ON AN INTEGRAL OPERATOR ON THE UNIT BALL IN $\mathbb{C}^n$

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Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. In this paper, we investigate the integral operator $T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)(dt/t)$, $f \in H(B)$, $z \in B$, where $g \in H(B)$ and $\Re g(z) = \sum_{j=1}^n z_j(\partial g/\partial z_j)(z)$ is the radial derivative of g. The operator can be considered as an extension of the Cesàro operator on the unit disk. The boundedness of the operator on a-Bloch spaces is considered.

1. Introduction

Let U be the unit disc in the complex plane $\mathbb{C}$ and $H(U)$ the space of all analytic functions in U.

For each complex γ with $\operatorname{Re}\gamma > -1$ and k nonnegative integer, let A_k^γ be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} = \sum_{k=0}^{\infty} A_k^\gamma x^k, \tag{1.1}$$

so that $A_k^\gamma = (\gamma+1)\cdots(\gamma+k)/k!$.

For an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ on U, the generalized Cesàro operator is defined by

$$\mathscr{C}^\gamma(f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\gamma+1}}\sum_{k=0}^{n} A_{n-k}^\gamma a_k\right)z^n. \tag{1.2}$$

For $\gamma = 0$ we obtain the Cesàro operator on U. The boundedness of the operator on some spaces of analytic functions was considered by a number of authors, see, for example, [8, 10, 13], and the references therein.

The integral form of $\mathscr{C}^0 = \mathscr{C}$ is

$$\mathscr{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{1}{(1-\zeta)}d\zeta = \frac{1}{z}\int_0^z f(\zeta)\left(\ln\frac{1}{(1-\zeta)}\right)' d\zeta, \tag{1.3}$$

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or, taking simply as a path the segment joining 0 and z,

$$\mathscr{C}(f)(z) = \int_0^1 f(tz)\left(\ln\frac{1}{(1-\zeta)}\right)'\bigg|_{\zeta=tz} dt. \tag{1.4}$$

On most holomorphic function spaces the boundedness of the previous operator is equivalent to the boundedness of the operator

$$z\mathscr{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta}\, d\zeta. \tag{1.5}$$

Hence, Aleman and Siskakis [2] have introduced and investigated the following natural generalization of operator (1.5):

$$T_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta. \tag{1.6}$$

In [1, 2, 3] were investigated the boundedness and the compactness of the operator on Hardy and Bergman spaces. A natural question is to define a similar integral operator which acts on $H(B)$ (the space of all holomorphic functions in the unit ball B).

Let $z = (z_1,\dots,z_n)$ and $w = (w_1,\dots,w_n)$ be points in complex vector space $\mathbb{C}^n$ and

$$\langle z,w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n. \tag{1.7}$$

Let dV_N stand for the normalized Lebesgue measure on $\mathbb{C}^n$. For a holomorphic function f we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right). \tag{1.8}$$

Let $\Re f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z)$ stand for the radial derivative of $f \in H(B)$ (see [7]). It is easy to see that if $f \in H(B)$, $f(z) = \sum_\alpha a_\alpha z^\alpha$, where α is a multi-index, then

$$\Re f(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha. \tag{1.9}$$

Let $a > 0$. The a-Bloch space $\mathscr{B}^a = \mathscr{B}^a(B)$ is the space of all $f \in H(B)$ such that

$$b_a(f) = \sup_{z\in B}\left(1-|z|^2\right)^a \left|\Re f(z)\right| < \infty. \tag{1.10}$$

The little a-Bloch space $\mathscr{B}_0^a = \mathscr{B}_0^a(B)$ consists of all $f \in H(B)$ such that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)^a \left|\Re f(z)\right| = 0. \tag{1.11}$$

On $\mathscr{B}^a$ the norm is introduced by

$$\|f\|_{\mathscr{B}^a} = \left|f(0)\right| + b_a(f). \tag{1.12}$$

With this norm $\mathcal{B}^a$ is a Banach space and $\mathcal{B}_0^a$ is a closed subspace of $\mathcal{B}^a$. If $a = 1$, we denote $\mathcal{B}^a$ and $\mathcal{B}_0^a$ simply by $\mathcal{B}$ and $\mathcal{B}_0$.

