

# THE EXTENSION OF MONTGOMERY IDENTITY VIA FINK IDENTITY WITH APPLICATIONS

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The new extension of the weighted Montgomery identity is given by using Fink identity and is used to obtain some Ostrowski-type inequalities and estimations of the difference of two integral means.

## 1. Introduction

The following Ostrowski inequality is well known [10]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right] (b-a)L, \quad x \in [a, b], \quad (1.1)$$

where  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable function such that  $|f'(x)| \leq L$ , for every  $x \in [a, b]$ .

The Ostrowski inequality has been generalized over the last years in a number of ways. Milovanović and Pečarić [8] and Fink [6] have considered generalizations of (1.1) in the form

$$\left| \frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq K(n, p, x) \|f^{(n)}\|_p \quad (1.2)$$

which is obtained from the identity

$$\frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} k(t, x) f^{(n)}(t) dt, \quad (1.3)$$

where

$$F_k(x) = \frac{n-k}{k!} \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t, x) = \begin{cases} t-a, & a \leq t \leq x \leq b, \\ t-b, & a \leq x < t \leq b. \end{cases} \quad (1.4)$$

In fact, Milovanović and Pečarić have proved that

$$K(n, \infty, x) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)}, \quad (1.5)$$

while Fink gave the following generalizations of this result.

**THEOREM 1.1.** *Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and let  $f^{(n)} \in L_p[a, b]$ . Then inequality (1.2) holds with*

$$K(n, p, x) = \frac{[(x-a)^{nq+1} + (b-x)^{nq+1}]^{1/q}}{n!(b-a)} B((n-1)q+1, q+1)^{1/q}, \quad (1.6)$$

where  $1 < p \leq \infty$ ,  $1/p + 1/q = 1$ ,  $B$  is the Beta function, and

$$K(n, 1, x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)} \max[(x-a)^n, (b-x)^n]. \quad (1.7)$$

Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable on  $[a, b]$  and  $f' : [a, b] \rightarrow \mathbb{R}$  integrable on  $[a, b]$ . Then the Montgomery identity holds [9]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \int_a^b P(x, t) f'(t) dt, \quad (1.8)$$

where  $P(x, t)$  is the Peano kernel defined by

$$P(x, t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x, \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases} \quad (1.9)$$

Now, we suppose  $w : [a, b] \rightarrow [0, \infty)$  is some probability density function, that is, an integrable function satisfying  $\int_a^b w(t) dt = 1$ , and  $W(t) = \int_a^t w(x) dx$  for  $t \in [a, b]$ ,  $W(t) = 0$  for  $t < a$ , and  $W(t) = 1$  for  $t > b$ . The following identity (given by Pečarić in [12]) is the weighted generalization of the Montgomery identity:

$$f(x) = \int_a^b w(t) f(t) dt + \int_a^b P_w(x, t) f'(t) dt, \quad (1.10)$$

where the weighted Peano kernel is

$$P_w(x, t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases} \quad (1.11)$$

The aim of this paper is to give the extension of the weighted Montgomery identity (1.10) using identity (1.2), and further, obtain some new Ostrowski-type inequalities, as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [1, 3, 7, 11]).

**2. The extension of Montgomery identity via Fink identity**

**THEOREM 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is an absolutely continuous function on  $[a, b]$  for some  $n \geq 1$ . If  $w : [a, b] \rightarrow [0, \infty)$  is some probability density function, then the following identity holds:*

$$\begin{aligned}
 f(x) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} F_k(x) + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
 &+ \frac{1}{(n-1)!(b-a)} \int_a^b (x-y)^{n-1}k(y,x)f^{(n)}(y)dy \\
 &- \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
 \end{aligned}
 \tag{2.1}$$

