\|_\psi \ge \varepsilon$, which implies that

$$\frac{1}{2}\|x+y\|_\psi \le 1 - \delta_\psi(\varepsilon). \tag{3.4}$$

This in turn implies

$$\frac{1}{2}\|x+y\|_{\psi,\varphi} \le \frac{1}{2}M_{\varphi,\psi}\|x+y\|_\psi \le M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)), \tag{3.5}$$

thus

$$1 - \frac{1}{2}\|x+y\|_{\psi,\varphi} \ge 1 - M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)). \tag{3.6}$$

Case 2. Now take x, y as above, but with $x - y \in Q_2^\circ \cup Q_4^\circ$. By Lemma 3.2(iii), $x + y \in Q_1 \cup Q_3$. Since $\|x-y\|_{\psi,\varphi} \ge \varepsilon$,

$$\|x-y\|_\psi \ge \frac{\|x-y\|_{\psi,\varphi}}{M_{\varphi,\psi}} \ge \frac{\varepsilon}{M_{\varphi,\psi}}. \tag{3.7}$$

Then

$$\frac{1}{2}\|x+y\|_{\psi,\varphi} = \frac{1}{2}\|x+y\|_{\psi} \le 1-\delta_{\psi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right), \tag{3.8}$$

and so

$$1-\frac{1}{2}\|x+y\|_{\psi,\varphi} \ge \delta_{\psi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right). \tag{3.9}$$

Hence we obtain (3.1). By Proposition 1.1(vi), (3.2) follows. □

The following corollary shows that we can have equality in (3.2).

Corollary 3.4 [4, 8]. *If $1 \le q \le p < \infty$ and $p \ge 2$, then*

$$J(\ell_p\text{-}\ell_q) \le 2\left(\frac{2^{p/q}}{2^{p/q}+2}\right)^{1/p}. \tag{3.10}$$

In particular, if $p=2$ and $q=1$, then $J(\ell_2\text{-}\ell_1)=\sqrt{8/3}$.

Proof. It follows that since

$$M_{\psi_q,\psi_p} = 2^{1/q-1/p}, \qquad \delta_{\ell_p\text{-}\ell_p}(\varepsilon) = 1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}. \tag{3.11}$$

Moreover, if $p=2$ and $q=1$, then $J(\ell_2\text{-}\ell_1) \le \sqrt{8/3}$. Now we put

$$x_0 = \left(\frac{2+\sqrt{2}}{2\sqrt{3}}, \frac{2-\sqrt{2}}{2\sqrt{3}}\right), \qquad y_0 = \left(\frac{2-\sqrt{2}}{2\sqrt{3}}, \frac{2+\sqrt{2}}{2\sqrt{3}}\right). \tag{3.12}$$

Then

$$||x_0||_{2,1} = ||y_0||_{2,1} = 1, \qquad ||x_0 \pm y_0||_{2,1} = \sqrt{\frac{8}{3}}. \tag{3.13}$$

□

Theorem 3.5. *Let $\psi,\varphi \in \Psi_2$ with $\psi(t) \le \varphi(t)$ for all $t \in [0,1]$, let $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$, and let $\delta_\varphi(\cdot)$ be the modulus of convexity of $\ell_\varphi\text{-}\ell_\varphi$. Then for $\varepsilon \in [0,2]$,*

$$\delta_{\psi,\varphi}(\varepsilon) \ge 1-M_{\varphi,\psi}\left(1-\delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right), \tag{3.14}$$

where $\delta_{\psi,\varphi}(\cdot)$ is the modulus of convexity of $\ell_\psi\text{-}\ell_\varphi$. Consequently,

$$J(\ell_\psi\text{-}\ell_\varphi) \le \sup\left\{\varepsilon \in (0,2) : \varepsilon \le 2M_{\varphi,\psi}\left(1-\delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right)\right\}. \tag{3.15}$$

