SELF-SIMILAR SINGULAR SOLUTION OF DOUBLY SINGULAR PARABOLIC EQUATION WITH GRADIENT ABSORPTION TERM

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Received 6 February 2004; Revised 8 January 2005; Accepted 8 January 2005

We deal with the self-similar singular solution of doubly singular parabolic equation with a gradient absorption term $u_t = \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) - |\nabla u|^q$ for $p > 1$, $m(p-1) > 1$ and $q > 1$ in $\mathbb{R}^n \times (0,\infty)$. By shooting and phase plane methods, we prove that when $p > 1 + n/(1+mn)q + mn/(mn+1)$ there exists self-similar singular solution, while $p \leq n + 1/(1+mn)q + mn/(mn+1)$ there is no any self-similar singular solution. In case of existence, the self-similar singular solution is the self-similar very singular solutions which have compact support. Moreover, the interface relation is obtained.

1. Introduction and main results

In this paper we consider the self-similar singular solution of the doubly singular parabolic equation with nonlinear gradient absorption terms

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - |\nabla u|^q \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

where $p > 1$, $m > 0$, $m(p-1) > 1$ and $q > 1$. When $m = 1$ and $p = 2$ the corresponding conclusions have given in [14, 15], respectively. Here by singular solution we mean a nonnegative and nontrivial solution $u(x,t)$ which is continuous in $\mathbb{R}^n \times [0,\infty)\setminus\{(0,0)\}$ and satisfies

$$\lim_{t\to 0} \sup_{|x|>\varepsilon} u(x,t) = 0, \quad \forall \varepsilon > 0. \tag{1.2}$$

A singular solution $u(x,t)$ is called a very singular solution provided that it satisfies

$$\lim_{t\to 0} \int_{|x|<\varepsilon} u(x,t)dx = \infty, \quad \forall \varepsilon > 0. \tag{1.3}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 20438, Pages 1–21
DOI 10.1155/JIA/2006/20438

By self-similar solution we mean the function $u(x,t)$ which has the following form

$$u(x,t) = \left(\frac{\alpha}{t}\right)^{\alpha} f\left(|x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}\right), \tag{1.4}$$

where

$$\alpha = \frac{p-q}{q(1-m(p-1))+p(q-1)}, \qquad \beta = \frac{q-m(p-1)}{p-q}. \tag{1.5}$$

To guarantee the constants α and β are positive, here we consider the case

$$p > q, \quad q > m(p-1) > 1. \tag{1.6}$$

Since $q(1-m(p-1))+p(q-1) > (p-q)(q-1) > 0$, the self-similar singular solution to (1.1), if it exists, satisfies the following ODE boundary problem

$$\begin{gathered}(|(f^m)'|^{p-2}(f^m)')' + \frac{n-1}{r}|(f^m)'|^{p-2}(f^m)' + \beta r f' + f - |f'|^q = 0, \quad \forall r > 0, \\ f(0) = a > 0, \qquad \lim_{r\to\infty} r^{1/\beta} f(r) = 0,\end{gathered} \tag{1.7}$$

where $f = f(r)$ with the self-similar variable $r = |x|(\alpha/t)^{\alpha\beta}$, the prime denotes the differentiation with respect to r.

Throughout this paper we set

$$\begin{gathered}\nu = p + (m(p-1)-1)/\beta = q + (q-1)/\beta > 1, \qquad \sigma = m(p-1)-1, \\ \gamma = q - m(p-1).\end{gathered} \tag{1.8}$$

Singular solutions were first discovered for the semilinear heat equation

$$u_t = \Delta u - u^p. \tag{1.9}$$

Brézis and Friedman [1] proved that (1.9) admits a unique singular solution for every $c \in (0,\infty)$ when $1 < p < 1+2/n$ such that

$$\lim_{t\to 0}\int_{|x|<\varepsilon} u(x,t)dx = c, \quad \forall \varepsilon > 0, \tag{1.10}$$

which is called a fundamental solution with initial mass c, while it has no for $p \ge 1+2/n$. Shortly, Brézis et al. [2] had proved that (1.9) posses a unique very singular solution when $1 < p < 1+2/n$. In recent years, many authors studied the self-similar singular solutions (see [4, 7, 9–11, 13] and the references therein) of the following equations:

$$\begin{gathered}u_t = \Delta(u^m) - u^p, \quad 0 < m < \infty,\ p > 1, \\ u_t = \Delta(u^m) - |\nabla u|^p, \quad 1 < m < \infty,\ p > 1, \\ u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - u^q, \quad 0 < m < \infty,\ p > 1,\ q > 1.\end{gathered} \tag{1.11}$$

The large time behavior of solutions to the Cauchy problems corresponding to the above equations with absorption u^p or u^q (with $m = 1$) can also be characterized by their corresponding self-similar solutions, singular solutions, fundamental solutions and very singular solution, see [3, 5, 6, 8, 12, 16] and the references therein.

To study the boundary value problem (1.7), we consider the initial value problem

$$\begin{gathered}(|(f^m)'|^{p-2}(f^m)')' + \frac{n-1}{r}|(f^m)'|^{p-2}(f^m)' + \beta r f' + f - |f'|^q = 0, \quad r > 0, \\ f(0) = a > 0, \qquad f'(0) = 0.\end{gathered} \tag{1.12}$$

Let $f(r;a)$ be the solution of (1.12) and $(0, R(a))$ be the maximal existence interval where $f(r;a) > 0$. Our main results read as follows.

THEOREM 1.1. *Assume that $p > q > m(p-1) > 1$, α and β satisfy (1.5). For each $a > 0$, let $f(r;a)$ be the solution of (1.12). Then the statements hold:*

(I) *If $n\beta \geq 1$, namely, $p \leq ((n+1)/(mn+1))q + mn/(mn+1)$, then $f(r;a) > 0$, $f'(r;a) < 0$ for $r \in (0,\infty)$ and $\lim_{r\to\infty} r^{1/\beta} f(r;a) = k(a) > 0$ for some constant $k(a)$. Moreover, for $r \gg 1$,*

$$f(r;a) = k(a)r^{-1/\beta}\left\{1 + \frac{1}{\beta^2}\left(\frac{k(a)}{\beta}\right)^{\sigma}\left[m^{p-1}\left(1 - \frac{n\beta - 1}{\nu\beta}\right) - \frac{1}{\nu}\left(\frac{k(a)}{\beta}\right)^{\gamma}\right]r^{-\nu} + o(r^{-\nu})\right\}. \tag{1.13}$$

(II) *If $n\beta < 1$. Then there exist one closed set $\mathcal{B}$ and two open sets $\mathcal{A}$ and $\mathcal{C}$ which are nonempty and disjoint and satisfy $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0,\infty)$ such that the followings hold.*

(i) *There is $a_1 > 0$ such that $(0, a_1) \subset \mathcal{A}$. Moreover, when $a \in \mathcal{A}$, then $R(a) < \infty$ and $f(r;a) > 0$, $f'(r;a) < 0$ $\forall r \in (0, R(a))$, $f(R(a);a) = 0$ and $f'(R(a);a) < 0$.*

(ii) *There exists $a_2 \geq a_1$ such that $(a_2, \infty) \subset \mathcal{C}$. If $a \in \mathcal{C}$ then $f(r;a) > 0$, $f'(r;a) < 0$ for all $r \in (0,\infty)$, and there is $k(a) > 0$ such that $\lim_{r\to\infty} r^{1/\beta} f(r;a) = k(a)$ and (1.13) holds for $r \gg 1$.*

(iii) *If $a \in \mathcal{B} \subset [a_1, a_2]$, then $R(a) < \infty$ and $f(r;a) > 0$, $f'(r;a) < 0$ for $0 < r < R(a)$, $f(R(a);a) = f'(R(a);a) = 0$. Moreover, the interface relation*

$$\lim_{r\to R(a)} \frac{(f^m)'(r;a)}{f^{1/(p-1)}(r;a)} = -(\beta R(a))^{1/(p-1)} \tag{1.14}$$

holds.

This theorem shows that when $n\beta < 1$ and $a \in \mathcal{B}$, the solution $f(r;a)$ of (1.12) has compact support, hence $\lim_{r\to\infty} r^{1/\beta} f(r;a) = 0$. Moreover, we have the following.

THEOREM 1.2. *Let the conditions of Theorem 1.1 fulfill. Then the sufficient and necessary condition that (1.7) has at least one nonnegative and nontrivial solution is $n\beta < 1$. In case of existence, the function $u(x,t)$, defined by (1.4), is a self-similar very singular solution to*

(1.1) and satisfies

$$\int_{R^n} u^{n\beta}(x,t)dx = constant, \qquad \lim_{t\to 0}\int_{|x|<\varepsilon} u(x,t)dx = \infty, \quad \forall \varepsilon > 0. \tag{1.15}$$

In fact, applying (1.4), for every $t > 0$ and $\varepsilon > 0$, we have

$$\begin{gathered}\int_{|x|<\varepsilon} u(x,t)dx = (\alpha/t)^{\alpha(1-n\beta)}\int_{|y|<\varepsilon(\alpha/t)^{\alpha\beta}} f(|y|)dy,\\ \int_{R^n} u^{n\beta}(x,t)dx = \int_{R^n} f^{n\beta}(|y|)dy.\end{gathered} \tag{1.16}$$

Recall that (1.6) and f has compact support, the integrands at the right-hand side of (1.16) are integrable as $t \to 0$. Then the result follows. Therefore, the condition $p > ((n+1)/(mn+1))q + mn/(mn+1)$ implies that the self-similar very singular solution of (1.1) exists and has compact support.