*Proof.* We apply identity (1.3) with  $f'(t)$ :

$$\begin{aligned}
 f'(t) &= - \sum_{k=1}^{n-1} \frac{n-k}{k!} \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} + n \frac{f(b) - f(a)}{b-a} \\
 &+ \frac{1}{(n-1)!(b-a)} \int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy \\
 &= - \sum_{k=0}^{n-1} \frac{n-k}{k!} \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} \\
 &+ \frac{1}{(n-1)!(b-a)} \int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy.
 \end{aligned}
 \tag{2.2}$$

Now, by putting this formula in the weighted Montgomery identity (1.10), we obtain

$$\begin{aligned}
 f(x) &= \int_a^b w(t)f(t)dt \\
 &- \sum_{k=0}^{n-1} \frac{n-k}{k!} \int_a^b P_w(x,t) \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} dt \\
 &+ \frac{1}{(n-1)!(b-a)} \int_a^b P_w(x,t) \left( \int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)dy \right) dt.
 \end{aligned}
 \tag{2.3}$$

Further,

$$\begin{aligned}
 & \int_a^b P_w(x,t) \frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a} dt \\
 &= \frac{f^{(k)}(a)(x-a)^{k+1} - f^{(k)}(b)(x-b)^{k+1}}{(b-a)(k+1)} \\
 &\quad - \int_a^b w(t) \frac{f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}}{(b-a)(k+1)} dt, \tag{2.4} \\
 & \int_a^b P_w(x,t)(t-y)^{n-1}k(y,t)dt \\
 &= \frac{1}{n}(x-y)^nk(y,x) - \frac{1}{n} \int_a^b w(t)(t-y)^nk(y,t)dt.
 \end{aligned}$$

Now, if we replace  $n$  with  $n - 1$ , we will get (2.1). This identity is valid for  $n - 1 \geq 1$ , that is,  $n > 1$ . □

*Remark 2.2.* We could also obtain identity (2.1) by applying identity (1.3) such that we multiply this identity by  $w(x)$  and than integrate it to obtain

$$\begin{aligned}
 \int_a^b w(x)f(x)dx &= - \sum_{k=1}^{n-1} \int_a^b w(x)F_k(x)dx + \left( \int_a^b w(x)dx \right) \frac{n}{b-a} \int_a^b f(t)dt \\
 &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(x)(x-t)^{n-1}k(t,x)dx \right) f^{(n)}(t)dt. \tag{2.5}
 \end{aligned}$$

If we subtract this identity from (1.3) we will obtain (2.1).

*Remark 2.3.* In the special case, if we take  $w(t) = 1/(b - a)$ ,  $t \in [a, b]$ , we will have

$$\begin{aligned}
 \frac{1}{b-a} \sum_{k=1}^{n-1} \int_a^b F_k(t)dt &= \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{n-k}{k!} \int_a^b \frac{f^{(k-1)}(a)(t-a)^k - f^{(k-1)}(b)(t-b)^k}{b-a} dt \\
 &= \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}], \\
 \frac{1}{b-a} \int_a^b (t-y)^{n-1}k(y,t)dt &= k(y,b) \frac{(b-y)^n}{n(b-a)} - k(y,a) \frac{(a-y)^n}{n(b-a)} \\
 &= \frac{(y-a)(b-y)^n}{n(b-a)} - \frac{(y-b)(a-y)^n}{n(b-a)}. \tag{2.6}
 \end{aligned}$$

We denote

$$I_n = \frac{1}{n!(b-a)^2} \int_a^b [(y-a)(b-y)^n - (y-b)(a-y)^n] f^{(n)}(y) dy. \quad (2.7)$$

Then we have

$$I_n = \frac{1}{n!(b-a)^2} \int_a^b [(a-y)^n - (b-y)^n] f^{(n-1)}(y) dy + I_{n-1} = J_n + I_{n-1}, \quad (2.8)$$

where

$$I_0 = \frac{1}{(b-a)^2} \int_a^b (b-a) f(y) dy = \frac{1}{b-a} \int_a^b f(y) dy. \quad (2.9)$$

