HYERS-ULAM STABILITY OF BUTLER-RASSIAS FUNCTIONAL EQUATION

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We will prove the Hyers-Ulam stability of the Butler-Rassias functional equation following an idea by M. T. Rassias.

1. Introduction

In 1940, Ulam [9] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [5] under the assumption that G_1 and G_2 are Banach spaces.

Taking this fact into account, the additive Cauchy functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability. This terminology is also applied to the case of other functional equations. For a more detailed definition of such terminology, one can refer to [4, 6, 7].

In 2003, Butler [3] posed the following problem.

Problem 1.1 (Butler [3]). Show that for $d < -1$, there are exactly two solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation $f(x+y) - f(x)f(y) = d \sin x \sin y$.

Recently, Rassias excellently answered this problem by proving the following theorem (see [8]).

Theorem 1.2 (Rassias [8]). *Let $d < -1$ be a constant. The functional equation*

$$f(x+y) - f(x)f(y) = d \sin x \sin y \tag{1.1}$$

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has exactly two solutions in the class of functions $f : \mathbb{R} \to \mathbb{R}$*. More precisely, if a function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the Butler-Rassias functional equation for all* $x, y \in \mathbb{R}$*, then* f *has one of the forms*

$$f(x) = c\sin x + \cos x, \quad f(x) = -c\sin x + \cos x, \tag{1.2}$$

where $c = \sqrt{-d-1}$ *is set.*

In this paper, we will prove the Hyers-Ulam stability of the Butler-Rassias functional equation (1.1).

2. Preliminaries

We follow an idea of Rassias [8] to prove the following lemma. In Section 3, we apply this lemma to the proof of the Hyers-Ulam stability of the Butler-Rassias functional equation (1.1).

Lemma 2.1. *Let* d *be a nonzero real number and* $0 < \varepsilon < |d|$*. If a function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the functional inequality*

$$|f(x+y) - f(x)f(y) - d\sin x \sin y| \le \varepsilon \tag{2.1}$$

for all $x, y \in \mathbb{R}$*, then* $M_f := \sup_{x\in\mathbb{R}} |f(x)|$ *is finite and*

$$\left| f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x \right| \le \frac{2(1+M_f)}{|d|}\varepsilon \tag{2.2}$$

for all $x \in \mathbb{R}$.

Proof. If we replace x by $x+z$ in (2.1), then we have

$$|f(x+y+z) - f(x+z)f(y) - d\sin(x+z)\sin y| \le \varepsilon \tag{2.3}$$

for any $x, y, z \in \mathbb{R}$. Similarly, if we replace y by $y+z$ in (2.1), then we get

$$|f(x+y+z) - f(x)f(y+z) - d\sin x \sin(y+z)| \le \varepsilon \tag{2.4}$$

for $x, y, z \in \mathbb{R}$.

Using (2.3) and (2.4), we obtain

$$\begin{aligned} &|f(x)f(y+z) - f(x+z)f(y) + d\sin x\sin(y+z) - d\sin(x+z)\sin y| \\ &\quad = |[f(x+y+z) - f(x+z)f(y) - d\sin(x+z)\sin y] \\ &\qquad - [f(x+y+z) - f(x)f(y+z) - d\sin x\sin(y+z)]| \le 2\varepsilon \end{aligned} \tag{2.5}$$

for all $x, y, z \in \mathbb{R}$. It follows from (2.5) that

$$\begin{aligned} &|f(x)[f(y+z)-f(y)f(z)-d\sin y\sin z]+f(x)f(y)f(z)+df(x)\sin y\sin z \\ &\quad -[f(x+z)-f(x)f(z)-d\sin x\sin z]f(y)-f(x)f(y)f(z)-df(y)\sin x\sin z \\ &\quad +d\sin x\sin(y+z)-d\sin(x+z)\sin y| \\ &\qquad = |f(x)f(y+z)-f(x+z)f(y)+d\sin x\sin(y+z)-d\sin(x+z)\sin y| \le 2\varepsilon \end{aligned} \tag{2.6}$$

for all $x, y, z \in \mathbb{R}$.

It is easy to check that

$$\begin{aligned} &|df(x)\sin y\sin z+d\sin x\sin(y+z)-df(y)\sin x\sin z-d\sin(x+z)\sin y| \\ &\quad = |f(x)[f(y+z)-f(y)f(z)-d\sin y\sin z] \\ &\qquad +f(x)f(y)f(z)+df(x)\sin y\sin z \\ &\qquad -[f(x+z)-f(x)f(z)-d\sin x\sin z]f(y) \\ &\qquad -f(x)f(y)f(z)-df(y)\sin x\sin z \\ &\qquad +d\sin x\sin(y+z)-d\sin(x+z)\sin y \\ &\qquad -f(x)[f(y+z)-f(y)f(z)-d\sin y\sin z] \\ &\qquad +[f(x+z)-f(x)f(z)-d\sin x\sin z]f(y)|. \end{aligned} \tag{2.7}$$