The organization of this paper is as follows. In Section 2, some properties of the solutions of (1.12) are studied. In particular, the behavior of the positive solution is obtained. In Sections 3 and 4, we prove the first part and the second part of Theorem 1.1, respectively.

2. Preliminary

In this section we consider (1.12). Let $z = f^m$, $a^m = b$, it follows from (1.12) that

$$\begin{gathered}\left(|z'|^{p-2}z'\right)' + \frac{n-1}{r}|z'|^{p-2}z' + \beta r\left(z^{1/m}\right)' + z^{1/m} - \left|\left(z^{1/m}\right)'\right|^q = 0, \quad r > 0,\\ z(0) = b, \qquad z'(0) = 0.\end{gathered} \tag{2.1}$$

Writing initial value problem (2.1) as an equivalent integral equation and using the standard Picard's iteration, we may prove that for each $b > 0$, (2.1) has a unique solution $z(r) = z(r;b)$, at least locally. In addition, (2.1) can be rewritten as

$$\begin{gathered}z' = |v|^{-(p-2)/(p-1)}v,\\ v' = -\frac{n-1}{r}v - \frac{\beta r}{m}z^{(1-m)/m}|v|^{-(p-2)/(p-1)}v - z^{1/m} + \frac{1}{m^q}z^{(1-m)q/m}|v|^{q/(p-1)}.\end{gathered} \tag{2.2}$$

For each $r_0 > 0$ and $z(r_0) = z_0 > 0$, $v(r_0) = v_0$, the above first order system admits an unique local solution at r_0 by the locally Lipschits continuous condition. Let $(0, R(b))$ be the maximal existence interval where $z(r;b) > 0$, it is easy to see that $R(a) = R(b)$.

Lemma 2.1. *Let $p > q > 1$, $m > 0$, and $q > m(p-1) > 1$. Equation (1.5) holds. For every $b > 0$, let $z(r)$ be the solution to (2.1), then the following statements hold:*

(i) *$z'(r) < 0$ for all $r \in (0, R(b))$;*

(ii) *If $R(b) = \infty$, then $\lim_{r\to\infty} z'(r) = 0$.*

Proof. (i) Since $(|z'|^{p-2}z')' = (p-1)|z'|^{p-2}z''$ and $\lim_{r\to 0}(|z'|^{p-2}z')' = -b^{1/m}/n < 0$, we deduce that there is $\tau > 0$ such that $z''(r) < 0$ for each $r \in (0,\tau)$. By $z'(0) = 0$, we get

$z'(r) < 0$ in $(0,\tau)$. If there is $r_1 \in (0,R(b))$ such that $z'(r_1) = 0$ and $z'(r) < 0$ in $(0,r_1)$, we have $\lim_{r\to r_1}(|z'|^{p-2}z')' = -z^{1/m}(r_1) < 0$. It follows that there exists $\delta > 0$ such that $z'(r) > 0$ in $(r_1 - \delta, r_1)$. This is a contradiction.

(ii) Since $z(r)$ is strictly decreasing in $(0,R(b))$ and $0 \le z(r) \le b$, we see that

$$\lim_{r\to R(b)} z(r) = \ell \tag{2.3}$$

for some $\ell \ge 0$. We claim that

$$\lim_{r\to R(b)} z'(r) = -\ell_1 \tag{2.4}$$

with $\ell_1 \ge 0$. In fact, we set

$$E(r) = \frac{p-1}{p}|z'(r)|^p + \frac{m}{m+1}z^{(m+1)/m}(r), \tag{2.5}$$

then it follows from (2.1) that

$$E'(r) = -\frac{n-1}{r}|z'(r)|^p - \frac{\beta r}{m}z^{(1-m)/m}(r)|z'(r)|^2 - \frac{1}{m^q}z^{(1-m)q/m}(r)|z'(r)|^{q+1} < 0 \tag{2.6}$$

for all $r \in (0,R(b))$. Thus, $E(r)$ is strictly decreasing in $(0,R(b))$ and $0 \le E(r) \le m/(m+1)b^{(1+m)/m}$. From (2.3) and the existence of $\lim_{r\to R(b)} E(r)$, (2.4) follows. Moreover,

$$|z'(r)| \le \left(\frac{mp}{(m+1)(p-1)}b^{(m+1)/m}\right)^{1/p}, \quad r \in [0,R(b)). \tag{2.7}$$

If $R(b) = \infty$, by (2.3) and (2.4), it is easy to see that $\ell_1 = 0$. This completes the proof. □

Lemma 2.2. *Assume that $p > q > q > m(p-1) > 1$, $m > 0$ and $R(b) = \infty$, α and β satisfy (1.5). Then, for each μ satisfying $0 < \mu < \beta$, there exists a $r_*(\mu)$ depending on μ such that $(\mu/m)rz'(r) + z(r) > 0$ when $r > r_*(\mu)$.*

Proof. Set $h(r) = (\mu/m)rz'(r) + z(r)$, we first show that there is a $r_*(\mu) > 0$ such that $h(r)$ does not change signs for all $r > r_*(\mu)$. In fact, if there is $r_0 > 0$ such that $h(r_0) = 0$, namely,

$z'(r_0) = -(m/\mu r_0)z(r_0)$. As $\sigma > 0$, we have

$$h'(r_0) = \left(\frac{\mu}{m}+1\right)z'(r_0) + \frac{\mu}{m}r_0 z''(r_0) = \left(\frac{\mu}{m}+1\right)z'(r_0)$$
$$-\frac{\mu r_0}{m(p-1)}\left[\frac{n-1}{r_0}z'(r_0) + \frac{\beta r_0}{m}z^{(1-m)/m}(r_0)z'(r_0)\,|z'(r_0)|^{2-p}\right.$$
$$\left.+z^{1/m}(r_0)\,|z'(r_0)|^{2-p} - \frac{1}{m^q}z^{(1-m)q/m}(r_0)\,|z'(r_0)|^{2+q-p}\right] \tag{2.8}$$
$$= \frac{1}{r_0}z^{1-\sigma/m}(r_0)\left[\frac{\mu^{p-2}(\beta-\mu)}{(p-1)m^{p-1}}r_0^p - (m/\mu+1)z^{\sigma/m}(r_0)\right]$$
$$+\frac{n-1}{(p-1)r_0}z(r_0) + \frac{1}{(p-1)m^q}\left(\frac{m}{\mu}\right)^{1+q-p} r_0^{p-q-1}z^{1+\gamma/m}(r_0) > 0$$

provided that

$$m^{1-p}\mu^{p-2}(\beta-\mu)r_0^p - (p-1)(1+m/\mu)z^{\sigma/m}(r_0) \geq 0. \tag{2.9}$$

Hence, when $r_0 \geq r_*(\mu) := [(p-1)((m+\mu)/(\beta-\mu))(m/\mu)^{p-1}b^{\sigma/m}]^{1/p}$, we have $h'(r_0) > 0$. This implies that there is a $\delta > 0$ such that $h(r) > 0$ in $(r_0, r_0+\delta)$ whenever $h(r_0) = 0$ and $r_0 > r_*(\mu)$. If there exists a $r_1 > r_0$ such that $h(r_1) = 0$, we may assume that r_1 is the first one. It follows that $h(r) > 0$ in (r_0, r_1). On the other hand, from $h'(r_1) > 0$ we see that there exists a $\delta' > 0$ such that $h(r) < 0$ in $(r_1-\delta', r_1)$. It is a contradiction. The above arguments show that, for $r > r_*(\mu)$,

$$\text{either } h(r) < 0 \quad \text{or} \quad h(r) > 0. \tag{2.10}$$

In the following we prove that it is impossible for the case of $h(r) < 0$ to occur. In fact, if $h(r) < 0$ then $z(r) < (\mu/m)r|z'(r)|$ or $z^{1/m}(r) < \mu r|(z^{1/m})'(r)|$, it follows that

$$z(r) \leq Cr^{-m/\mu}, \quad \text{i.e., } z(r) \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{2.11}$$

By the first equation of (2.1) and Lemma 2.1, we have

$$\begin{aligned}(|z'|^{p-2}z')'(r) &= \frac{n-1}{r}\,|z'(r)|^{p-1} + |(z^{1/m})'(r)|^q - \beta r(z^{1/m})'(r) - z^{1/m}(r)\\ &> -(\beta-\mu)(z^{1/m}(r))' = \frac{\beta-\mu}{m}rz^{(1-m)/m}|z'|.\end{aligned} \tag{2.12}$$

(a) If $m \geq 1$, then $z^{(1-m)/m}(r) \geq ((\mu/m)r|z'(r)|)^{(1-m)/m}$. By (2.12), we have

$$(|z'|^{p-2}z')'(r) > \frac{\beta-\mu}{m}\left(\frac{\mu}{m}\right)^{(1-m)/m} r^{1/m}\,|z'(r)|^{1/m}. \tag{2.13}$$

This implies that

$$\left(|z'|^{\sigma/m-1}z'\right)'(r) > \frac{\sigma(\beta-\mu)}{m^2(p-1)}\left(\frac{\mu}{m}\right)^{(1-m)/m} r^{1/m} := \delta r^{1/m}. \tag{2.14}$$

Integrating (2.14) over $(r_*(\mu), r)$ gives

$$\begin{aligned}\left(|z'|^{\sigma/m-1}z'\right)(r) &> \left(|z'|^{\sigma/m-1}z'\right)(r_*(\mu)) \\ &\quad + \frac{m}{m+1}\delta\left(r^{(m+1)/m} - r_*^{(m+1)/m}(\mu)\right) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty,\end{aligned} \tag{2.15}$$

which contradicts to Lemma 2.1(ii) since $\sigma > 0$.