The aim of this paper is to investigate the boundedness of the following operator:

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad f \in H(B),\ z \in B, \tag{1.13}$$

where $g \in H(B)$, on the a-Bloch spaces. This operator can be considered as a natural extension of operator (1.6) on $H(B)$ (when $n = 1$ we indeed obtain (1.6)). Operator (1.13) has appeared, for the first time, in [6] where its boundedness and compactness are investigated.

Closely related operators to the above mentioned on the unit polydisc were investigated in [4, 5, 9, 11, 12].

In this paper, we prove the following results.

Theorem 1.1. *Let $g \in H(B)$ and $a \in (0,1)$. Then the following statements are equivalent:*

(a) *T_g is bounded on $\mathcal{B}^a$;*
(b) $\sup_{z\in B} |\Re g(z)|(1-|z|^2)^a < \infty$.

Moreover $\|T_g\| \asymp \sup_{z\in B} |\Re g(z)|(1-|z|^2)^a$.

Theorem 1.2. *Let $g \in H(B)$. Then the following statements are equivalent:*

(a) *T_g is bounded on $\mathcal{B}$;*
(b) *T_g is bounded on $\mathcal{B}_0$;*
(c) $\sup_{z\in B} |\Re g(z)|(1-|z|^2)\ln 1/(1-|z|^2) < \infty$;

and the relationship $\|T_g\| \asymp \sup_{z\in B} |\Re g(z)|(1-|z|^2)\ln 1/(1-|z|^2)$ *holds.*

Theorem 1.3. *Let $g \in H(B)$ and $a > 1$. Then the following statements are equivalent:*

(a) *T_g is bounded on $\mathcal{B}^a$;*
(b) $\sup_{z\in B} |\Re g(z)|(1-|z|^2) < \infty$.

Moreover $\|T_g\| \asymp \sup_{z\in B} |\Re g(z)|(1-|z|^2)$.

2. Auxiliary results

In order to prove our results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 2.1. *For every $f, g \in H(B)$, it holds that*

$$\Re[T_g(f)](z) = f(z)\Re g(z). \tag{2.1}$$

Proof. Assume that the holomorphic function $f\Re g$ has the expansion $\sum_\alpha a_\alpha z^\alpha$. Then

$$\Re[T_g(f)](z) = \Re\int_0^1 \sum_\alpha a_\alpha (tz)^\alpha \frac{dt}{t} = \Re\left(\sum_\alpha \frac{a_\alpha}{|\alpha|} z^\alpha\right) = \sum_\alpha a_\alpha z^\alpha, \tag{2.2}$$

which is what we wanted to prove. □

Lemma 2.2. *Let $f \in \mathcal{B}^a(B)$, $0 < a < \infty$. Then*

$$|f(z)| \le C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^a}, & a \in (0,1), \\ |f(0)| + \|f\|_{\mathcal{B}^a} \ln \dfrac{e}{1-|z|^2}, & a = 1, \\ |f(0)| + \dfrac{\|f\|_{\mathcal{B}^a}}{(1-|z|^2)^{a-1}}, & a > 1, \end{cases} \tag{2.3}$$

for some $C > 0$ independent of f.