Proof. By Lemma 3.1(iii), we have

$$\frac{1}{M_{\varphi,\psi}}\|\cdot\|_\varphi \le \|\cdot\|_{\psi,\varphi} \le \|\cdot\|_\varphi. \tag{3.16}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. Let

$$\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1 \quad \text{with } \|x-y\|_{\psi,\varphi} \geq \varepsilon. \tag{3.17}$$

Then

$$\begin{gathered} \frac{1}{M_{\varphi,\psi}}\|x\|_\varphi \leq 1, \qquad \frac{1}{M_{\varphi,\psi}}\|y\|_\varphi \leq 1, \\ \frac{1}{M_{\varphi,\psi}}\|x-y\|_\varphi \geq \frac{1}{M_{\varphi,\psi}}\|x-y\|_{\psi,\varphi} \geq \frac{\varepsilon}{M_{\varphi,\psi}}, \end{gathered} \tag{3.18}$$

which implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1-\delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right). \tag{3.19}$$

This in turn implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_{\psi,\varphi} \leq \frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1-\delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right), \tag{3.20}$$

thus

$$1-\frac{1}{2}\|x+y\|_{\psi,\varphi} \geq 1-M_{\varphi,\psi}\left(1-\delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right). \tag{3.21}$$

Hence we obtain (3.14). By Proposition 1.1(vi), (3.15) follows. □

COROLLARY 3.6. *If* $2 \leq q \leq p < \infty$, *then*

$$J(\ell_p\text{-}\ell_q) \leq 2^{1-1/p}. \tag{3.22}$$

It is easy to see that the estimate (3.22) is better than one obtained in [4, Example 2.4(3)].

For some generalized Day-James spaces, [8, Corollary 4] of Kato et al. gives only rough result for the estimate of the James constant, that is, for $\psi \in \Psi_2$,

$$\frac{2}{M} \leq J(\ell_\psi\text{-}\ell_\infty) \leq 2M, \tag{3.23}$$

where $M = \max_{0\leq t\leq 1} \psi_\infty(t)/\psi(t)$.

However, the following theorem gives the exact value of the James constant of these spaces.

THEOREM 3.7. *Let* $\psi \in \Psi_2$. *Then*

$$J(\ell_\psi\text{-}\ell_\infty) = 1+\frac{1/2}{\psi(1/2)}. \tag{3.24}$$

Proof. For our convenience, we write $\|\cdot\|$ instead of $\|\cdot\|_{\psi,\psi_\infty}$. Let $x,y \in S_{\ell_\psi\text{-}\ell_\infty}$. We prove that

$$\text{either } \|x+y\| \le 1+\frac{1/2}{\psi(1/2)} \quad \text{or} \quad \|x-y\| \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.25}$$

Let us consider the following cases.

Case 1. $x,y \in Q_1$. Let $x=(a,b)$ and $y=(c,d)$ where $a,b,c,d \in [0,1]$. By Lemma 3.2(i), we have $x-y \in Q_2 \cup Q_4$. Then

$$\|x-y\| = \max\{|a-c|,\ |b-d|\} \le 1 \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.26}$$

Case 2. $x,y \in Q_2$. If x,y lies in the same segment, then $\|x-y\| \le 1$. We now suppose that $x=(-1,a)$ and $y=(-c,1)$ where $a,c \in [0,1]$.

Subcase 2.1. $a \le (1/2)/\psi(1/2)$ and $c \le (1/2)/\psi(1/2)$. Then

$$\|x+y\| = \big\|(-1-c,1+a)\big\|_\infty = \max\{1+c,\ 1+a\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.27}$$

Subcase 2.2. $a \ge (1/2)/\psi(1/2)$ or $c \ge (1/2)/\psi(1/2)$. Put $z=(-1,1)$, then

$$\|x-y\| \le \|x-z\|+\|z-y\| = 1-a+1-c \le 1+1-\frac{1/2}{\psi(1/2)} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.28}$$

From now on, we may assume without loss of generality that there is $\beta \in [1/2,1]$ such that $\psi(\beta) \le \psi(t)$ for all $t \in [0,1]$. Indeed, $J(\ell_\psi\text{-}\ell_\infty) = J(\ell_{\tilde\psi}\text{-}\ell_\infty)$ where $\tilde\psi(t) = \psi(1-t)$ for all $t \in [0,1]$.

