# ON MODULI OF CONVEXITY IN BANACH SPACES

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Let $X$ be a normed linear space, $x \in X$ an element of norm one, and $\varepsilon > 0$ and $\delta(x,\varepsilon)$ the local modulus of convexity of $X$. We denote by $\varrho(x,\varepsilon)$ the greatest $\varrho \geq 0$ such that for each closed linear subspace $M$ of $X$ the quotient mapping $Q : X \to X/M$ maps the open $\varepsilon$-neighbourhood of $x$ in $U$ onto a set containing the open $\varrho$-neighbourhood of $Q(x)$ in $Q(U)$. It is known that $\varrho(x,\varepsilon) \geq (2/3)\delta(x,\varepsilon)$. We prove that there is no universal constant $C$ such that $\varrho(x,\varepsilon) \leq C\delta(x,\varepsilon)$, however, such a constant $C$ exists within the class of Hilbert spaces $X$. If $X$ is a Hilbert space with $\dim X \geq 2$, then $\varrho(x,\varepsilon) = \varepsilon^2/2$.

## 1. Introduction

Let $X$ be a real normed linear space of dimension $\dim X \geq 1$ and let $U$ be the closed unit ball of $X$.

Let $\varepsilon > 0$. The modulus of local convexity $\delta(x,\varepsilon)$, where $x \in U$, is defined by

$$\delta(x,\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : y \in U,\ \|x-y\| \geq \varepsilon\right\} \tag{1.1}$$

and the modulus of convexity is

$$\delta(\varepsilon) = \inf\{\delta(x,\varepsilon) : x \in U\}. \tag{1.2}$$

If $\dim X \geq 2$, one can use an equivalent definition (see, e.g., [1]),

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x,y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} \tag{1.3}$$

and if $\|x\| = 1$,

$$\delta(x,\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : y \in X,\ \|y\| = 1, \|x-y\| = \varepsilon\right\}. \tag{1.4}$$

The space $X$ is said to be uniformly convex (locally uniformly convex) if for each $\varepsilon > 0$, $\delta(\varepsilon) > 0$ ($\delta(x,\varepsilon) > 0$ for $x \in U$, resp.).


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The moduli $\delta(\varepsilon)$ of the spaces $L_p(\mu)$ have been found in [2]; they behave for $\varepsilon \to 0$ as $(p-1)\varepsilon^2/8 + o(\varepsilon^2)$ when $1 < p \le 2$, and as $p^{-1}(\varepsilon/2)^p + o(\varepsilon^p)$ when $2 < p < \infty$. In case of a Hilbert space $X$ with $\dim X \ge 2$, $\delta(\varepsilon) = 1 - (1 - \varepsilon^2/4)^{1/2}$ for $\varepsilon \in (0,2]$.

We denote by $\mathcal{T}$ the family of the canonical quotient maps $Q : X \to X/M$, where $M$ ranges over all closed linear subspaces of $X$. For any $\varepsilon > 0$ and $x \in U$, let $\varrho(x,\varepsilon) = \sup\{r : r \ge 0$ and for each $Q \in \mathcal{T}$, $Q$ maps the open $\varepsilon$-neighbourhood of $x$ in $U$ onto a set containing the open $r$-neighbourhood of $Q(x)$ in $Q(U)\}$, and let $\varrho(\varepsilon)$ be defined by

$$\varrho(\varepsilon) = \inf\{\varrho(x,\varepsilon) : x \in U\}. \tag{1.5}$$

We note that if $T$ is an open linear mapping from $X$ onto a normed linear space $Y$ such that $T^{-1}(0)$ is closed and $T(U)$ contains a $c$-neighbourhood of 0 in $Y$, then for each $x \in U$ and $\varepsilon > 0$, $T$ maps the $\varepsilon$-neighbourhood of $x$ in $U$ onto a set containing the $c\varrho(x,\varepsilon)$-neighbourhood of $T(x)$ in $T(U)$. Thus the "$\varrho$-moduli" help to estimate relative openness of $T$ on $U$ in a quantitative way. Relative openness of affine maps on convex sets has been treated in literature in various contexts, a list of references is presented in [3]. For each $\varepsilon > 0$, the following holds [3]:

$$\varrho(x,\varepsilon) \ge \frac{2}{3}\delta(x,\varepsilon) \quad \text{for each } x \text{ of norm one,} \tag{1.6}$$

$$\varrho(\varepsilon) \ge \frac{2}{3}\delta(\varepsilon), \tag{1.7}$$

$$\varrho(x,\varepsilon) \le \frac{4}{\lambda - 1}\delta(x,\lambda\varepsilon) \quad \text{for each } x \in U \text{ and } \lambda \in (1,3], \tag{1.8}$$

$$\varrho(\varepsilon) \le \frac{4}{\lambda - 1}\delta(\lambda\varepsilon) \quad \text{for each } \lambda \in (1,3]. \tag{1.9}$$

These relations suggest the following questions.

*Question 1.1.* Is there a constant $c_1$ such that

$$\varrho(x,\varepsilon) \le c_1\delta(x,\varepsilon) \tag{1.10}$$

for all $X$, $x \in X$ of norm one, and $\varepsilon \in (0,2]$?

*Question 1.2.* Is there a constant $c_2$ such that

$$\varrho(\varepsilon) \le c_2\delta(\varepsilon) \tag{1.11}$$

for all $X$ and $\varepsilon \in (0,2]$?

We give a negative answer to Question 1.1, yet Question 1.2 remains unsolved. We believe that evaluations of $\varrho(\varepsilon)$ for (some) spaces $L_p(\mu)$ might yield a negative answer to Question 1.2.

In Proposition 2.7 we prove that for any $X$,

$$\varrho(\varepsilon) = \inf\{\varrho(x,\varepsilon) : x \in X,\ \|x\| = 1\}. \tag{1.12}$$

It follows from this that if a constant $c$ works in (1.6) instead of the number 2/3, then it also does in (1.7) and we conjecture that $c = 2$ can be used for (1.6), hence also for (1.7).

Finally, we prove that if $X$ is a Hilbert space, $\dim X \geq 2$, $x \in X$ with $\|x\| = 1$ and $\varepsilon \in (0,2]$, then

$$\varrho(x,\varepsilon) = \varrho(\varepsilon) = \frac{\varepsilon^2}{2}. \tag{1.13}$$

Thus, in this case, the ratio $\varrho(x,\varepsilon)/\delta(x,\varepsilon) = \varrho(\varepsilon)/\delta(\varepsilon)$ ranges over the interval $(2,4]$.

## 2. Results

We start with auxiliary statements. The first one is very simple.

Lemma 2.1. *Let $X$ be a two-dimensional normed linear space, $z \in X$, $\|z\| = 1$, $0 < \varepsilon \leq 2$, and let $\rho_1 = \sup\{r : r \geq 0$ and for each $f \in X^*$ with $\|f\| = 1$ and each $y \in [-1,1]$ with $|y - f(z)| < r$ there is $u \in U$ such that $\|u - z\| < \varepsilon$ and $f(u) = y\}$. Then $\rho_1 = \rho(z,\varepsilon)$.*

*Proof.* As $\dim X = 2$, the set of linear functionals on $X$ of norm one can be identified with the family of quotient maps $Q_M : X \to X/M$, where $M$ ranges throughout the set of all one-dimensional linear subspaces of $X$. So, it suffices to show that if $M = X$ or $M = \{0\}$, $Q_M$ maps the $\varepsilon$-neighbourhood of $z$ in $U$ onto a set containing the $\rho_1$-neighbourhood of $Q_M(z)$ in $Q_M(U)$.