Further,

$$\begin{aligned} J_n &= \frac{1}{n!} [f^{(n-2)}(a)(b-a)^{n-2} + f^{(n-2)}(b)(a-b)^{n-2}] + J_{n-1}, \\ J_1 &= \frac{1}{(b-a)^2} \int_a^b (a-b) f(y) dy = -\frac{1}{b-a} \int_a^b f(y) dy. \end{aligned} \quad (2.10)$$

So,

$$J_n = \sum_{k=1}^{n-1} \frac{1}{(k+1)!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] + J_1, \quad (2.11)$$

and then

$$\begin{aligned} I_n &= \sum_{m=2}^n J_m + nJ_1 + I_0 \\ &= \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\ &\quad - \frac{n-1}{b-a} \int_a^b f(y) dy. \end{aligned} \quad (2.12)$$

Consequently, identity (2.1) reduces to identity (1.3). So we may regard it as a weighted Fink identity.

*Remark 2.4.* Applying identity (2.1) with  $x = a$  and  $x = b$ , we get

$$\begin{aligned}
 f(a) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} f^{(k-1)}(b)(a-b)^{k-1} + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
 &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (a-y)^{n-1}(y-b)f^{(n)}(y)dy \\
 &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy, \\
 f(b) &= \int_a^b w(t)f(t)dt - \sum_{k=1}^{n-1} \frac{n-k}{k!} f^{(k-1)}(a)(b-a)^{k-1} + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
 &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b (b-y)^{n-1}(y-a)f^{(n)}(y)dy \\
 &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
 \end{aligned} \tag{2.13}$$

So, we get the generalized trapezoid identity

$$\begin{aligned}
 \frac{1}{2}[f(a) + f(b)] &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
 &\quad - \frac{1}{2} \sum_{k=1}^{n-1} \frac{n-k}{k!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\
 &\quad + \frac{1}{2(n-1)!(b-a)} \int_a^b [(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a)]f^{(n)}(y)dy \\
 &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
 \end{aligned} \tag{2.14}$$

Similarly, applying identity (2.1) with  $x = (a+b)/2$ , we get

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &= \int_a^b w(t)f(t)dt + \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt \\
 &\quad - \sum_{k=1}^{n-1} \frac{n-k}{2^k k!} [f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}] \\
 &\quad + \frac{1}{(n-1)!(b-a)} \int_a^b \left(\frac{a+b}{2} - y\right)^{n-1} k\left(y, \frac{a+b}{2}\right) f^{(n)}(y)dy \\
 &\quad - \frac{1}{(n-1)!(b-a)} \int_a^b \left( \int_a^b w(t)(t-y)^{n-1}k(y,t)dt \right) f^{(n)}(y)dy.
 \end{aligned} \tag{2.15}$$

We can regard this as the second Euler-Maclaurin formula (the generalized midpoint identity).

### 3. Ostrowski-type inequalities

We denote, for  $n \geq 2$ ,

$$T_{w,n}(x) = \sum_{k=1}^{n-1} F_k(x) - \sum_{k=1}^{n-1} \int_a^b w(t)F_k(t)dt. \tag{3.1}$$

**THEOREM 3.1.** *Assume  $(p, q)$  is a pair of conjugate exponents, that is,  $1 \leq p, q \leq \infty, 1/p + 1/q = 1$ . Let  $|f^{(n)}|^p : [a, b] \rightarrow \mathbb{R}$  be an  $R$ -integrable function for some  $n > 1$ . Then, for  $x \in [a, b]$ , the following inequality holds:*

$$\begin{aligned} & \left| f(x) - \int_a^b w(t)f(t)dt + T_{w,n}(x) \right| \\ & \leq \frac{1}{(n-2)!(b-a)} \left( \int_a^b \left| \int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt \right|^q dy \right)^{1/q} \|f^{(n)}\|_p. \end{aligned} \tag{3.2}$$

The constant  $(1/(n-2)!(b-a))(\int_a^b \int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt|{}^q dy)^{1/q}$  is sharp for  $1 < p \leq \infty$  and is the best possible for  $p = 1$ .