Case 3. $x \in Q_1$ and $y \in Q_2$. Let $x=(a,b)$, $y=(-c,1)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 3.1. $a \le (1/2)/\psi(1/2)$ or $c \le (1/2)/\psi(1/2)$. Then

$$\|x-y\| = \big\|(a+c,b-1)\big\|_\infty = \max\{a+c,\ 1-b\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.29}$$

Subcase 3.2. $(1/2)/\psi(1/2) \le a \le c$. Then $b \le (1/2)/\psi(1/2)$ and

$$\|x+y\| = \big\|(a-c,b+1)\big\|_\infty = \max\{c-a,\ 1+b\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.30}$$

Subcase 3.3. $(1/2)/\psi(1/2) < c \le a$. We write $a=(1-t_0)/\psi(t_0)$, $b=t_0/\psi(t_0)$ where $t_0 = b/(a+b)$ and $0 \le t_0 \le 1/2$. By the convexity of ψ and $\psi(t) \ge \psi(\beta)$ for all $0 \le t \le 1$, we have $\psi(t_0) \geq \psi(1/2)$ and so $1/\psi(t_0) \leq 1/\psi(1/2)$. By Lemma 3.1(i),

$$\begin{aligned} \|x+y\| &= \|(a,b)+(-c,1)\| \leq \|(a-c,b+1)\|_1 \\ &= a-c+b+1 = \frac{1}{\psi(t_0)}+1-c \\ &\leq \frac{1}{\psi(1/2)}+1-\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}. \end{aligned} \tag{3.31}$$

Case 4. $x \in Q_1$ and $y \in Q_2$. Let $x=(a,b)$, $y=(-1,c)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 4.1. $b \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$\|x+y\| = \|(a-1,b+c)\|_\infty = \max\{1-a,\ b+c\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.32}$$

Subcase 4.2. $(1/2)/\psi(1/2) < b \leq c$. Then $a \leq (1/2)/\psi(1/2)$ and

$$\|x-y\| = \|(1+a,b-c)\|_\infty = \max\{1+a,c-b\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.33}$$

Subcase 4.3. $(1/2)/\psi(1/2) < c \leq b$. We write $a=(1-t_0)/\psi(t_0)$, $b=t_0/\psi(t_0)$, where $t_0 = b/(a+b)$ and $1/2 \leq t_0 \leq 1$. We choose $\alpha = b/(a+2b-1)$, then

$$\frac{1}{2} \leq \alpha \leq 1, \qquad a = \frac{1-2\alpha}{\alpha}b+1. \tag{3.34}$$

Since $b-c \leq 1+a$ and $b \leq 1$,

$$\frac{b-c}{1+a+b-c} \leq \frac{1}{2} \leq t_0 \leq \alpha. \tag{3.35}$$

Let

$$\psi_\alpha(t) = \begin{cases} \dfrac{\alpha-1}{\alpha}t+1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases} \tag{3.36}$$

We see that $\psi_\alpha(t_0) = \psi(t_0)$. By the convexity of ψ, we have

$$\psi(t) \leq \psi_\alpha(t) \quad \forall t \leq t_0. \tag{3.37}$$

Therefore,

$$\begin{aligned}
\|x-y\| &= \|(a+1,b-c)\|_\psi = (1+a+b-c)\psi\left(\frac{b-c}{1+a+b-c}\right)\\
&\le (1+a+b-c)\psi_\alpha\left(\frac{b-c}{1+a+b-c}\right) = \frac{\alpha-1}{\alpha}(b-c)+1+a+b-c\\
&= 1+a+\frac{2\alpha-1}{\alpha}b-\frac{2\alpha-1}{\alpha}c = 1+1-\frac{2\alpha-1}{\alpha}c\\
&< 1+1-\frac{2\alpha-1}{\alpha}\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}+1-\frac{3\alpha-1}{2\alpha}\frac{1}{\psi(1/2)}\\
&= 1+\frac{1/2}{\psi(1/2)}+1-\frac{\psi_\alpha(1/2)}{\psi(1/2)} \le 1+\frac{1/2}{\psi(1/2)}.
\end{aligned} \tag{3.38}$$