(b) If $0 < m < 1$. Since $r|z'(r)| > (m/\mu)z(r)$, applying (2.12), we have

$$\left(|z'|^{p-2}z'\right)' > \frac{\beta-\mu}{\mu} z^{1/m}(r). \tag{2.16}$$

It follows that

$$\left(|z'|^{p-1}z'\right)' > -\frac{p(\beta-\mu)}{\mu(p-1)} z^{1/m}(r)z'(r) := -\delta_1 z^{1/m}(r)z'(r). \tag{2.17}$$

Using (2.11) and Lemma 2.1(ii) and integrating (2.17) from $r \geq r_*(\mu)$ to ∞ gives

$$\left|z'(r)\right|^p > \frac{m\delta_1}{m+1} z^{(m+1)/m}(r), \quad r > r_*(\mu), \tag{2.18}$$

that is,

$$-z'z^{-(m+1)/(mp)} > \left(\frac{m\delta_1}{m+1}\right)^{1/p}, \quad \forall r > r_*(\mu). \tag{2.19}$$

Since $(m+1)/(mp) < 1$, integrating (2.19) from $r_*(\mu)$ to r gives

$$-z^{\sigma/(mp)}(r) > -z^{\sigma/(mp)}(r_*(\mu)) + \frac{\sigma}{mp}\left(\frac{m\delta_1}{m+1}\right)^{1/p}(r - r_*(\mu)) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \tag{2.20}$$

It is a contradiction. This completes the proof. □

For $0 < \mu < \beta$, the following estimates are the direct conclusion of Lemma 2.2

$$\left|z'(r)\right| < \frac{m}{\mu r} z(r), \quad z(r) > C_0 r^{-m/\mu} \quad \forall r \gg 1. \tag{2.21}$$

LEMMA 2.3. *Assume that the conditions of Lemma 2.1 fulfill. Then* $\lim_{r\to R(b)} z(r;b) = 0$.

Proof. The conclusion is obvious when $R(a) < \infty$. We only prove the result for $R(b) = \infty$. By Lemma 2.1(ii), we have $\lim_{r\to\infty} |z'|^{p-2}z' = 0$. We divide the proof into four steps.

Step 1. We first claim that

$$\liminf_{r\to\infty} \left|\left(|z'|^{p-2}z'\right)'\right|(r) = 0. \tag{2.22}$$

In fact, if (2.22) is not true, there must exist $\delta_0 > 0$ and $r_0 > 0$ such that $|(|z'|^{p-2}z')'| > \delta_0$ for $r > r_0$. We will obtain a contradiction from the following two facts.

(a) If there is a $\hat{t} > r_0$ such that $|z'|^{p-2}z'(r)$ is monotonic in $r > \hat{t}$, then it must be increasing (because $z'' \le 0$ in $(\hat{t},\infty)$ contradicts with $R(b) = \infty$). Therefore, $(p-1)|z'|^{p-2}z'' = (|z'|^{p-2}z')' = |(z'|^{p-2}z')'| > \delta_0$ in $(\hat{t},\infty)$. Integrating the above over $(\hat{t},r)$ gives

$$(|z'|^{p-2}z')(r) > (|z'|^{p-2}z')(\hat{t}) + \delta_0(r-\hat{t}) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \tag{2.23}$$

It is impossible.

(b) If $|z'|^{p-2}z'(r)$ is not monotonic in $r > t$ for any $t > r_0$, which implies that $(|z'|^{p-2}z')(r)$ ultimately oscillates infinite times. Let $\{t_j\}$ be the sequence realizing the minima and satisfying $\lim_{j\to\infty} t_j = \infty$. Then $(|z'|^{p-2}z')'(t_j) = 0$, which contradicts to what we assume previously.

Step 2. We will show that $\lim_{r\to\infty} |(z^{1/m})'(r)| = 0$. In fact, if $0 < m \le 1$, it is a direct conclusion of Lemma 2.1(ii). While $m > 1$, using Lemma 2.2 yields

$$|(z^{1/m})'(r)| = \frac{1}{m} z^{(1-m)/m}(r)|z'(r)| < \frac{1}{\mu r} z^{1/m}(r) \le \frac{1}{\mu r} b^{1/m} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{2.24}$$

Step 3. We claim that

$$\lim_{r\to\infty} r(z^{1/m})'(r) = -\ell^{1/m}/\beta, \tag{2.25}$$

where ℓ is defined in (2.3). To prove this, we consider the following two cases.

(a) If there is $\tilde{r} > 0$ such that $r(z^{1/m})'(r)$ is monotonic in $r \in (\tilde{r},\infty)$, then it must be increasing (if it decrease then $r(z^{1/m})'(r) \le \tilde{r}(z^{1/m})'(\tilde{r}) := -C_0 < 0$ for all $r \ge \tilde{r}$, which gives $z^{1/m}(r) \le z^{1/m}(\tilde{r}) - C_0 \ln(r/\tilde{r}) \to -\infty$ as $r \to \infty$). Since $r(z^{1/m})'(r) < 0$, we see that $\lim_{r\to\infty} r(z^{1/m})'(r)$ exists. From (2.22) we can take $\{\hat{r}_j\}$ such that $\lim_{j\to\infty} \hat{r}_j = \infty$ and $\lim_{j\to\infty}(|z'|^{p-2}z')'(\hat{r}_j) = 0$. Applying (2.1), we get $\lim_{r\to\infty} r(z^{1/m})'(r) = \lim_{j\to\infty} \hat{r}_j(z^{1/m})'(\hat{r}_j) = -\ell^{1/m}/\beta$.

(b) If $r(z^{1/m})'(r)$ oscillates infinite times in $(\bar{r},\infty)$ for each $\bar{r} > 0$, then we take the sequences $\{r_j\}$ and $\{\hat{r}_j\}$ realizing the minima and the maxima, respectively, such that $\lim_{j\to\infty}(r_j,\hat{r}_j) = (\infty,\infty)$ and $r_j < \hat{r}_j < r_{j+1} < \hat{r}_{j+1}$ for all j. Therefore, $0 = (r(z^{1/m})')'(r_j) = (z^{1/m})'(r_j) + r_j(z^{1/m})''(r_j)$, that is, $z''(r_j) = -z'(r_j)/r_j + ((m-1)/m)(z'(r_j))^2/z(r_j)$. In view of (2.1), we have

$$\begin{gathered} (p-n)|z'|^{p-1}(r_j)/r_j + (p-1)(m-1)|z'(r_j)|^p/(mz(r_j)) + \beta r_j (z^{1/m})'(r_j) \\ + z^{1/m}(r_j) - |(z^{1/m})'|^q(r_j) = 0. \end{gathered} \tag{2.26}$$

By Lemma 2.2, we get $|z'(r)|^p/z(r) < (m/\mu r)^p z^{p-1}(r) \to 0$. Putting $j \to \infty$, it follows from (2.26) and Lemma 2.2 that $\lim_{j\to\infty} r_j(z^{1/m})'(r_j) = -\ell^{1/m}/\beta$. In the similar way, $\lim_{j\to\infty} \hat{r}_j(z^{1/m})'(\hat{r}_j) = -\ell^{1/m}/\beta$. Since for each sufficiently large r, either $r_j \le r < \hat{r}_j$ or $\hat{r}_j \le r < r_{j+1}$, we obtain (2.25).

Step 4. We prove $\ell = 0$. Assume by contradiction that $\ell > 0$, then from (2.25) there is $r_1 > 0$ such that $r(z^{1/m})'(r) < -\ell^{1/m}/(2\beta) < 0$ in $r \in (r_1,\infty)$. Integrating this inequality gives

$$\begin{aligned} b^{1/m} &> z^{1/m}(r_1) - z^{1/m}(r) \\ &= \int_{r_1}^{r} \left(- \left(z^{1/m}\right)'(s)\right) ds > \frac{\varrho^{1/m}}{2\beta} \ln (r/r_1) \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \end{aligned} \tag{2.27}$$

It is impossible. The proof is completed. □

Lemma 2.4. *Assume that the conditions of Lemma 2.1 fulfill and $R(a) = \infty$. Then*

$$\lim_{r\to\infty} rz'(r)/z(r) = -m/\beta. \tag{2.28}$$

Moreover, for each small $\varepsilon > 0$,

$$z(r) \le C_1 r^{-m/(\beta+\varepsilon)}, \quad \left|z'(r)\right| \le C_2 r^{-1-m/(\beta+\varepsilon)}, \quad \forall r \gg 1, \tag{2.29}$$

where C_1 and C_2 are positive constants.