*Proof.* From Theorem 2.1 we have

$$(x-y)^{n-1}k(y,x) - \int_a^b w(t)(t-y)^{n-1}k(y,t)dt = (n-1) \int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt. \tag{3.3}$$

We denote  $C_1(y) = (1/(n-2)!(b-a)) \int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)dt$ . We use identity (2.1) and apply the Hölder inequality to obtain

$$\begin{aligned} & \left| f(x) - \int_a^b w(t)f(t)dt + T_{w,n}(x) \right| \\ & = \left| \int_a^b C_1(y)f^{(n)}(y)dy \right| \leq \left( \int_a^b |C_1(y)|^q dy \right)^{1/q} \|f^{(n)}\|_p. \end{aligned} \tag{3.4}$$

For the proof of the sharpness of the constant  $(\int_a^b |C_1(y)|^q dy)^{1/q}$ , we will find a function  $f$  for which the equality in (3.2) is obtained.

For  $1 < p < \infty$ , take  $f$  to be such that

$$f^{(n)}(y) = \text{sgn } C_1(y) \cdot |C_1(y)|^{1/(p-1)}. \tag{3.5}$$

For  $p = \infty$ , take

$$f^{(n)}(y) = \text{sgn } C_1(y). \tag{3.6}$$

For  $p = 1$ , we will prove that

$$\left| \int_a^b C_1(y) f^{(n)}(y) dy \right| \leq \max_{y \in [a, b]} |C_1(y)| \left( \int_a^b |f^{(n)}(y)| dy \right) \quad (3.7)$$

is the best possible inequality. Suppose that  $|C_1(y)|$  attains its maximum at  $y_0 \in [a, b]$ . First we assume that  $C_1(y_0) > 0$ . For  $\varepsilon$  small enough, define  $f_\varepsilon(y)$  by

$$f_\varepsilon(y) = \begin{cases} 0, & a \leq y \leq y_0, \\ \frac{1}{\varepsilon n!} (y - y_0)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\ \frac{1}{n!} (y - y_0)^{n-1}, & y_0 + \varepsilon \leq y \leq b. \end{cases} \quad (3.8)$$

Then, for  $\varepsilon$  small enough,

$$\left| \int_a^b C_1(y) f^{(n)}(y) dy \right| = \left| \int_{y_0}^{y_0 + \varepsilon} C_1(y) \frac{1}{\varepsilon} dy \right| = \frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) dy. \quad (3.9)$$

Now, from inequality (3.7) we have

$$\frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) dy \leq C_1(y_0) \int_{y_0}^{y_0 + \varepsilon} \frac{1}{\varepsilon} dy = C_1(y_0). \quad (3.10)$$

Since

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{y_0}^{y_0 + \varepsilon} C_1(y) dy = C_1(y_0), \quad (3.11)$$

the statement follows. In case  $C_1(y_0) < 0$ , we take

$$f_\varepsilon(y) = \begin{cases} \frac{1}{n!} (y - y_0 - \varepsilon)^{n-1}, & a \leq y \leq y_0, \\ -\frac{1}{\varepsilon n!} (y - y_0 - \varepsilon)^n, & y_0 \leq y \leq y_0 + \varepsilon, \\ 0, & y_0 + \varepsilon \leq y \leq b, \end{cases} \quad (3.12)$$

and the rest of the proof is the same as above.  $\square$



*Remark 3.2.* For  $w(t) = 1/(b - a)$ ,  $n = 2$ , and  $q = 1$  in Theorem 3.1, we get