If $M = X$, we have $Q_M(X) = \{0\}$, thus the image of any neighbourhood of $z$ in $U$ coincides with $Q_M(U)$. Now, let $M = \{0\}$; then $Q_M$ is the identity map on $X$, so we must show that $\rho_1 \leq \varepsilon$. Pick an $f \in X^*$ such that $\|f\| = f(z) = 1$. Then, for any $u \in U$ such that $\|u - z\| < \varepsilon$, we have

$$f(u) = 1 + f(u - z) \geq 1 - \|u - z\| > 1 - \varepsilon, \tag{2.1}$$

hence $\rho_1 \leq \varepsilon$ by the definition of $\rho_1$. □

Lemma 2.2. *Let $1 < p < \infty$ and let $X = \mathbb{R}^2$ be given the $l_p$-norm $\|(x,y)\| = (|x|^p + |y|^p)^{1/p}$ for any $(x,y) \in X$. Then for the element $z = (0,1)$ of $X$ and $\varepsilon > 0$ we have $\rho(z,\varepsilon) = (p-1)p^{-1}\varepsilon^p + o(\varepsilon^p)$ for $\varepsilon \to 0$.*

*Proof.* For $\varepsilon \in (0,1)$, let $t = t(\varepsilon) \in (0,1)$ be defined by the equation

$$t^p + 1 - (1-t)^p = \varepsilon^p \tag{2.2}$$

and let $r = 1 - (1-t)^{p-1}$. Clearly, for $\varepsilon \to 0$ we have $t \to 0$, (2.2) yields $pt + o(t) = \varepsilon^p$, hence

$$t = p^{-1}\varepsilon^p + o(\varepsilon^p), \tag{2.3}$$

so that $r = (p-1)t + o(t) = (p-1)p^{-1}\varepsilon^p + o(\varepsilon^p)$.

Thus, by Lemma 2.1, it suffices to show that for small $\varepsilon$ and for $\rho_1$ defined in Lemma 2.1 we have $\rho_1 = r$. Define $y_1 = 1 - t$ and $x_1 = (1 - y_1^p)^{1/p}$. The element $z_1 = (x_1, y_1)$ of $X$ has norm one and (2.2) implies

$$||z_1 - z|| = \varepsilon. \tag{2.4}$$

Represent $X^*$ by $\mathbb{R}^2$ with the $l_q$-norm, where $1/q+1/p=1$, and consider the functional $f_1 \in X^*$ represented by $f_1=(x_1^{p-1},y_1^{p-1})$. Then $f_1(z_1)=1$ and, since $q(p-1)=p$, $f_1$ is of norm one. As the space $X$ is strictly convex, there is no point $u$ in the closed unit ball $U$ of $X$ such that $u \neq z_1$ and $f_1(u)=1$. Hence, taking (2.4) into account, we get

$$\rho_1 \leq 1-f_1(z)=1-y_1^{p-1}=r. \tag{2.5}$$

Now we will prove the inequality $\rho_1 \geq r$ for small $\varepsilon$. To show this, let $f \in X^*$ be a functional of norm one. Represent $f$ by $(v,w) \in \mathbb{R}^2$ with $|v|^q+|w|^q=1$. We will prove that, for small $\varepsilon$, $f$ maps the set $U_\varepsilon=\{u \in U : \|u-z\|<\varepsilon\}$ onto a set containing the interval $[-1,1]\cap(f(z)-r,f(z)+r)$.

Let $g,h \in X^*$ be the functionals with the representations $g=(-v,w)$ and $h=(v,-w)$. Since, for any $(x,y)\in\mathbb{R}^2$, $(x,y)$ is in $U_\varepsilon$ if and only if $(-x,y)$ is in $U_\varepsilon$, we have $g(U_\varepsilon)=f(U_\varepsilon)$ and $h(U_\varepsilon)=-f(U_\varepsilon)$. Trivially, $g(z)=f(z)$ and $h(z)=-f(z)$. It follows readily from this that we can assume without loss of generality that $v,w\geq 0$. Since $X$ is strictly convex, there is exactly one point $z_f=(x_f,y_f)\in X$ such that $\|z_f\|=f(z_f)=1$. It is easy to see that $x_f\geq 0, y_f\geq 0$ and that

$$v=x_f^{p/q}=x_f^{p-1}, \qquad w=y_f^{p/q}=y_f^{p-1}. \tag{2.6}$$

As $\|z_f\|=\|z_1\|$, we have

$$x_f^p+y_f^p=x_1^p+y_1^p. \tag{2.7}$$

We consider two cases. Suppose first that $x_f<x_1$; then, by (2.7), $y_f>y_1$. Therefore, $\|z_f-z\|<\|z_1-z\|$, hence by (2.4), $z_f$ is in the $\varepsilon$-neighbourhood of $z$. As $f(z_f)=1$, it suffices to find a $u\in U$ such that $\|u-z\|<\varepsilon$ and $f(u)\leq f(z)-r$. Define $u=(1-\varepsilon/2)z$. Then $u\in U$, $\|u-z\|=\varepsilon/2$, and