Proof. We first prove that $H(r) = (\mu/m)rz'(r) + z(r)$ does not change signs as $r \gg 1$ for every $\mu > \beta$. Using the arguments of Lemma 2.2, if there is a $r_0 \ge 1$ such that $H(r_0) = 0$, it follows that $z'(r_0) = -(m/\mu r_0)z(r_0)$. Then we have

$$\begin{aligned} H'(r_0) &= \left(\frac{\mu}{m} + 1\right) z'(r_0) + \frac{\mu}{m} r_0 z''(r_0) \\ &= \frac{1}{m(p-1)} r_0^{p-1} z^{1-\sigma/m}(r_0) \\ &\quad \times \left[(\beta-\mu)\left(\frac{\mu}{m}\right)^{p-2} + (n-1)m r_0^{-p} z^{\sigma/m}(r_0) + \mu m^{-q} \left(\frac{m}{\mu}\right)^{2+q-p} r_0^{-q} z^{(q-1)/m}(r_0) \right] \\ &\quad - \frac{m+\mu}{\mu r_0} z(r_0) < 0 \end{aligned} \tag{2.30}$$

provided that

$$(\beta-\mu)\left(\frac{m}{\mu}\right)^{2-p} + (n-1)m r_0^{-p} z^{\sigma/m}(r_0) + \mu m^{-q}\left(\frac{m}{\mu}\right)^{2+q-p} r^{-q} z^{(q-1)/m}(r_0) < 0. \tag{2.31}$$

Recall that $p > q > m(p-1) > 1$ and $r_0 \ge 1$, set

$$\bar{r}(\mu) = \left[\frac{1}{\mu-\beta} \left(\frac{\mu}{m}\right)^{2-p} \left((n-1)m + \mu m^{-q} \left(\frac{m}{\mu}\right)^{2+q-p} \right) b^{\sigma/m} \right]^{1/q}. \tag{2.32}$$

Let $r^*(\mu) = \max\{1, \bar{r}(\mu)\}$. We see that $H'(r_0) < 0$ whenever $H(r_0) = 0$ and $r_0 \ge r^*(\mu)$, which implies that there exists a $\delta > 0$ such that $H(r) < 0$ in $(r_0, r_0+\delta)$. If there is a $r_1 > r_0$ such that $H(r_1) = 0$, we may assume that r_1 is the first one. Then $H(r) < 0$ in (r_0, r_1).

On the other hand, by $H'(r_1) < 0$ we see $H(r) > 0$ in $(r_1 - \delta', r_1)$ for some $\delta' > 0$. It is a contradiction. Thus,

$$\text{either } H(r) < 0 \quad \text{or} \quad H(r) > 0, \quad \forall r > r^*(\mu). \tag{2.33}$$

If $H(r) > 0$ then $|z'(r)| < (m/\mu r)z(r)$ and $\beta r(z^{1/m})'(r) + (\beta/\mu)z^{1/m}(r) > 0$. By (2.1), we have

$$\begin{aligned}(|z'|^{p-2}z')'(r) &= \frac{n-1}{r}|z'(r)|^{p-1} + |(z^{1/m})'(r)|^q - \beta r(z^{1/m})'(r) - z^{1/m}(r)\\ &< \left[\frac{n-1}{r^p}\left(\frac{m}{\mu}\right)^{p-1} z^{\sigma/m}(r) + \frac{1}{(\mu r)^q} z^{(q-1)/m} + \frac{\beta}{\mu} - 1\right] z^{1/m}.\end{aligned} \tag{2.34}$$

Since $\sigma > 0$, $q > 1$ and $\mu > \beta$, we see

$$\lim_{r\to\infty}\left[\frac{n-1}{r^p}\left(\frac{m}{\mu}\right)^{p-1} z^{\sigma/m} + \frac{1}{(\mu r)^q} z^{(q-1)/m}\right] = 0. \tag{2.35}$$

It follows from (2.34) that there is $\hat{r} \geq r^*(\mu)$ such that $(|z'|^{p-2}z')'(r) < 0$, that is, $z''(r) < 0$ in $(\hat{r}, \infty)$. It contradicts to $R(b) = \infty$. This implies that for each $\varepsilon > 0$, there exists a $r^*(\beta+\varepsilon) > 0$ such that

$$\frac{\beta+\varepsilon}{m} r z'(r) + z(r) < 0, \quad \forall r > r^*(\beta+\varepsilon). \tag{2.36}$$

Integrating (2.36) over $(r^*(\beta+\varepsilon), r)$ and applying (2.21), we obtain (2.29).

Using Lemma 2.2 and (2.36), we have, for each $\varepsilon > 0$,

$$-\frac{m}{\beta-\varepsilon} < \frac{rz'(r)}{z(r)} < -\frac{m}{\beta+\varepsilon}, \quad \forall r > \max\{r_*(\beta-\varepsilon),\ r^*(\beta+\varepsilon)\}. \tag{2.37}$$

This completes the proof. □

3. The case of $n\beta \geq 1$

In this section, we will prove the following lemma.

Lemma 3.1. *Assume that $p > q > m(p-1) > 1$, α and β satisfy (1.5). Then for each $b > 0$ there exists a $k(b) > 0$ such that when $n\beta \geq 1$, $R(b) = \infty$ and $\lim_{r\to\infty} r^{1/\beta} z^{1/m}(r) = k(b)$. Moreover, for $r \gg 1$,*

$$z^{1/m}(r;b) = k(b)r^{-1/\beta}\left\{1 + \frac{1}{\beta^2}\left(\frac{k(b)}{\beta}\right)^{\sigma}\left[m^{p-1}\left(1 - \frac{n\beta-1}{\beta\nu}\right) - \frac{1}{\nu}\left(\frac{k(a)}{\beta}\right)^{\gamma}\right] r^{-\nu} + o(r^{-\nu})\right\}. \tag{3.1}$$

Proof. By (2.1), we have

$$(r^{1/\beta-1}|z'|^{p-2}z' + \beta r^{1/\beta} z^{1/m})' = (n-1/\beta) r^{1/\beta-2}|z'|^{p-1} + r^{1/\beta-1}|(z^{1/m})'|^q > 0. \tag{3.2}$$

So the function $G(r) := r^{1/\beta-1}|z'|^{p-2}z' + \beta r^{1/\beta}z^{1/m}$ is strictly increasing in $(0,R(b))$. In the proof of Lemma 2.1(i) we see that $|z'(r)|^{p-1} = O(r)$ as $r \ll 1$. Thus $\lim_{r\to 0} G(r) = 0$ and $G(r) > 0$ in $(0,R(b))$, which implies $R(b) = \infty$. Consequently,

$$\begin{aligned} &r^{1/\beta-1}(|z'|^{p-2}z')(r) + \beta r^{1/\beta}z^{1/m}(r) \\ &\quad = (n-1/\beta)\int_0^r s^{1/\beta-2}|z'(s)|^{p-1}ds + \int_0^r s^{1/\beta-1}|(z^{1/m})'(s)|^q ds. \end{aligned} \tag{3.3}$$

It follows from (2.21) and (2.29) that

$$r^{1/\beta-2}|z'(r)|^{p-1} \le Cr^{-m_1}, \qquad r^{1/\beta-1}|(z^{1/m})'(r)|^q \le Cr^{-m_2} \tag{3.4}$$

with

$$m_1 = 1+\nu - \frac{m(p-1)\varepsilon}{\beta(\beta+\varepsilon)}, \qquad m_2 = 1+\nu - \frac{q\varepsilon}{\beta(\beta+\varepsilon)}. \tag{3.5}$$

Hence, $\lim_{r\to\infty} r^{1/\beta-1}(|z'|^{p-2}z')(r) = 0$ and the integrals at the right-hand side of (3.3) make sense over $(0,\infty)$ if ε is suitably small. Therefore, from (3.3) we derive

$$\begin{aligned} &\lim_{r\to\infty} r^{1/\beta}z^{1/m}(r) \\ &\quad = \Big((n-1/\beta)\int_0^\infty s^{1/\beta-2}|z'(s)|^{p-1}ds + \int_0^\infty s^{1/\beta-1}|(z^{1/m})'(r)|^q ds\Big)/\beta := k(b). \end{aligned} \tag{3.6}$$

Consequently, by Lemma 2.4 we have

$$z^{1/m}(r;b) = k(b)r^{-1/\beta}(1+o(1)), \quad (z^{1/m})'(r;b) = -\frac{k(b)}{\beta}r^{-1-1/\beta}(1+o(1)), \quad \forall r \gg 1. \tag{3.7}$$