Proof. Let $|z| > 1/2$, $z = r\zeta$, and $\zeta \in \partial B$. We have

$$\begin{aligned} \left| f(z) - f\left(\frac{r\zeta}{2}\right) \right| = \left| \int_{1/2}^{1} \langle \nabla f(tz), z \rangle dt \right| &\le \int_{1/2}^{1} \left| \frac{\Re f(tz)}{t} \right| dt \\ &\le 4\|f\|_{\mathcal{B}^a} \int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^a}. \end{aligned} \tag{2.4}$$

Let $I_a = \int_0^1 (|z|dt/(1-t^2|z|^2)^a)$. If $a \in (0,1)$, then

$$I_a \le \int_0^1 \frac{|z|dt}{(1-t|z|)^a} = \frac{1-(1-|z|)^{1-a}}{1-a} \le \frac{1}{1-a}. \tag{2.5}$$

If $a = 1$, then

$$\int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^a} = \frac{1}{2}\ln\frac{1+|z|}{1-|z|} \le \frac{1}{2}\ln\frac{4}{1-|z|^2}. \tag{2.6}$$

Finally, if $a > 1$, then

$$I_a \le \int_0^1 \frac{|z|dt}{(1-t|z|)^a} = \frac{1}{a-1}\left(\frac{1}{(1-|z|)^{a-1}} - 1\right) \le \frac{2^{a-1}}{(a-1)(1-|z|^2)^{a-1}}. \tag{2.7}$$

From all of the above we have

$$|f(z)| \le \begin{cases} M\left(\dfrac{1}{2}\right) + \dfrac{4\|f\|_{\mathcal{B}^a}}{1-a}, & a \in (0,1), \\ M\left(\dfrac{1}{2}\right) + 2\|f\|_{\mathcal{B}^a} \ln \dfrac{4}{1-|z|^2}, & a = 1, \\ M\left(\dfrac{1}{2}\right) + \dfrac{2^{a+1}\|f\|_{\mathcal{B}^a}}{(a-1)(1-|z|^2)^{a-1}}, & a > 1, \end{cases} \tag{2.8}$$

where $M(1/2) = \max_{|z|\le 1/2} |f(z)|$.

Let $|z| \le 1/2$, then, by the mean value property of the function $f(z) - f(0)$ (see [7]), and Jensen's inequality, we obtain

$$\begin{aligned}\max_{|z|\le 1/2} |f(z)-f(0)|^2 &\le 4^n \int_{|z|\le 3/4} |f(w)-f(0)|^2 dV_N(w)\\ &\le 4^n \int_{|z|\le 3/4} |\Re f(w)|^2 dV_N(w)\\ &\le 3^n \max_{|z|\le 3/4} |\Re f(z)|^2.\end{aligned} \tag{2.9}$$

The second inequality can be easily proved by using the homogeneous expansion of f. Hence,

$$M\left(\frac{1}{2}\right) \le |f(0)| + (\sqrt{3})^n \max_{|z|\le 3/4} |\Re f(z)| \le |f(0)| + \frac{2^{4a}(\sqrt{3})^n}{7^a} \|f\|_{\mathcal{B}^a}. \tag{2.10}$$

From (2.8) and (2.10), the result follows easily when $a \ne 1$. If $a = 1$, then we have

$$\begin{aligned}|f(z)| &\le |f(0)| + \frac{16(\sqrt{3})^n}{7} \|f\|_{\mathcal{B}} + 2\|f\|_{\mathcal{B}} \ln \frac{4}{1-|z|^2}\\ &\le \left(\frac{16(\sqrt{3})^n}{7} + \ln 16\right)\left(|f(0)| + \|f\|_{\mathcal{B}} \ln \frac{e}{1-|z|^2}\right),\end{aligned} \tag{2.11}$$

thus finishing the proof. □

3. Proofs of the main results

Proof of Theorem 1.1. Assume that T_g is bounded on $\mathcal{B}^a$. Choose $f_0(z) \equiv 1$. It is clear that $f_0 \in \mathcal{B}^a_0$ and that $\|f_0\|_{\mathcal{B}^a} = 1$. The boundedness of T_g implies

$$(1-|z|^2)^a |\Re[T_g(f_0)](z)| = (1-|z|^2)^a |\Re g(z)| \le ||T_g|| \, ||f_0||_{\mathcal{B}^a} = ||T_g|| < \infty. \tag{3.1}$$

Hence $g \in \mathcal{B}^a$, as desired.