$$\begin{aligned} f(z)-f(u)&=\frac{\varepsilon}{2}f(z)=\frac{\varepsilon}{2}w=\frac{\varepsilon}{2}y_f^{p-1}\\ &>\frac{\varepsilon}{2}y_1^{p-1}=\frac{\varepsilon}{2}(1-t)^{p-1}=\frac{\varepsilon}{2}(1-r). \end{aligned} \tag{2.8}$$

Since $r=o(\varepsilon)$ for $\varepsilon\to 0$, the last expression is greater than $r$ for small $\varepsilon$.

Consider now the second case, that is, let

$$x_f\geq x_1; \tag{2.9}$$

then (2.7) yields

$$y_f\leq y_1. \tag{2.10}$$

For any $x\in(0,x_1]$, let $a(x)$ be the uniquely determined positive number such that the elements $u(x)$, $\bar{u}(x)$ of $X$, defined by

$$u(x)=(x,a(x)), \qquad \bar{u}(x)=(-x,a(x)), \tag{2.11}$$

are of norm one. Clearly, $u(x_1) = z_1$. The function $d(x) = \|u(x) - z\|$ is strictly increasing on $(0, x_1]$ and, by (2.4), $d(x_1) = \varepsilon$. Thus, for each $x \in (0, x_1)$, $u(x)$ (and hence also $\bar{u}(x)$) is in the $\varepsilon$-neighbourhood of $z$. Furthermore,

$$\begin{aligned} f(z) - f(\bar{u}(x)) &= w + vx - wa(x) \\ &\geq vx + wa(x) - w \\ &= f(u(x)) - f(z). \end{aligned} \tag{2.12}$$

Therefore, it suffices to show that, for each $\alpha > 0$, there is $x \in (0, x_1)$ such that $f(u(x)) - f(z) > r - \alpha$. Since the functions $f$ and $u$ are continuous, it will suffice to prove that $f(u(x_1)) - f(z) \geq r$. If follows from (2.6), (2.9), and (2.10) that $v \geq x_1^{p-1}$ and $w \leq y_1^{p-1}$.

Consequently, $f(u(x_1)) - f(z) = vx_1 + w(y_1 - 1) \geq x_1^p + y_1^{p-1}(y_1 - 1) = 1 - y_1^{p-1} = 1 - (1 - t)^{p-1} = r$, which concludes the proof. □

Lemma 2.3. *Let $X$ and $z$ be as in Lemma 2.2 and let $\varepsilon > 0$. Then*

$$\delta(z, \varepsilon) = p^{-1}(2^{-1} - 2^{-p})\varepsilon^p + o(\varepsilon^p) \quad \textit{for } \varepsilon \longrightarrow 0. \tag{2.13}$$

*Proof.* Let $0 < \varepsilon < 1$. By the results of [1],

$$\delta(z, \varepsilon) = 1 - \left\| \frac{z_1 + z}{2} \right\| \tag{2.14}$$

for a point $z_1 = (x_1, y_1) \in X$ of norm one such that

$$\|z_1 - z\| = \varepsilon. \tag{2.15}$$

The symmetry of the unit ball of $X$ and the inequality $\varepsilon < 1$ enable us to assume that $x_1, y_1 > 0$. Define $t = 1 - y_1$. Since $\|z_1\| = 1$, we have

$$x_1^p = 1 - y_1^p = 1 - (1 - t)^p. \tag{2.16}$$

The equality (2.15) can be written as (2.2) and, for $\varepsilon \to 0$, (2.3) is true. Using (2.16), we have

$$\begin{aligned} \left\| \frac{z_1 + z}{2} \right\|^p &= \left( \frac{x_1}{2} \right)^p + \left( \frac{(y_1 + 1)}{2} \right)^p \\ &= 2^{-p}(1 - (1 - t)^p) + \left( 1 - \frac{t}{2} \right)^p \\ &= 2^{-p}pt + 1 - 2^{-1}pt + o(t) \quad \text{for } t \longrightarrow 0. \end{aligned} \tag{2.17}$$