$$\begin{aligned}
 & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)(f(b) - f(a)) \right| \\
 & \leq \frac{1}{b-a} \left( \int_a^b \left| (x-y)k(y,x) - \frac{1}{b-a} \int_a^b (t-y)k(y,t)dt \right| dy \right) \|f''\|_\infty \\
 & = \frac{1}{b-a} \left( \int_a^b \left| (x-y)k(y,x) - \frac{(y-a)(b-y)}{2} \right| dy \right) \|f''\|_\infty \\
 & = \frac{1}{2(b-a)} \left( \int_a^x |(y-a)(2x-y-b)| dy + \int_x^b |(b-y)(-2x+y+a)| dy \right) \|f''\|_\infty \\
 & = \left( \frac{4}{3} \delta^3(x) - \frac{1}{2} \delta^2(x) + \frac{1}{24} \right) \|f''\|_\infty,
 \end{aligned}
 \tag{3.13}$$

where  $\delta(x) = |x - (a+b)/2|$ .

If instead of  $q = 1$  ( $p = \infty$ ) we put  $p = 1$ , then, similarly we have

$$\begin{aligned}
 & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)(f(b) - f(a)) \right| \\
 & \leq \frac{1}{2(b-a)} \max \left\{ \max_{y \in [a,x]} |(y-a)(2x-y-b)|, \max_{y \in [x,b]} |(b-y)(-2x+y+a)| \right\} \|f''\|_1 \\
 & = \frac{1}{4} \left[ \frac{1}{4} + \left| \frac{1}{4} - 2 \left(x - \frac{a+b}{2}\right)^2 \right| \right] \|f''\|_1.
 \end{aligned}
 \tag{3.14}$$

These two inequalities are proved in [5].

**COROLLARY 3.3.** *Suppose that all the assumptions of Theorem 3.1 hold. Then the following inequality holds:*

$$\begin{aligned}
 & \left| f(x) - \int_a^b w(t)f(t)dt + T_{w,n}(x) \right| \\
 & \leq \frac{1}{(n-1)!(b-a)} \left( \int_a^b [(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1}]^q dy \right)^{1/q} \|f^{(n)}\|_p.
 \end{aligned}
 \tag{3.15}$$

*Proof.* Since  $0 \leq W(t) \leq 1$ ,  $t \in [a, b]$ , so  $|P_w(x, t)| \leq 1$ . Then, for every  $y \in [a, b]$ , we have

$$\begin{aligned}
 & \left| \int_a^b P_w(x, t)(t - y)^{n-2}k(y, t)dt \right| \\
 & \leq \int_a^b |P_w(x, t)| |(t - y)^{n-2}k(y, t)| dt \\
 & \leq \int_a^b |(t - y)^{n-2}k(y, t)| dt \\
 & = \left[ \int_a^y (y - t)^{n-2}(b - y)dt + \int_y^b (t - y)^{n-2}(y - a)dt \right] \\
 & = \frac{1}{n - 1} [(b - y)(y - a)^{n-1} + (y - a)(b - y)^{n-1}].
 \end{aligned} \tag{3.16}$$

So,

$$\begin{aligned}
 & \left( \int_a^b \left| \int_a^b P_w(x, t)(t - y)^{n-1}k(y, t)dt \right|^q dy \right)^{1/q} \\
 & \leq \frac{1}{n - 1} \left( \int_a^b [(b - y)(y - a)^{n-1} + (y - a)(b - y)^{n-1}]^q dt \right)^{1/q}
 \end{aligned} \tag{3.17}$$

and, by applying (3.2), the inequality is proved. □

*Remark 3.4.* Inequality (3.15) reduces to the following: for  $n = 2$ ,

$$\begin{aligned}
 & \left| f(x) - \int_a^b w(t)f(t)dt + T_{w,2}(x) \right| \\
 & \leq \frac{2}{b - a} \left( \int_a^b (b - y)^q(y - a)^q dy \right)^{1/q} \|f''\|_p \\
 & = 2(b - a)^{(q+1)/q} \left( \int_0^1 (1 - s)^q s^q ds \right)^{1/q} \|f''\|_p \\
 & = 2(b - a)^{(q+1)/q} B(q + 1, q + 1)^{1/q} \|f''\|_p.
 \end{aligned} \tag{3.18}$$