Moreover, applying the first equation of (2.1) yields

$$\begin{aligned} \beta(r^{1/\beta}z^{1/m}(r))' &= r^{1/\beta-1}\Big[|(z^{1/m})'(r)|^q + \frac{n-1}{r}|z'(r)|^{p-1} - (|z'(r)|^{p-2}z'(r))'\Big] \\ &= \Big(\frac{k(b)}{\beta}\Big)^q r^{-1-\nu}(1+o(1)) + (n-1)m^{p-1}\Big(\frac{k(b)}{\beta}\Big)^{m(p-1)} r^{-1-\nu}(1+o(1)) \\ &\quad - r^{1/\beta-1}(|z'(r)|^{p-2}z'(r))'. \end{aligned} \tag{3.8}$$

Integrating (3.8) over (r,∞) we obtain

$$\begin{aligned}
&\beta k(b)-\beta r^{1/\beta}z^{1/m}(r)\\
&\quad=\frac{1}{\nu}\left(\frac{k(b)}{\beta}\right)^{q}r^{-\nu}(1+o(1))+\frac{(n-1)m^{p-1}}{\nu}\left(\frac{k(b)}{\beta}\right)^{m(p-1)}r^{-\nu}(1+o(1))\\
&\quad\quad-\int_r^{\infty}s^{1/\beta-1}\left(|z'|^{p-2}z'\right)'(s)ds.
\end{aligned}\tag{3.9}$$

Since $\lim_{r\to\infty}r^{1/\beta-1}(|z'|^{p-2}z')(r)=0$, it follows that

$$\begin{aligned}
\int_r^{\infty}s^{1/\beta-1}\left(|z'|^{p-2}z'\right)'(s)ds&=r^{1/\beta-1}|z'|^{p-1}(r)+(1/\beta-1)\int_r^{\infty}s^{1/\beta-2}|z'|^{p-1}(s)ds\\
&=m^{p-1}\left(\frac{k(b)}{\beta}\right)^{m(p-1)}r^{-\nu}(1+o(1))\\
&\quad+(1/\beta-1)m^{p-1}\left(\frac{k(b)}{\beta}\right)^{m(p-1)}\int_r^{\infty}s^{-1-\nu}(1+o(1))ds\\
&=m^{p-1}\left(\frac{k(b)}{\beta}\right)^{m(p-1)}\left(1+\frac{1-\beta}{\beta\nu}\right)r^{-\nu}(1+o(1)).
\end{aligned}\tag{3.10}$$

By (3.9) we see

$$\beta k(b)-\beta r^{1/\beta}z^{1/m}(r)=\left\{\frac{1}{\nu}\left(\frac{k(b)}{\beta}\right)^{q}+m^{p-1}\left(\frac{k(b)}{\beta}\right)^{m(p-1)}\left(\frac{n\beta-1}{\beta\nu}-1\right)\right\}r^{-\nu}(1+o(1)).\tag{3.11}$$

Hence, we have

$$z^{1/m}(r)=k(b)r^{-1/\beta}\left\{1+\frac{1}{\beta^2}\left(\frac{k(b)}{\beta}\right)^{\sigma}\left[m^{p-1}\left(1-\frac{n\beta-1}{\beta\nu}\right)-\frac{1}{\nu}\left(\frac{k(b)}{\beta}\right)^{\gamma}\right]r^{-\nu}(1+o(1))\right\}.\tag{3.12}$$

This completes the proof. □

Lemma 3.1 gives the proof of the first part of Theorem 1.1, which indicates that when $n\beta\geq 1$ there is no self-similar singular solution.

4. The case of $n\beta<1$

In this section we will prove the second part of Theorem 1.1.

Lemma 4.1. *Assume that $p>q>m(p-1)>1$, equation (1.5) holds. Let $b>0$, $z(r)$ be the solution of (2.1). Then*

(i) *If $mq+1-q>0$, then $|z'(r)|\leq mb^{(mq+1-q)/(mq)}$, $\forall r\in(0,R(b))$.*
(ii) *If $mq+1-q\leq 0$, then $|z'(r)|\leq mz^{(mq+1-q)/(mq)}(r)$, $\forall r\in(0,R(b))$.*

Proof. Notice that $\lim_{r\to 0}(|z'|^{p-2}z')'(r) = -b^{1/m}/n < 0$, it is easy to see that there is a $\hat{r}$ such that $(|z'|^{p-2}z')'(r) \le 0$, namely, $z''(r) \le 0$ in $(0,\hat{r})$. Combining (2.1) with $z'(r) < 0$ yields $z^{1/m}(r) - |(z^{1/m})'(r)|^q \ge 0$ in $(0,\hat{r})$. Therefore,

$$|z'(r)| \le mz^{(mq+1-q)/(mq)}(r), \quad \forall r \in (0,\hat{r}). \tag{4.1}$$

If $(|z'|^{p-2}z')'(\tilde{r}) > 0$ for some $\tilde{r} > 0$, then there exists $r_0 < \tilde{r}$ such that $(|z'|^{p-2}z')'(r_0) = 0$ and $(|z'|^{p-2}z')'(r) > 0$, that is, $z''(r) > 0$ in $(r_0,\tilde{r})$. So $z'(r_0) < z'(\tilde{r}) < 0$, and $|z'(r_0)| \le mz^{(mq+1-q)/(mq)}(r_0)$, which implies

$$|z'(\tilde{r})| < |z'(r_0)| \le mz^{(mq+1-q)/(mq)}(r_0). \tag{4.2}$$

If $mq+1-q > 0$, then for every $r \in (0,R(b))$ combining (4.1) with (4.2) gives (i). While $mq+1-q \le 0$, since $z(r)$ is strictly decreasing, it follows from (4.2) that

$$|z'(\tilde{r})| \le mz^{(mq+1-q)/(mq)}(\tilde{r}). \tag{4.3}$$

Hence, (ii) follows from (4.1) and (4.3). This completes the proof. □

If $mq+1-q > 0$, then $0 < (mq+1-q)/(mq) < (m+1)/(mp) < 1$ and $1/[m(p-1)] < (m+1)/(mp)$ since $1 < m(p-1) < q < p$. We can choose θ such that

$$\max\left\{\frac{mq+1-q}{mq}, \frac{1}{m(p-1)}\right\} < \theta < \frac{m+1}{mp} \tag{4.4}$$

whether or not $mq+1-q > 0$. It follows from that (4.4) that $1-\theta > (p-1)\theta - 1/m$. We define, for each $\lambda > 0$ and $\eta > 0$,

$$\begin{aligned} \mathcal{S}_{\lambda,\eta} &= \{(z,z') \mid 0 < z < \eta,\ -\lambda z^\theta < z' < 0\}, \\ \mathcal{S}_{\lambda} &= \{(z,z') \mid z > 0,\ -\lambda z^\theta < z' < 0\}. \end{aligned} \tag{4.5}$$

LEMMA 4.2. *Assume that $p > q > m(p-1) > 1$, (1.5) holds. Let*

$$r_{\lambda,\eta} := m\big(\theta(p-1)\lambda^p \eta^{(p-1)\theta-1/m} + \eta^{1-\theta}\big)/(\lambda\beta). \tag{4.6}$$

Then, for any given $\lambda > 0$ and $\eta > 0$, $\mathcal{S}_{\lambda,\eta}$ is positively invariant for $r > r_{\lambda,\eta}$, namely, if $(z(r_{\lambda,\eta}), z'(r_{\lambda,\eta})) \in \mathcal{S}_{\lambda,\eta}$ then the orbit $(z(r),z'(r))$ of (2.2) remains in $\mathcal{S}_{\lambda,\eta}$ for all $r > r_{\lambda,\eta}$.

Proof. Since the vector field enters $\mathcal{S}_{\lambda,\eta}$ from the positive z-axis, we only need to show that it also enters $\mathcal{S}_{\lambda,\eta}$ from the parabola

$$l_{\lambda,\eta} := \{(z,z') \mid 0 < z < \eta,\ z' = -\lambda z^\theta\}. \tag{4.7}$$

On $l_{\lambda,\eta}$,

$$\frac{z''}{(z^\theta)'} = \frac{1}{p-1}\bigg[-\frac{n-1}{\theta r}z^{1-\theta} - \frac{\beta r}{m\theta}|z'|^{2-p}z^{1/m-\theta} + \frac{1}{\theta}|z'|^{1-p}z^{(m+1)/m-\theta} - \frac{1}{m^q\theta}|z'|^{1+q-p}z^{1-\theta+(1-m)q/m}\bigg] < \frac{1}{m(p-1)\theta}\big[-\beta r\lambda^{2-p}z^{1/m-(p-1)\theta} + m\lambda^{1-p}z^{(m+1)/m-p\theta}\big] < -\lambda \tag{4.8}$$

provided that

$$r > m\big(\theta(p-1)\lambda^p z^{(p-1)\theta-1/m}(r) + z^{1-\theta}(r)\big)/(\lambda\beta). \tag{4.9}$$

Notice that $m(p-1)\theta > 1$, $1-\theta > 0$ and $0 < z < \eta$. Consequently, (4.8) holds when $r > r_{\lambda,\eta} = m(\theta(p-1)\lambda^p\eta^{(p-1)\theta-1/m} + \eta^{1-\theta})/(\lambda\beta)$. This implies that $(z' + \lambda z^\theta)' > 0$ on the parabola $l_{\lambda,\eta}$ when $r > r_{\lambda,\eta}$ and that the orbit enters $\mathcal{S}_{\lambda,\eta}$ again unless the orbit is not in $\mathcal{S}_{\lambda,\eta}$ all the time. This completes the proof. □

We define three sets:

$$\mathcal{A} = \{b > 0 \mid R(b) < \infty \text{ and } z'(R(b)) < 0\},$$
$$\mathcal{B} = \{b > 0 \mid R(b) < \infty \text{ and } z'(R(b)) = 0\}, \tag{4.10}$$
$$\mathcal{C} = \{b > 0 \mid \text{the orbit } (z,z') \text{ starting from } (b,0) \text{ enters } \mathcal{S}_1 \text{ eventually}\}.$$

Remark 4.3. For any $b \in \mathcal{C}$, the corresponding solution $z(r;b)$ satisfies $z' + z^\theta > 0$ when $r < R(b)$ and close to $R(b)$, which implies $R(b) = \infty$. On the other hand, if $R(b) = \infty$ and $b \notin \mathcal{C}$, by Lemma 4.2 there is $r_0 > 0$ such that $z' + z^\theta \le 0$ for $r > r_0$. This implies that

$$z^{1-\theta}(r) \le z^{1-\theta}(r_0) - (1-\theta)(r - r_0) \longrightarrow -\infty, \quad \text{as } r \longrightarrow \infty. \tag{4.11}$$

It is impossible. Therefore the three sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are disjoint with $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0,\infty)$.