From this we obtain $\|(z_1 + z)/2\| = 1 + 2^{-p}t - 2^{-1}t + o(t)$, and in combination with (2.14) and (2.3), it concludes the proof. □

Proposition 2.4. *Let $c$ be a real constant such that for every normed linear space $X$ there is $\varepsilon_o > 0$ such that*

$$\rho(x, \varepsilon) \geq c\delta(x, \varepsilon) \tag{2.18}$$

*for each $x \in X$ of norm one and $\varepsilon \in (0, \varepsilon_0)$. Then $c \leq 2/\log 2$.*

*Proof.* It follows from Lemmas 2.2 and 2.3 that if $c$ satisfies the assumptions of the proposition,

$$c \le (p-1)(2^{-1}-2^{-p})^{-1} \quad \forall p > 1. \tag{2.19}$$

One can easily observe that the limit of the right side of this inequality for $p \to 1$ (or, infimum over $p > 1$) is $2/\log 2$. □

Proposition 2.5. *Let $\lambda, C$ be real constants, $\lambda > 1$, such that for every normed linear space $X$ there is $\varepsilon_0 > 0$ such that*

$$\rho(x,\varepsilon) \le C\delta(x,\lambda\varepsilon) \tag{2.20}$$

*for each $x \in X$ of norm one and $\varepsilon \in (0,\varepsilon_0)$. Then $C > 2(e\lambda\log\lambda)^{-1}$.*

*Proof.* Let $\lambda$ and $C$ satisfy the assumptions of the proposition. By Lemmas 2.2 and 2.3, for each $p > 1$ we have

$$C \ge (p-1)(2^{-1}-2^{-p})^{-1}\lambda^{-p} > 2(p-1)\lambda^{-p}. \tag{2.21}$$

Choosing $p = 1+\log^{-1}\lambda$, we obtain from this the desired inequality. □

Corollary 2.6. *There is no constant $C$ such that for every normed linear space $X$ there is $\varepsilon_0 > 0$ such that*

$$\rho(x,\varepsilon) \le C\delta(x,\varepsilon) \tag{2.22}$$

*for each $x \in X$ of norm one and $\varepsilon \in (0,\varepsilon_0)$.*

*Proof.* If $C$ were such a constant, Proposition 2.5 and the inequality $\delta(x,\varepsilon) \le \delta(x,\lambda\varepsilon)$ for $\lambda > 1$ would yield $C > 2(e\lambda\log\lambda)^{-1}$ for each $\lambda > 1$, a contradiction. □

Proposition 2.7. *For every normed linear space $X$ and $\varepsilon > 0$ we have*

$$\rho(\varepsilon) = \inf\{\rho(x,\varepsilon) : x \in X,\ \|x\| = 1\}. \tag{2.23}$$

*Proof.* It follows from the definition that we need only prove the inequality

$$\rho(\varepsilon) \ge \inf\{\rho(x,\varepsilon) : x \in X,\ \|x\| = 1\}. \tag{2.24}$$

Let $r$ be a real number such that

$$r > \rho(\varepsilon). \tag{2.25}$$

It suffices to show that, for each such a number $r$, there is $x_1 \in X$ of norm one such that

$$\rho(x_1,\varepsilon) \le r. \tag{2.26}$$

By (2.25), there is $x_0 \in U$ with $\rho(x_0,\varepsilon) < r$. Therefore, there exists a closed linear subspace $M$ of $X$ with the associated quotient map $Q : X \to X/M$ and a $y \in Q(U)$ such that

$\|y - Q(x_0)\| < r$ and $\|x - x_0\| \geq \varepsilon$ for each $x \in U$ with $Q(x) = y$. Let $x$ be a fixed inverse image of $y$ in $U$. Then

$$||Q(x - x_0)|| = ||y - Q(x_0)|| < r \tag{2.27}$$

and, for all $m \in M$,

$$||x + m - x_0|| \geq \varepsilon \quad \text{whenever } x + m \in U. \tag{2.28}$$

Applying (2.28) to $m = 0$, we get

$$||x - x_0|| \geq \varepsilon, \tag{2.29}$$

which, particularly, implies that $\varepsilon \leq 2$ and that the space $X$ is not trivial, that is, $X \neq \{0\}$.