Finally, we conclude that

$$J(\ell_\psi\text{-}\ell_\infty) \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.39}$$

Now, we put $x_0 = ((1/2)/\psi(1/2),(1/2)/\psi(1/2))$ and $y_0 = (-1,1)$, then

$$\|x_0\| = \|y_0\| = 1, \qquad \|x_0 \pm y_0\| = 1+\frac{1/2}{\psi(1/2)}. \tag{3.40}$$

Thus,

$$J(\ell_\psi\text{-}\ell_\infty) \ge \min\{\|x_0-y_0\|,\ \|x_0+y_0\|\} = 1+\frac{1/2}{\psi(1/2)}. \tag{3.41}$$

This together with (3.39) completes the proof. □

Corollary 3.8 [4, Example 2.4(2)]. *Let $1 \le p \le \infty$, then*

$$J(\ell_p\text{-}\ell_\infty) = 1+\left(\frac{1}{2}\right)^{1/p}. \tag{3.42}$$

Indeed, $\psi_p(1/2) = 2^{1/p-1}$.

We now obtain the bounds for $J(\ell_\psi\text{-}\ell_1)$.

Corollary 3.9. *Let $\psi \in \Psi_2$. Then*

$$2\min_{0\le t\le 1}\psi(t) \le J(\ell_\psi\text{-}\ell_1) \le \frac{3}{2}+\frac{1}{2}\min_{0\le t\le 1}\psi(t). \tag{3.43}$$

Proof. Note that $\psi^*(1/2) = \max_{0\le t\le 1}(1/2)/\psi(t) = 1/2\min_{0\le t\le 1}\psi(t)$. By Theorem 3.7, we have $J(\ell_{\psi^*}\text{-}\ell_\infty) = 1+\min_{0\le t\le 1}\psi(t)$. Applying Proposition 1.1(iv), the assertion is obtained. □

We now improve the upper bound for $J(\ell_p\text{-}\ell_1)$ (see also Corollary 3.4).

Corollary 3.10. *Let* $1 \le p < \infty$. *Then*

$$J(\ell_p\text{-}\ell_1) \le \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}. \tag{3.44}$$

In particular, if $p \ge 2$, *then*

$$J(\ell_p\text{-}\ell_1) \le \min\left\{\frac{4}{(2^p+2)^{1/p}}, \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}\right\}. \tag{3.45}$$

The following corollary follows by Theorem 3.7 and Corollary 3.9.

Corollary 3.11. *Let* $\psi \in \Psi_2$. *Then*
(i) ℓ_ψ-ℓ_∞ *is uniformly nonsquare if and only if* $\psi \ne \psi_\infty$,
(ii) ℓ_ψ-ℓ_1 *is uniformly nonsquare if and only if* $\psi \ne \psi_1$.

We can say more about the uniform nonsquareness of ℓ_ψ-ℓ_φ.

Theorem 3.12. *Let* $\psi, \varphi \in \Psi_2$. *Then all* ℓ_ψ-ℓ_φ *except* ℓ_1-ℓ_1 *and* ℓ_∞-ℓ_∞ *are uniformly nonsquare.*

Proof. If $\psi = \varphi$, we are done by [10, Corollary 3]. Assume that $\psi \ne \varphi$. We prove that ℓ_ψ-ℓ_φ is uniformly nonsquare. Suppose not, that is, there are $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$ such that $\|x \pm y\|_{\psi,\varphi} = 2$. We consider three cases.