For  $n = 3$ ,

$$\begin{aligned}
 & \left| f(x) - \int_a^b w(t)f(t)dt + T_{w,3}(x) \right| \\
 & \leq \frac{1}{2(b - a)} \left( \int_a^b (b - y)^q(y - a)^q(b - a)^q dy \right)^{1/q} \|f'''\|_p \\
 & = \frac{1}{2} (b - a)^{(2q+1)/q} B(q + 1, q + 1)^{1/q} \|f'''\|_p.
 \end{aligned} \tag{3.19}$$

*Remark 3.5.* If we use the identities (2.14) and (2.15) for  $n = 2$  and  $w(t) = 1/(b - a)$ ,  $t \in [a, b]$ , and then apply the Hölder inequality with  $p = \infty, q = 1$ , we will obtain

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \right| &\leq \frac{(b - a)^2}{12} \|f''\|_\infty, \\ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(t) dt \right| &\leq \frac{(b - a)^2}{24} \|f''\|_\infty. \end{aligned} \tag{3.20}$$

By doing the same for  $n = 3$ , we have

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt - \frac{b - a}{12} [f'(b) - f'(a)] \right| &\leq \frac{(b - a)^3}{192} \|f'''\|_\infty, \\ \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(t) dt + \frac{b - a}{24} [f'(b) - f'(a)] \right| &\leq \frac{(b - a)^3}{192} \|f'''\|_\infty. \end{aligned} \tag{3.21}$$

The first two inequalities were obtained in [4] and the last two in [2].

#### 4. Estimations of the difference of two weighted integral means

In this section, we will denote, for  $n > 1$ ,

$$T_{w,n}^{[a,b]}(x) = \sum_{k=1}^{n-1} F_k^{[a,b]}(x) - \sum_{k=1}^{n-1} \int_a^b w(t) F_k^{[a,b]}(t) dt, \tag{4.1}$$

for a function  $f : [a, b] \rightarrow \mathbb{R}$  such that  $f^{(n-1)}$  is an absolutely continuous function on  $[a, b]$ .

The following results are generalizations of the results from [3] in two cases. The first case is when  $[c, d] \subseteq [a, b]$  and the second is when  $[a, b] \cap [c, d] = [c, b]$ . Other two possible cases, when  $[a, b] \cap [c, d] \neq \emptyset$  ( $[a, b] \subset [c, d]$  and  $[a, b] \cap [c, d] = [a, d]$ ) are simply got by change  $a \leftrightarrow c, b \leftrightarrow d$ .

**THEOREM 4.1.** *Let  $f : [a, b] \cup [c, d] \rightarrow \mathbb{R}$  be such that  $f^{(n-1)}$  is an absolutely continuous function on  $[a, b]$  for some  $n > 1$ , and let  $w : [a, b] \rightarrow [0, \infty)$  and  $u : [c, d] \rightarrow [0, \infty)$  be some probability density functions. Then, if  $[a, b] \cap [c, d] \neq \emptyset$  and  $x \in [a, b] \cap [c, d]$ ,*

$$\int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x, y) f^{(n)}(y) dy, \tag{4.2}$$

where, in case  $[c, d] \subseteq [a, b]$ ,

$$K_n(x, y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x, t)(t-y)^{n-2} k^{[a,b]}(y, t) dt \right], & y \in [a, c], \\ \frac{-1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x, t)(t-y)^{n-2} k^{[a,b]}(y, t) dt \right] \\ + \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(x, t)(t-y)^{n-2} k^{[c,d]}(y, t) dt \right], & y \in \langle c, d \rangle, \\ \frac{-1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x, t)(t-y)^{n-2} k^{[a,b]}(y, t) dt \right], & y \in \langle d, b \rangle, \end{cases} \quad (4.3)$$