LEMMA 4.4. *Set $\mathcal{A}$ is nonempty and open. Moreover, $(0,b_1) \subset \mathcal{A}$ if $0 < b_1 \ll 1$.*

Proof. Let $w_\varepsilon(t) = \varepsilon^{-1}z(r;\varepsilon)$ with $t = r\varepsilon^{-\sigma/(mp)}$, applying (2.1), w_ε satisfies

$$\big(|w_\varepsilon'|^{p-2}w_\varepsilon'\big)' + \frac{n-1}{t}\big(|w_\varepsilon'|^{p-2}w_\varepsilon'\big) + \beta t\big(w_\varepsilon^{1/m}\big)' + w_\varepsilon^{1/m} - \varepsilon^{\sigma_1}\big|\big(w_\varepsilon^{1/m}\big)'\big|^q = 0, \quad r > 0,$$
$$w_\varepsilon(0) = 1, \qquad w_\varepsilon'(0) = 0, \tag{4.12}$$

where $\sigma_1 = [p(q-1) - q\sigma]/(mp) > 0$ since $p(q-1) + q(1 - m(p-1)) > (p-q)(q-1) > 0$, we see that $\sigma_1 > 0$. Set $E_\varepsilon(t) = ((p-1)/p)|w_\varepsilon'(t)|^p + (m/(m+1))w_\varepsilon^{(m+1)/m}(t)$, then it follows that

$$E_\varepsilon'(t) = -\frac{n-1}{t}\,|\,w_\varepsilon'(t)\,|^p - \frac{\beta t}{m} w_\varepsilon^{(1-m)/m}(t)\,|\,w_\varepsilon'(t)\,|^2 - \frac{\varepsilon^{\sigma_1}}{m^q} w_\varepsilon^{(1-m)q/m}(t)\,|\,w_\varepsilon'(t)\,|^{q+1} < 0. \tag{4.13}$$

Consequently, $E_\varepsilon(t) \le m/(m+1)$ for each $\varepsilon > 0$, both $w_\varepsilon(t)$ and $w_\varepsilon'(t)$ are uniformly bounded with respect to $t \ge 0$ and $\varepsilon > 0$. Moreover,

$$|\,w_\varepsilon'(t)\,| \le \left(\frac{mp}{(m+1)(p-1)}\right)^{1/p}. \tag{4.14}$$

Denote by $(0, T_\varepsilon)$ the maximal existence interval where $w_\varepsilon(t) > 0$, then $w_\varepsilon'(t) < 0$ in $(0, T_\varepsilon)$.

Considering the problem

$$\begin{gathered}(|w'|^{p-2}w')' + \frac{n-1}{t}|w'|^{p-2}w' + \beta t(w^{1/m})' + w^{1/m} = 0, \quad r > 0,\\ w(0) = 1, \qquad w'(0) = 0.\end{gathered} \tag{4.15}$$

We claim that there exists some $t_0 > 0$ such that the solution $w(t)$ of (4.15) satisfies $w(t_0) = 0$, $w'(t_0) < 0$ and $w(t) > 0$, $w'(t) < 0$ for every $t \in (0, t_0)$. In fact, by the contradiction that if the solution $w(t)$ of (4.15) is strictly positive, then we have, since $n\beta < 1$,

$$(t^{n-1}|w'|^{p-2}w' + \beta t^n w^{1/m})' = -(1-n\beta)t^{n-1}w^{1/m} < 0, \quad \forall t > 0. \tag{4.16}$$

The function $t^{n-1}(|w'|^{p-2}w')(t) + \beta t^n w^{1/m}(t)$ is strictly decreasing in $(0, \infty)$, thus

$$w'(t) + \beta^{1/(p-1)} w^{1/(m(p-1))}(t) t^{1/(p-1)} < 0, \quad \forall t > 0. \tag{4.17}$$

It follows from (4.17) that

$$1 - w^{\sigma/(m(p-1))}(t) > \frac{\sigma}{mp}\beta^{1/(p-1)} t^{p/(p-1)} \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{4.18}$$

It is a contradiction. Hence, there is some finite $t_0 > 0$ such that $w(t) > 0$, $w'(t) < 0$ for $t \in (0, t_0)$ and $w(t_0) = 0$. Moreover, we can show $w'(t_0) < 0$. In fact, let $t_0 > t > t_0/2$ and integrate (4.16) over $(0, t)$, we have

$$\begin{aligned}&t^{n-1}(w'\ |^{p-2}\ w')(t) + \beta t^n w^{1/m}(t)\\ &\quad = -(1-n\beta)\int_0^t s^{n-1}w^{1/m}(s)ds < -(1-n\beta)\int_0^{t_0} s^{n-1}w^{1/m}(s)ds := -C_0 < 0\end{aligned} \tag{4.19}$$

for some $C_0 > 0$. Sending $t \to t_0$ yields $t_0^{n-1}(|w'|^{p-2}w')(t_0) \le -C_0 < 0$, namely, $w'(t_0) < 0$.

Let $\eta_0 > 0$ be so small that when t_1 fulfills $0 < t_0 - t_1 \ll 1$, the followings hold

$$w(t_1) < \eta_0, \qquad w'(t_1) < w'(t_0)/2,$$
$$\frac{1}{2^{p-1}}(|w'|^{p-2}w')(t_0) + \beta t_0 \eta_0^{1/m} + \frac{1}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p} \eta_0^{\sigma_2} < 0, \tag{4.20}$$

with $\sigma_2 = [(1-m)q + m]/m > [q - m(p-1)]/m > 0$. By the continuous dependence of the solution on the parameter ε we have

$$T_\varepsilon > t_1, \quad w_\varepsilon(t_1) = \eta < \eta_0, \quad w'_\varepsilon(t_1) < w'(t_0)/2, \quad \forall 0 < \varepsilon \ll 1. \tag{4.21}$$

It follows from (4.12) and (4.14) that, for $t > t_1$,

$$\begin{aligned}&(t^{n-1}|w'_\varepsilon|^{p-2}w'_\varepsilon + \beta t^n w_\varepsilon^{1/m})' \\ &\quad < -(1-n\beta)t^{n-1}w_\varepsilon^{1/m} - \frac{\varepsilon^{\sigma_1}}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p} t^{n-1}w^{(1-m)q/m}w'_\varepsilon.\end{aligned} \tag{4.22}$$

Let $t_2 > t_1$ and satisfy

$$t_2^{p/(p-1)} = t_1^{p/(p-1)} + \frac{mp}{m(p-1)-1}\beta^{-1/(p-1)}\eta_0^{\sigma/(m(p-1))}, \qquad \varepsilon \le \left(\frac{t_1}{t_2+1}\right)^{(n-1)/\sigma_1}. \tag{4.23}$$

Integrating (4.22) from t_1 to $t < \min\{T_\varepsilon, t_2+1\}$, we get,

$$\begin{aligned}&t^{n-1}\left(|w'_\varepsilon|^{p-2}w'_\varepsilon\right)(t) + \beta t^n w_\varepsilon^{1/m}(t) \\ &\quad < t_1^{n-1}(|w'_\varepsilon|^{p-2}w'_\varepsilon)(t_1) + \beta t_1^n w_\varepsilon^{1/m}(t_1) \\ &\qquad - \frac{\varepsilon^{\sigma_1}}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p}\int_{t_1}^t s^{n-1}w_\varepsilon^{(1-m)q/m}w'_\varepsilon ds \\ &\quad < t_1^{n-1}(|w'_\varepsilon|^{p-2}w'_\varepsilon)(t_1) + \beta t_1^n w_\varepsilon^{1/m}(t_1) + \frac{\varepsilon^{\sigma_1}t^{n-1}}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p} w_\varepsilon^{\sigma_2}(t_1) \\ &\quad \le t_1^{n-1}\left\{(|w'_\varepsilon|^{p-2}w'_\varepsilon)(t_1) + \beta t_1 w_\varepsilon^{1/m}(t_1) + \frac{1}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p} w_\varepsilon^{\sigma_2}(t_1)\right\} \\ &\quad < t_1^{n-1}\left\{\frac{1}{2^{p-1}}(|w'|^{p-2}w')(t_0) + \beta t_0\eta_0^{1/m} + \frac{1}{m^q}\left(\frac{mp}{(m+1)(p-1)}\right)^{(q-1)/p}\eta_0^{\sigma_2}\right\} < 0.\end{aligned} \tag{4.24}$$