Suppose first that $M = \{0\}$. Then $\|x - x_0\| = \|Q(x - x_0)\|$ and, combining this with (2.27) and (2.29), we obtain $\varepsilon < r$. Choose any $x_1 \in X$ of norm one. Since $Q$ is an isometry and, as we have showed, $\varepsilon \leq 2$ and $\varepsilon < r$, $Q$ does not map the open $\varepsilon$-neighbourhood of $x_1$ in $U$ onto a set containing the open $r$-neighbourhood of $Q(x_1)$ in $Q(U)$, so that (2.26) holds.

Suppose now $M \neq \{0\}$. By (2.27), we can choose a nonzero $m_0 \in M$ such that

$$||x - x_0 + m_0|| < r. \tag{2.30}$$

Let $S = [s_1, s_2]$ and $T = [t_1, t_2]$ be the intervals of real numbers defined by

$$S = \{s : x + sm_0 \in U\} \tag{2.31}$$

and

$$T = \{t : x_0 + tm_0 \in U\}. \tag{2.32}$$

As $x_0 \in U$, we have $0 \in T$, that is,

$$t_1 \leq 0 \leq t_2. \tag{2.33}$$

Denote

$$u_s = x + sm_0 \quad \text{for } s \in S \tag{2.34}$$

and

$$v_i = x_0 + t_i m_0 \quad \text{for } i = 1, 2. \tag{2.35}$$

Clearly, $\|v_i\| = 1$ for $i = 1, 2$. We will show that (2.26) is true for either $x_1 = v_1$ or $x_1 = v_2$.

Let $M_0$ denote the one-dimensional linear subspace of $X$ containing $m_0$ and let $Q_0 : X \to X/M_0$ be the quotient map associated with $M_0$. We have $Q_0(x) - Q_0(v_i) = Q_0(x - x_0)$ for $i = 1, 2$, hence, by (2.30),

$$||Q_0(x) - Q_0(v_i)|| < r \quad \text{for } i = 1, 2. \tag{2.36}$$

Let $u \in U$ be such that $Q_0(u) = Q_0(x)$; then $u - x \in M_0$, hence $u = u_s$ for some $s \in S$. Thus, it suffices to show that for some $i \in \{1,2\}$,

$$||u_s - v_i|| \geq \varepsilon \quad \forall s \in S. \tag{2.37}$$

Suppose on the contrary that there are some $r_i \in S$ ($i = 1,2$) such that

$$||u_{r_i} - v_i|| < \varepsilon \quad \text{for } i = 1,2. \tag{2.38}$$

By the definitions of $u_s$ and $v_i$, it follows that

$$||x - x_0 + p_i m_0|| < \varepsilon \quad \text{for } i = 1,2, \tag{2.39}$$

where $p_i = r_i - t_i$ ($i = 1,2$). Observe that (2.33) implies

$$p_1 \geq r_1, \qquad p_2 \leq r_2, \tag{2.40}$$

and, since $r_i \in S$ for $i = 1,2$, we get

$$p_1 \geq s_1, \qquad p_2 \leq s_2. \tag{2.41}$$

Suppose first that $p_1 \leq s_2$. Then (2.41) yields $p_1 \in S$ so that $x + p_1 m_0 \in U$ by the definition of $S$. Therefore, (2.39) is in contradiction with (2.28).

Suppose now that $p_1 > s_2$. Then, by (2.41), the element $s_2$ is in $[p_2, p_1)$. Since the function $f(s) = \|x - x_0 + s m_0\|$ is convex, we get from (2.39) that $f(s_2) < \varepsilon$. But, since $s_2 \in S$, we have $x + s_2 m_0 \in U$, which contradicts (2.28). □

Turning our attention to the case of a Hilbert space $X$, we start with a lemma.