Assume now that $g \in \mathcal{B}^a$. Then, by Lemma 2.2 we have

$$\begin{aligned}(1-|z|^2)^a |\Re[T_g(f)](z)| &= (1-|z|^2)^a |f(z)| \, |\Re g(z)|\\ &\le \|g\|_{\mathcal{B}^a} C(|f(0)| + \|f\|_{\mathcal{B}^a})\\ &\le 2C\|g\|_{\mathcal{B}^a}\|f\|_{\mathcal{B}^a}.\end{aligned} \tag{3.2}$$

Taking supremum $z \in B$ in (3.2), we obtain

$$||T_g(f)||_{\mathcal{B}^a} \le 2C\|g\|_{\mathcal{B}^a}\|f\|_{\mathcal{B}^a}. \tag{3.3}$$

Hence

$$||T_g|| \le 2C\|g\|_{\mathcal{B}^a}, \tag{3.4}$$

as desired. □

Proof of Theorem 1.2. First, assume that T_g is bounded on $\mathcal{B}$. From the proof which follows we will see that we also consider the case when T_g is bounded on $\mathcal{B}_0$. For $w \in B$, put $f_w(z) = \ln 1/(1 - \langle z,w\rangle)$. Since

$$(1-|z|^2)\,|\Re f_w(z)| \le (1-|z|^2)\,|\nabla f_w(z)| = (1-|z|^2)\left|\frac{w}{1-\langle z,w\rangle}\right| \le \frac{(1-|z|^2)}{|1-\langle z,w\rangle|} \le 2, \tag{3.5}$$

we have $\|f_w\|_{\mathcal{B}} \le 2$, for each $w \in B$. On the other hand, we have

$$(1-|z|^2)\,|\Re f_w(z)| \le \frac{(1-|z|^2)}{|1-\langle z,w\rangle|} \le \frac{(1-|z|^2)}{1-|w|} \longrightarrow 0, \tag{3.6}$$

as $|z| \to 1$. Hence $f_w \in \mathcal{B}_0$, for each $w \in B$.

By Lemma 2.1 we have

$$\begin{aligned}(1-|w|^2)\,|\Re g(w)|\ln\frac{1}{1-|w|^2} &= |f_w(w)\Re g(w)|\,(1-|w|^2)\\ &= |\Re(T_g f_w)(w)|\,(1-|w|^2)\\ &\le \|T_g f_w\|_{\mathcal{B}} \le 2\|T_g\|.\end{aligned} \tag{3.7}$$

Taking supremum in (3.7) over $w \in B$, we obtain that conditions (a) and (b) imply (c).

Assume that (c) holds. Since $|f(0)| \le \|f\|_{\mathcal{B}}$, and by Lemma 2.2, we have

$$|f(z)| \le C\|f\|_{\mathcal{B}}\left(1+\ln\frac{1}{1-|z|^2}\right), \tag{3.8}$$

for some $C > 0$.

Hence

$$\begin{aligned}|\Re[T_g(f)](z)|\,(1-|z|^2) &= |f(z)|\,|\Re g(z)|\,(1-|z|^2)\\ &\le C\|f\|_{\mathcal{B}}\left(1+\ln\frac{1}{1-|z|^2}\right)|\Re g(z)|\,(1-|z|^2)\\ &\le C\|f\|_{\mathcal{B}}\sup_{|z|\le 1/2}\left(1+\ln\frac{1}{1-|z|^2}\right)|\Re g(z)|\,(1-|z|^2)\\ &\quad + C\|f\|_{\mathcal{B}}\sup_{1/2<|z|<1}\left(1+\ln\frac{1}{1-|z|^2}\right)|\Re g(z)|\,(1-|z|^2)\\ &\le C_1\|f\|_{\mathcal{B}}\sup_{z\in B}\ln\frac{1}{1-|z|^2}\,|\Re g(z)|\,(1-|z|^2),\end{aligned} \tag{3.9}$$

since (c) implies

$$\sup_{z\in B}|\Re g(z)|\,(1-|z|^2) < \infty. \tag{3.10}$$

From (3.9) and since $T_g(f)(0) = 0$, (a) follows.