It follows from (4.24) that

$$w'_\varepsilon(t) + (\beta t)^{1/(p-1)}w_\varepsilon^{1/(m(p-1))}(t) < 0, \quad t_1 < t < \min\{T_\varepsilon, t_2+1\}. \tag{4.25}$$

Integrating (4.25) from t_1 to t_2, we have

$$\begin{aligned} w_\varepsilon^{\sigma/(m(p-1))}(t_2) &< w_\varepsilon^{\sigma/(m(p-1))}(t_1) - \frac{m(p-1)-1}{mp}\beta^{1/(p-1)}\left(t_2^{p/(p-1)} - t_1^{p/(p-1)}\right) \\ &< w_\varepsilon^{\sigma/(m(p-1))}(t_1) - \eta_0^{\sigma/(m(p-1))} < 0. \end{aligned} \tag{4.26}$$

This shows that $T_\varepsilon < t_2$. By (4.24) it follows that, for some constant C,

$$(|w_\varepsilon'|^{p-2}w_\varepsilon')(t) + \beta t w_\varepsilon^{1/m}(t) < C < 0, \quad \forall t \in (t_1, T_\varepsilon). \tag{4.27}$$

Sending $t \to T_\varepsilon$ gives $w_\varepsilon'(T_\varepsilon) < 0$.

The above arguments show that whenever $b = \varepsilon \ll 1$, $(0,b) \subset \mathscr{A}$. By the continuous dependence of the solution on the initial value b, it is easy to see that $\mathscr{A}$ is open. The proof is completed. □

Lemma 4.5. *Set $\mathscr{C}$ is nonempty and open. Moreover, $(b,\infty) \subset \mathscr{C}$ if $b \gg 1$.*

Proof. We first show that if the initial value b is suitably large then the corresponding orbit (z,z') must remain in $\mathscr{S}_1$ for all $r > 0$. This implies that $b \in \mathscr{C}$.

To do this, let $r_0 > 0$ be the first value where the orbit intersects with the boundary of $\mathscr{S}_1$. Then $z'(r_0) = -z^\theta(r_0)$ because the orbit enters $\mathscr{S}_1$ from the positive z-axis.

(a) Applying Lemma 4.1(i) we have

$$z(r_0) = |z'(r_0)|^{1/\theta} \le m^{1/\theta} b^{(mq+1-q)/(mq\theta)} \tag{4.28}$$

provided that $mq + 1 - q > 0$. On the other hand,

$$z(r_0) = z(0) + \int_0^{r_0} z'(s)ds \ge b + \int_0^{r_0} \left(-mb^{(mq+1-q)/(mq)}\right)ds = b - mb^{(mq+1-q)/(mq)}r_0. \tag{4.29}$$

Consequently, we obtain

$$r_0 \ge \frac{1}{m} b^{1-(mq+1-q)/(mq)}\left(1 - m^{1/\theta}b^{(mq+1-q)/(mq\theta)-1}\right) := \phi_1(b). \tag{4.30}$$

By Lemma 4.2, $r_{1,b} = m(1+\theta(p-1)b^{p\theta-(m+1)/m})b^{1-\theta}/\beta$. According to (4.4) the choice of θ, we have

$$\frac{\phi_1(b)}{r_{1,b}} \longrightarrow \infty \quad \text{as } b \longrightarrow \infty. \tag{4.31}$$

It contradicts to Lemma 4.2.

(b) If $mq + 1 - q \le 0$, then $m < 1$ and $(q-1)(1-m) \ge m$. Since $|z'(r_0)| = z^\theta(r_0)$, applying Lemma 4.1(ii), we have $z^{\theta+(q-1-mq)/(mq)}(r_0) \le m$ and

$$\begin{aligned} r_0 &\ge \frac{q}{q-1}\left(b^{(q-1)/(mq)} - z^{(q-1)/(mq)}(r_0)\right) \\ &\ge \frac{q}{q-1}\left(b^{(q-1)/(mq)} - C(m)\right) := \phi_2(b) \end{aligned} \tag{4.32}$$

with $C(m) = m^{q(q-1)(1-m)/(mq\theta+q-1-mq)}$. Notice that

$$\frac{q-1}{mq} > 1-\theta, \qquad \frac{\phi_2(b)}{r_{1,b}} \longrightarrow \infty \quad \text{as } b \longrightarrow \infty. \tag{4.33}$$

This also contradicts to Lemma 4.2. This fact shows that when $b \gg 1$, $(b,\infty) \subset \mathscr{C}$.

Now, we prove that $\mathscr{C}$ is open. By the definition of $\mathscr{C}$ and Lemma 4.2, if $b_0 \in \mathscr{C}$ then there exists $r_0 > r_{1,2b_0} = m(1+\theta(p-1)(2b_0)^{p\theta-(m+1)/m})(2b_0)^{1-\theta}/\beta > r_{1,b_0}$ such that $(z(r_0;b_0), z'(r_0;b_0)) \in \mathscr{S}_1$. Hence, by the continuous dependence of the solution on the initial value there must be a neighborhood $\mathscr{N}$ of b_0 such that $r_0 > r_{1,b}$, $z(r;b) > 0$ on $[0, r_0]$ and $(z(r_0;b), z'(r_0;b)) \in \mathscr{S}_1$ for all $b \in \mathscr{N}$. Lemma 4.2 implies that $(z(r;b), z'(r;b)) \in \mathscr{S}_1$ for all $r_0 < r < R(b)$, and hence $b \in \mathscr{C}$. □

When $b \in \mathscr{C}$, the corresponding solution of (2.1) is strictly positive and $R(b) = \infty$.

Lemma 4.6. *Let $b \in \mathscr{C}$. Then there exists a $k(b) > 0$ such that* $\lim_{r\to\infty} r^{1/\beta} z^{1/m}(r;b) = k(b)$. *Moreover, for $r \gg 1$,*

$$z^{1/m}(r;b) = k(b) r^{-1/\beta}\left\{1 + \frac{1}{\beta^2}\left(\frac{k(a)}{\beta}\right)^{\sigma}\left[m^{p-1}\left(1-\frac{n\beta-1}{\beta\nu}\right) - \frac{1}{\nu}\left(\frac{k(a)}{\beta}\right)^{\gamma}\right] r^{-\nu} + o(r^{-\nu})\right\}. \tag{4.34}$$

Proof. Since the proof of the second part of this lemma is completely similar to Lemma 3.1, we only prove the first part. By the first equation of (2.1), we have

$$\left(r^{1/\beta-1}|z'|^{p-2}z' + \beta r^{1/\beta} z^{1/m}\right)'(r) = (n-1/\beta) r^{1/\beta-2}|z'(r)|^{p-1} + r^{1/\beta-1}|(z^{1/m})'(r)|^{q}. \tag{4.35}$$

Integrating the above over $(0,r)$, we have

$$\begin{aligned} &r^{1/\beta-1}(|z'|^{p-2}z')(r) + \beta r^{1/\beta} z^{1/m}(r) \\ &\quad = (n-1/\beta)\int_0^r s^{1/\beta-2}|z'(s)|^{p-1}ds + \int_0^r s^{1/\beta-1}|(z^{1/m})'(s)|^{q}ds. \end{aligned} \tag{4.36}$$

Since $n\beta < 1$, using the estimates (2.21) and (2.29), the integrals at the right-hand side of (4.36) make sense over $(0,\infty)$ and $\lim_{r\to\infty} r^{1/\beta-1}|z'(r)|^{p-2}f'(r) = 0$. Hence, $\lim_{r\to\infty} r^{1/\beta} z^{1/m}(r) = k(b) \geq 0$ exists.