Case 1. $x, y \in Q_1$. Then

$$\begin{aligned} \|x\|_{\psi,1} &= \|x\|_\psi = \|x\|_{\psi,\varphi} = 1, \\ \|y\|_{\psi,1} &= \|y\|_\psi = \|y\|_{\psi,\varphi} = 1. \end{aligned} \tag{3.46}$$

It follows by Lemma 3.2(i) that $x + y \in Q_1$ and $x - y \in Q_2 \cup Q_4$. Therefore

$$\begin{aligned} \|x+y\|_{\psi,1} &= \|x+y\|_{\psi,\varphi} = 2, \\ 2 = \|x-y\|_{\psi,\varphi} &\le \|x-y\|_1 = \|x-y\|_{\psi,1} \le 2. \end{aligned} \tag{3.47}$$

Hence $\|x \pm y\|_{\psi,1} = 2$ and this implies that ℓ_ψ-ℓ_1 is not uniformly nonsquare. By Corollary 3.11(ii), we have $\psi = \psi_1$. Again, since ℓ_ψ-$\ell_\varphi = \ell_1$-ℓ_φ is not uniformly nonsquare, $\varphi = \psi_1 = \psi$; a contradiction.

Case 2. $x, y \in Q_2$. It is similar to Case 1, so we omit the proof.

Case 3. $x := (a,b) \in Q_1$ and $y := (-c,d) \in Q_2$ where $a,b,c,d \in [0,1]$. Since $\|x+y\|_{\psi,\varphi} = 2$, the line segment joining x and y must lie in the sphere. In particular, there is $\alpha \in [0,1]$ such that

$$(0,1) = \alpha x + (1-\alpha)y. \tag{3.48}$$

It follows that $b = 1$ since $b, d \le 1$. Similarly consider x and $-y$ instead of x and y, we can also conclude that $a = 1$. Hence $\|(1,1)\|_\psi = \|(1,1)\|_{\psi,\varphi} = 1$, that is, $\psi(1/2) = 1/2$. Then $\psi = \psi_\infty$ and so ℓ_ψ-$\ell_\varphi = \ell_\infty$-$\ell_\varphi$ is not uniformly nonsquare. By Corollary 3.11(i), we have $\varphi = \psi_\infty = \psi$; a contradiction. □

Acknowledgments

The authors would like to thank the referee for suggestions which led to a presentation of the paper. The second author was supported by the Thailand Research Fund under Grant BRG 4780013.

References

[1] F. F. Bonsall and J. Duncan, *Numerical Ranges, Vol. II*, Cambridge University Press, New York, 1973.

[2] M. M. Day, *Uniform convexity in factor and conjugate spaces*, Annals of Mathematics. Second Series **45** (1944), no. 2, 375–385.

[3] S. Dhompongsa, A. Kaewkhao, and S. Saejung, *Uniform smoothness and U-convexity of ψ-direct sums*, Journal of Nonlinear and Convex Analysis **6** (2005), no. 2, 327–338.

[4] S. Dhompongsa, A. Kaewkhao, and S. Tasena, *On a generalized James constant*, Journal of Mathematical Analysis and Applications **285** (2003), no. 2, 419–435.

[5] J. Gao and K.-S. Lau, *On the geometry of spheres in normed linear spaces*, Journal of Australian Mathematical Society. Series A **48** (1990), no. 1, 101–112.

[6] O. Hanner, *On the uniform convexity of L^p and l^p*, Arkiv för Matematik **3** (1956), 239–244.

[7] R. C. James, *Inner product in normed linear spaces*, Bulletin of the American Mathematical Society **53** (1947), 559–566.

[8] M. Kato, L. Maligranda, and Y. Takahashi, *On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces*, Studia Mathematica **144** (2001), no. 3, 275–295.

[9] K.-I. Mitani and K.-S. Saito, *The James constant of absolute norms on* $\mathbb{R}^2$, Journal of Nonlinear and Convex Analysis **4** (2003), no. 3, 399–410.

[10] K.-S. Saito, M. Kato, and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, Journal of Mathematical Analysis and Applications **244** (2000), no. 2, 515–532.

Weerayuth Nilsrakoo: Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand
Current address: Department of Mathematics, Statistics and Computer, Ubon Rajathanee University, Ubon Ratchathani 34190, Thailand
E-mail address: nweerayuth@sci.ubu.ac.th

Satit Saejung: Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand
E-mail address: saejung@kku.ac.th