Lemma 2.8. *Let $X$ be a Hilbert space,* $\dim X \geq 2$, *$x$ an element of $X$ of norm one, and let $\varepsilon \in (0,2]$. Then $\rho(x,\varepsilon) \leq \varepsilon^2/2$.*

*Proof.* Choose a point $u \in X$ of norm one such that $\|x - u\| = \varepsilon$ and a point $m \in X$ such that $\{m, u\}$ is an orthonormal basis of the linear span of the points $x, u$. Let $M$ be the linear subspace of $X$ of dimension one containing $m$ and let $Q : X \to X/M$ be the quotient map associated with $M$. Then $x = tm + su$ for some real numbers $t, s$. We have

$$t^2 + s^2 = \|x\|^2 = 1 \tag{2.42}$$

and

$$t^2 + (s-1)^2 = \|x - u\|^2 = \varepsilon^2. \tag{2.43}$$

Subtracting these inequalities, we get $2s - 1 = 1 - \varepsilon^2$, hence $s = 1 - \varepsilon^2/2$. Since for any nonzero real number $r$ we have $\|u + rm\| > 1$, $u$ is the only inverse image of $Q(u)$ in $U$. These facts yield

$$\begin{aligned} \rho(x,\varepsilon) &\leq ||Q(x) - Q(u)|| = ||Q(tm + su - u)|| \\ &= \inf\{||(s-1)u + rm|| : r \in \mathbb{R}\} \\ &= |s-1| = \frac{\varepsilon^2}{2}. \end{aligned} \tag{2.44}$$

□

The reader is probably familiar with the following simple fact. We give a proof for the sake of completeness.

Lemma 2.9. *Let $X$ be a Hilbert space, $M$ a closed linear subspace of $X$, $Q : X \to X/M$ the quotient map associated with $M$, and let $y \in X/M$ be arbitrary. Then there exists $u \in X$ such that $Q(u) = y$, $\|u\| = \|y\|$, and $u$ is orthogonal to $M$.*

*Proof.* Choose any $x \in X$ such that $Q(x) = y$. As $X$ is reflexive, it follows readily that there is an $m_0 \in M$ such that $\|x + m_0\| = \|Q(x)\|$. Define $u = x + m_0$. Then $Q(u) = Q(x) = y$ and $\|u\| = \|y\|$. Let $m \in M$ be arbitrary; by the definitions of $u$ and $m_0$, for any real number $t$ we have $\|u + tm\| = \|x + m_0 + tm\| \geq \|Q(x)\| = \|u\|$, thus $u$ is orthogonal to $m$. □

Theorem 2.10. *Let $X$ be a Hilbert space, $x \in U$ and $\varepsilon > 0$. Then*

$$\rho(x,\varepsilon) \geq \frac{\varepsilon^2}{2}. \tag{2.45}$$

*Proof.* Let $M$ be a closed linear subspace of $X$, $Q : X \to X/M$ the quotient map associated with $M$, $x_0 \in U$ and $y_0 = Q(x_0)$. We show that $Q$ maps the $\varepsilon$-neighbourhood of $x_0$ in $U$ onto a set containing the $\varepsilon^2/2$-neighbourhood of $y_0$ in $Q(U)$.

Let $y \in Q(U)$ be such that $\|y - y_0\| = r$ with $r < \varepsilon^2/2$. We will find $x \in U$ such that $Q(x) = y$ and $\|x - x_0\|^2 \leq 2r$; observe that the last inequality implies that $\|x - x_0\| < \varepsilon$. By Lemma 2.9, there are elements $u_0, u$ of $X$ orthogonal to $M$ such that

$$Q(u_0) = y_0, \quad ||u_0|| = ||y_0||, \tag{2.46}$$

$$Q(u) = y, \quad \|u\| = \|y\|. \tag{2.47}$$

Clearly, $x_0 = u_0 + m_0$ for some $m_0 \in M$ and, since $x_0 \in U$, the orthogonality of $u_0$ and $m_0$ yields