Assume that $k(b) = 0$, it follows from (2.21) that

$$z(r) \leq C r^{-m/\beta}, \quad |z'(r)| \leq C r^{-1-m/\beta}, \quad \forall r \gg 1, \tag{4.37}$$

with some positive constant C. Then integrating (4.35) from r to ∞, we have

$$\begin{aligned} &r^{1/\beta-1}(|z'|^{p-2}z')(r) + \beta r^{1/\beta} z^{1/m}(r) \\ &\quad = (1/\beta-n)\int_r^\infty s^{1/\beta-2}|z'(s)|^{p-1}ds - \int_r^\infty s^{1/\beta-1}|(z^{1/m})'(s)|^{q}ds. \end{aligned} \tag{4.38}$$

Applying Lemma 2.2 and (4.37) we have, for $r \gg 1$,

$$r^{1/\beta}|z'(r)|^{p-1} \le C_1 r^{1-\nu},$$
$$r^{1+1/\beta}|(z^{1/m})'(r)|^q \le Cr^{1/\beta+1-q}z^{m/q}(r) \le C_2 r^{1-\nu} \tag{4.39}$$

with some constants C_1 and C_2. Thus we see that

$$\lim_{r\to\infty} r^{1/\beta}|z'(r)|^{p-2}z'(r)$$
$$= \lim_{r\to\infty} r\int_r^\infty s^{1/\beta-2}|z'(s)|^{p-1}ds = \lim_{r\to\infty} r\int_r^\infty s^{1/\beta-1}|(z^{1/m})'(s)|^q ds = 0. \tag{4.40}$$

It follows from (4.38) that $\lim_{r\to\infty} r^{1+1/\beta}z^{1/m}(r) = 0$ and the following estimates

$$z(r) \le Cr^{-m(1+/\beta)}, \qquad |z'(r)| \le Cr^{-1-m(1+/\beta)} \tag{4.41}$$

hold for all $r \gg 1$ and some $C > 0$. Repeating this argument, we have $\lim_{r\to\infty} r^M z(r) = 0$ for every positive number M. This implies that there is a constant C such that $z(r) \le Cr^{-M}$, which contradicts to (2.21). This completes the proof. □

Lemma 4.7. *$\mathcal{B}$ is nonempty and closed. Moreover, if $b \in \mathcal{B}$, the corresponding solution $z(r;b)$ of (2.1) satisfies the following interface condition:*

$$\lim_{r\to R(b)} \frac{z'(r;b)}{z^{1/(m(p-1))}(r;b)} = -(\beta R(b))^{1/(p-1)}. \tag{4.42}$$

Proof. Applying Lemmas 4.4 and 4.5 and the definitions of the three sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we see that $\mathcal{B}$ is nonempty and closed. By the first equation of (2.1) we have

$$(r^{n-1}|z'|^{p-2}z' + \beta r^n z^{1/m})' = -(1-n\beta)r^{n-1}z^{1/m} + r^{n-1}|(z^{1/m})'|^q. \tag{4.43}$$

Integrating the above from r to $R(b)$ yields

$$r^{n-1}(|z'|^{p-2}z')(r) + \beta r^n z^{1/m}(r)$$
$$= (1-n\beta)\int_r^{R(b)} s^{n-1}z^{1/m}(s)ds - \int_r^{R(b)} s^{n-1}|(z^{1/m})'(s)|^q ds. \tag{4.44}$$

It is easy to calculate that $\lim_{r\to R(a)} z^{-1/m}(r)\int_r^{R(a)} s^{n-1}z^{1/m}(s)ds = 0$. Divided (4.44) by $z^{1/m}(r)$ and putting $r \to R(b)$, by L'Hospital's rule we have

$$\lim_{r\to R(b)} \frac{(|z'|^{p-2}z')(r)}{z^{1/m}(r)} + \beta R(b) = \lim_{r\to R(b)} |(z^{1/m})'(r)|^{q-1}. \tag{4.45}$$

In the following we prove that

$$\lim_{r\to R(b)} |(z^{1/m})'(r)|^{q-1} = 0. \tag{4.46}$$

If $m \leq 1$, from the definition of $\mathscr{B}$ the conclusion is obviously. Thus we only need to prove (4.46) for $m > 1$. Since $1-(m-1)(p-1) = p - m(p-1) > 0$, namely, $(m-1)(p-1) \in (0,1)$, it follows that

$$\lim_{r\to R(b)} z^{-(m-1)(p-1)/m}(r)\int_r^{R(b)} s^{n-1}z^{1/m}(s)ds = 0. \tag{4.47}$$

Divided (4.44) by $z^{(m-1)(p-1)/m}(r)$ and putting $r \to R(b)$, by L'Hospital's rule we have

$$\lim_{r\to R(b)} \left|\left(z^{(1-m)/m}z'\right)(r)\right|^{p-1} = \frac{m^{1-q}}{(m-1)(p-1)} \lim_{r\to R(b)} \left|z^{(1-m)/m+m_1}(r)z'(r)\right|^{q-1}, \tag{4.48}$$

where $m_1 = [1-(m-1)(p-1)]/[m(q-1)] > 0$. Denote by $l_0 = (1-m)/m$, $l_1 = l_0 + m_1$. If $l_1 \geq 0$, by (4.48), then (4.46) holds. If $l_1 < 0$, since $1/m + (p-1)l_1 > [1-(m-1)(p-1)]/m > 0$, divided (4.44) by $z^{-(p-1)l_1}$ and sending $r \to R(b)$ and using L'Hospital's rule we get

$$\lim_{r\to R(b)} \left|z^{l_1}(r)z'(r)\right|^{p-1} = -\frac{m^{-q}}{(p-1)l_1} \lim_{r\to R(b)} \left|z^{l_2}(r)z'(r)\right|^{q-1}, \tag{4.49}$$

where

$$l_2 = \frac{1-m}{m} + \frac{p-1}{q-1}l_1 + \frac{1}{m(q-1)} = l_1 + \frac{p-1}{q-1}m_1. \tag{4.50}$$

If $l_2 \geq 0$, then it follows from (4.48) and (4.49) that (4.46) holds. If $l_2 \leq 0$ then repeating the above method, we obtain a sequence

$$l_k = l_{k-1} + \left(\frac{p-1}{q-1}\right)^{k-1} m_1, \quad k = 1,2,\ldots, \tag{4.51}$$

such that

$$\lim_{r\to R(b)} \left|z^{l_k}(r)z'(r)\right|^{p-1} = -\frac{m^{-q}}{(p-1)l_k} \lim_{r\to R(b)} \left|z^{l_{k+1}}(r)z'(r)\right|^{q-1} \tag{4.52}$$

provided that $l_k < 0$. From $p > q > 1$ and $m_1 > 0$, we see $l_k \to \infty$ as $k \to \infty$. There is $k_0 > 0$ such that $l_{k_0} < 0$, $l_{k_0+1} \geq 0$. Then by (4.52) we see that $\lim_{r\to R(b)} |z^{l_{k_0}} z'| = 0$. By a recursion relation (4.52), (4.46) follows. From (4.45) and (4.46) the conclusion holds. This prove (4.42). □

Proof of Theorem 1.1. Applying Lemmas 3.1, 4.4–4.7, the theorem follows. □

Acknowledgment

This work was supported by PRC Grant NSFC 19831060 and the "333" project of Jiangsu province.

References

[1] H. Brézis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, Journal de Mathématiques Pures et Appliquées. Neuvième Série **62** (1983), no. 1, 73–97.

[2] H. Brézis, L. A. Peletier, and D. Terman, *A very singular solution of the heat equation with absorption*, Archive for Rational Mechanics and Analysis **95** (1986), no. 3, 185–209.

[3] X. Chen, Y. Qi, and M. Wang, *Long time behavior of solutions to p-Laplacian equation with absorption*, SIAM Journal on Mathematical Analysis **35** (2003), no. 1, 123–134.

[4] ______, *Self-similar singular solutions of a p-Laplacian evolution equation with absorption*, Journal of Differential Equations **190** (2003), no. 1, 1–15.

[5] M. Escobedo, O. Kavian, and H. Matano, *Large time behavior of solutions of a dissipative semilinear heat equation*, Communications in Partial Differential Equations **20** (1995), no. 7-8, 1427–1452.

[6] L. Herraiz, *Asymptotic behaviour of solutions of some semilinear parabolic problems*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **16** (1999), no. 1, 49–105.

[7] S. Kamin and J. L. Vázquez, *Singular solutions of some nonlinear parabolic equations*, Journal d'Analyse Mathématique **59** (1992), 51–74.

[8] M. Kwak, *A porous media equation with absorption. I. Long time behaviour*, Journal of Mathematical Analysis and Applications **223** (1998), no. 1, 96–110.

[9] G. Leoni, *A very singular solution for the porous media equation $u_t = \Delta(u^m) - u^p$ when $0 < m < 1$*, Journal of Differential Equations **132** (1996), no. 2, 353–376.

[10] ______, *On very singular self-similar solutions for the porous media equation with absorption*, Differential and Integral Equations. An International Journal for Theory & Applications **10** (1997), no. 6, 1123–1140.

[11] L. A. Peletier and J. Y. Wang, *A very singular solution of a quasilinear degenerate diffusion equation with absorption*, Transactions of the American Mathematical Society **307** (1988), no. 2, 813–826.

[12] L. A. Peletier and J. N. Zhao, *Large time behaviour of solutions of the porous media equation with absorption: the fast diffusion case*, Nonlinear Analysis **17** (1991), no. 10, 991–1009.

[13] Y. Qi and M. Wang, *Singular solutions of doubly singular parabolic equations with absorption*, Electronic Journal of Differential Equations **2000** (2000), no. 67, 22.

[14] ______, *The self-similar profiles of generalized KPZ equation*, Pacific Journal of Mathematics **201** (2001), no. 1, 223–240.

[15] P. Shi, *Self-similar singular solution of a p-Laplacian evolution equation with gradient absorption term*, Journal of Partial Differential Equations **17** (2004), no. 4, 369–383.

[16] J. N. Zhao, *The asymptotic behaviour of solutions of a quasilinear degenerate parabolic equation*, Journal of Differential Equations **102** (1993), no. 1, 33–52.

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