and in case  $[a, b] \cap [c, d] = [c, b]$ ,

$$K_n(x, y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[ \int_a^b P_w(x, t)(t-y)^{n-2} k^{[a,b]}(y, t) dt \right], & y \in [a, c], \\ \frac{-1}{(n-2)!(b-a)} \left[ \int_a^b P_w(t)(t-y)^{n-2} k^{[a,b]}(y, t) dt \right] \\ + \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y, t) dt \right], & y \in \langle c, b \rangle, \\ \frac{1}{(n-2)!(d-c)} \left[ \int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y, t) dt \right], & y \in \langle b, d \rangle. \end{cases} \quad (4.4)$$

*Proof.* We subtract identity (2.1) for intervals  $[a, b]$  and  $[c, d]$  to get formula (4.2).  $\square$

**THEOREM 4.2.** Assume  $(p, q)$  is a pair of conjugate exponents, that is,  $1 \leq p, q \leq \infty, 1/p + 1/q = 1$ . Let  $|f^{(n)}|^p : [a, b] \rightarrow \mathbb{R}$  be an  $R$ -integrable function for some  $n > 1$ . Then

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ & \leq \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x, y)|^q dy \right)^{1/q} \|f^{(n)}\|_p \end{aligned} \quad (4.5)$$

for every  $x \in [a, b] \cap [c, d]$ . The constant  $(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x, y)|^q dy)^{1/q}$  in inequality (4.5) is sharp for  $1 < p \leq \infty$  and is the best possible for  $p = 1$ .

*Proof.* Use identity (4.2) and apply the Hölder inequality to obtain

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ & \leq \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x, y)| |f^{(n)}(y)| dy \leq \left( \int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x, y)|^q dy \right)^{1/q} \|f^{(n)}\|_p, \end{aligned} \quad (4.6)$$

which proves inequality (4.5). The proofs for sharpness and best possibility are as in Theorem 3.1.  $\square$

**COROLLARY 4.3.** *Suppose that all the assumptions of Theorem 4.2 hold. Then, for  $x \in [a, b] \cap [c, d]$ ,*

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ & \leq \frac{2}{(n-1)!} \left( \int_a^{\max\{b,d\}} \left| (y-a)^{n-1} + (\max\{b,d\} - y)^{n-1} \right|^q dy \right)^{1/q} \|f^{(n)}\|_p. \end{aligned} \tag{4.7}$$

*Proof.* We have

$$K_n(x, y) = \frac{-1}{(n-2)!} \int_{\min\{a,c\}}^{\max\{b,d\}} \left[ P_w(x, t) \frac{k^{[a,b]}(y, t)}{b-a} - P_u(x, t) \frac{k^{[c,d]}(y, t)}{d-c} \right] (t-y)^{n-2} dt \tag{4.8}$$

because  $P_w(x, t) = 0$ , for  $x \notin [a, b]$  and  $P_u(x, t) = 0$ , for  $x \notin [c, d]$ . Since

$$-1 \leq P_w(x, t), P_u(x, t), \frac{k^{[a,b]}(y, t)}{b-a}, \frac{k^{[c,d]}(y, t)}{d-c} \leq 1, \tag{4.9}$$

we get

$$\left| P_w(x, t) \frac{k^{[a,b]}(y, t)}{b-a} - P_u(x, t) \frac{k^{[c,d]}(y, t)}{d-c} \right| \leq 2, \tag{4.10}$$

and then we have

$$|K_n(x, y)| \leq \frac{2}{(n-2)!} \int_a^{\max\{b,d\}} |t-y|^{n-2} dt = \frac{2 \left( (y-a)^{n-1} + (\max\{b,d\} - y)^{n-1} \right)}{(n-1)!}. \tag{4.11}$$

$\square$

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