$$||u_0||^2 + ||m_0||^2 \leq 1. \tag{2.48}$$

As any $m \in M$ is orthogonal to $u$ and $u_0$ (and hence to $u - u_0$), we have $\|u - u_0 + m\| \geq \|u - u_0\|$ for each $m \in M$, thus

$$||u - u_0|| = ||Q(u - u_0)|| = ||y - y_0|| = r. \tag{2.49}$$

Suppose first that

$$\|u\|^2 + ||m_0||^2 \leq 1; \tag{2.50}$$

in this case define $x = u + m_0$. Then $Q(x) = Q(u) = y$, $x \in U$ by (2.50) and, using (2.49), we obtain

$$||x - x_0|| = ||(u + m_0) - (u_0 + m_0)|| = r \leq (2r)^{1/2}, \tag{2.51}$$

hence $x$ is the desired element of $U$.

Suppose now that

$$\|u\|^2+\|m_0\|^2>1. \tag{2.52}$$

Then, clearly, $m_0\neq 0$. Define real numbers $t, p$, and $x\in X$ by

$$\begin{aligned} t&=\|m_0\|^{-1}\left(1-\|u\|^2\right)^{1/2},\\ p&=(1-t)\|m_0\|,\\ x&=u+tm_0. \end{aligned} \tag{2.53}$$

We have $\|x\|^2=\|u\|^2+\|tm_0\|^2=1$, thus $x\in U$. Furthermore, $\|x-x_0\|^2=\|(u+tm_0)-(u_0+m_0)\|^2=\|u-u_0\|^2+(1-t)^2\|m_0\|^2$, hence, by (2.49) and by the definition of $p$,

$$\|x-x_0\|^2=r^2+p^2. \tag{2.54}$$

Also, (2.49) and triangle inequalities yield $\|u_0\|\geq |\|u\|-r|$. Thus, using (2.48), we have

$$\|m_0\|^2\leq 1-\left(\|u\|-r\right)^2. \tag{2.55}$$

We denote by $f$ the function

$$f(v,w)=\left(1-v^2\right)^{1/2}-\left(1-w^2\right)^{1/2}\quad \text{for } v,w\in[0,1]. \tag{2.56}$$

Observe that $p=\|m_0\|-(1-\|u\|^2)^{1/2}$; in combination with (2.52), (2.55) and with the definition of the function $f$, it yields

$$0<p\leq f\left(\left|\,\|u\|-r\,\right|,\|u\|\right). \tag{2.57}$$

We consider three cases.

*Case 1.* Let $\|u\|\geq r$. Since, for any fixed $r\geq 0$, $f(s-r,s)$ is an increasing function of the variable $s\in[r,1]$, we obtain from (2.54) and (2.57) that

$$\|x-x_0\|^2\leq r^2+f^2(1-r,1)=2r. \tag{2.58}$$

*Case 2.* Let $\|u\|<r\leq 1$. Now, since the function $f(v,w)$ is decreasing in the variable $v$ and increasing in the variable $w$, we get from (2.54) and (2.57) that

$$\|x-x_0\|^2\leq r^2+f^2(0,r)=2-2\left(1-r^2\right)^{1/2}\leq 2r. \tag{2.59}$$

*Case 3.* Let $r>1$. In this case, (2.54) with (2.57) and the inequality $\|u\|\leq 1$ yield

$$\|x-x_0\|^2\leq r^2+f^2(r-1,1)=2r, \tag{2.60}$$

which completes the proof. □

Theorem 2.11. *Let $X$ be a Hilbert space,* $\dim X \geq 2$, *and let* $\varepsilon \in (0,2]$. *Then*

$$\rho(\varepsilon) = \frac{\varepsilon^2}{2} \tag{2.61}$$

*and, for each $x \in X$ of norm one,*

$$\rho(x,\varepsilon) = \frac{\varepsilon^2}{2}. \tag{2.62}$$

*Proof.* The assertion follows immediately from Lemma 2.8, Theorem 2.10, and the definition of $\rho(\varepsilon)$. □

We note that since for one-dimensional space we have $\rho(\varepsilon) = \varepsilon$ for any $\varepsilon \in (0,2]$, the restriction $\dim X \geq 2$ in Theorem 2.11 is essential.

## References

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