We now prove that (c) implies (b). Since $\ln 1/(1-|z|) \to \infty$ as $|z| \to 1$, we have that $g \in \mathcal{B}_0$. Hence, by Lemma 2.1, we have that for each polynomial $p(z)$,

$$(1-|z|^2)\,|\Re[T_g(p)](z)| = (1-|z|^2)\,|p(z)|\,|\Re g(z)| \le M_p(1-|z|^2)\,|\Re g(z)|, \quad (3.11)$$

where $M_p = \sup_{z\in B}|p(z)|$. Since $M_p < \infty$ and $g \in \mathcal{B}_0$, we obtain that for each polynomial $p, T_g(p) \in \mathcal{B}_0$. The set of polynomials is dense in $\mathcal{B}_0$, thus for every $f \in \mathcal{B}_0$ there is a sequence of polynomials (p_n) such that $\|p_n - f\|_{\mathcal{B}} \to 0$. Hence

$$\|T_g p_n - T_g f\|_{\mathcal{B}} \le \|T_g\|\,\|p_n - f\|_{\mathcal{B}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad (3.12)$$

since the operator T_g is bounded. Hence $T_g(\mathcal{B}_0) \subset \mathcal{B}_0$, since $\mathcal{B}_0$ is closed subset of $\mathcal{B}$.

Finally, from (3.7) and (3.9) it follows that

$$\|T_g\| \asymp \sup_{z\in B}|\Re g(z)|\,(1-|z|^2)\ln\frac{1}{1-|z|^2}. \quad (3.13)$$

□

Proof of Theorem 1.3. Let T_g be bounded on $\mathcal{B}^a$. Let $w \in B$, and $f_w(z) = 1/(1-\langle z,w\rangle)^{a-1}$. It is clear that $f_w \in \mathcal{B}^a$ and that $\|f_w\|_{\mathcal{B}^a} \le (a-1)2^a$. The boundedness of T_g implies

$$\begin{aligned}(1-|w|^2)^a\,|\Re[T_g(f_w)](w)| &= (1-|w|^2)^a\,|\Re g(w)|\,|f_w(w)|\\ &= (1-|w|^2)\,|\Re g(w)|\\ &\le \|T_g\|\,\|f_w\|_{\mathcal{B}^a}\\ &= (a-1)2^a\|T_g\| < \infty.\end{aligned} \quad (3.14)$$

Hence $\sup_{w\in B}(1-|w|^2)|\Re g(w)| < \infty$, as desired.

Assume now that $g \in \mathcal{B}$. Then, by Lemma 2.2 we have

$$\begin{aligned}(1-|z|^2)^a\,|\Re[T_g(f)](z)| &= (1-|z|^2)^a\,|f(z)|\,|\Re g(z)|\\ &\le b_1(g)(1-|z|^2)^{a-1}C\left(|f(0)| + \frac{\|f\|_{\mathcal{B}^a}}{(1-|z|^2)^{a-1}}\right)\\ &\le 2Cb_1(g)\|f\|_{\mathcal{B}^a}.\end{aligned} \quad (3.15)$$

Hence

$$\|T_g(f)\|_{\mathcal{B}^a} \le 2Cb_1(g)\|f\|_{\mathcal{B}^a}, \quad (3.16)$$

and consequently $\|T_g\| \le 2Cb_1(g)$, as desired.

Form (3.14) and (3.16) it follows that $\|T_g\| \asymp \sup_{z\in B}|\Re g(z)|(1-|z|^2)$. □

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