Hence, in view of (2.6) and (2.1), we can now get

$$\begin{aligned} &|df(x)\sin y\sin z+d\sin x\sin(y+z)-df(y)\sin x\sin z-d\sin(x+z)\sin y| \\ &\quad \le |f(x)[f(y+z)-f(y)f(z)-d\sin y\sin z] \\ &\qquad +f(x)f(y)f(z)+df(x)\sin y\sin z \\ &\qquad -[f(x+z)-f(x)f(z)-d\sin x\sin z]f(y) \\ &\qquad -f(x)f(y)f(z)-df(y)\sin x\sin z \\ &\qquad +d\sin x\sin(y+z)-d\sin(x+z)\sin y| \\ &\qquad +|f(x)|\,|f(y+z)-f(y)f(z)-d\sin y\sin z| \\ &\qquad +|f(y)|\,|f(x+z)-f(x)f(z)-d\sin x\sin z| \\ &\quad \le (2+|f(x)|+|f(y)|)\varepsilon \end{aligned} \tag{2.8}$$

for all $x, y, z \in \mathbb{R}$. If we set $y = z = \pi/2$ in the above inequality, then

$$\left|df(x)-df\left(\frac{\pi}{2}\right)\sin x-d\cos x\right| \le \left(2+|f(x)|+\left|f\left(\frac{\pi}{2}\right)\right|\right)\varepsilon \tag{2.9}$$

for each $x \in \mathbb{R}$.

If we assume that f were unbounded, there should exist a sequence $\{x_n\} \subset \mathbb{R}$ such that $f(x_n) \neq 0$ for every $n \in \mathbb{N}$ and $|f(x_n)| \to \infty$ as $n \to \infty$. Set $x = x_n$ in (2.9), divide both sides of the resulting inequality by $|f(x_n)|$, and then let n diverge to infinity. Then, we have $|d| \le \varepsilon$ which is contrary to our hypothesis, say $\varepsilon < |d|$.

Therefore, f must be bounded, and hence $M_f := \sup_{x\in\mathbb{R}} |f(x)|$ has to be finite. Therefore, it follows from (2.9) that (2.2) holds for each $x \in \mathbb{R}$. □

3. Hyers-Ulam stability

In this section, using Lemma 2.1, we prove the Hyers-Ulam stability of the Butler-Rassias functional equation.

Theorem 3.1. *Let $d < -1$ be a constant. Then there exists a constant $K = K(d) \ge 0$ such that if $0 < \varepsilon < |d|$ and if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality (2.1) for all $x, y \in \mathbb{R}$, then*

$$|f(x) - f_0(x)| \le K(\varepsilon + \sqrt{\varepsilon}) \tag{3.1}$$

holds for all $x \in \mathbb{R}$ and for some solution function f_0 of the Butler-Rassias functional equation.

Proof. Let $0 < \varepsilon < |d|$ and f a real-valued function on $\mathbb{R}$ which satisfies inequality (2.1). It follows from Lemma 2.1 that $M_f := \sup_{x\in\mathbb{R}} |f(x)| < \infty$ and that (2.2) holds for all $x \in \mathbb{R}$. Put $x = \pi$ in (2.2) to get

$$|f(\pi) + 1| \le \frac{2(1 + M_f)}{|d|}\varepsilon. \tag{3.2}$$

Furthermore, set $x = y = \pi/2$ in (2.1) to obtain

$$\left| f(\pi) - f\left(\frac{\pi}{2}\right)^2 - d \right| \le \varepsilon. \tag{3.3}$$

By combining (3.2) and (3.3), we get

$$\left| f\left(\frac{\pi}{2}\right)^2 + d + 1 \right| \le \frac{2(1 + M_f) + |d|}{|d|}\varepsilon. \tag{3.4}$$

If we set

$$c = \sqrt{-d-1}, \qquad L = \frac{2(1 + M_f) + |d|}{|d|}, \tag{3.5}$$

then it follows from (3.4) that $|f(\pi/2)^2 - c^2| \le L\varepsilon$. Therefore, we can easily check that

$$\begin{gathered} \left| f\left(\frac{\pi}{2}\right) - c \right| \le \sqrt{L\varepsilon} \quad \left(\text{for } c > \sqrt{L\varepsilon},\ f\left(\frac{\pi}{2}\right) \ge 0\right), \\ \left| f\left(\frac{\pi}{2}\right) + c \right| \le \sqrt{L\varepsilon} \quad \left(\text{for } c > \sqrt{L\varepsilon},\ f\left(\frac{\pi}{2}\right) < 0\right), \\ \left| f\left(\frac{\pi}{2}\right) \right| \le \sqrt{L\varepsilon + c^2} \quad (\text{for } c \le \sqrt{L\varepsilon}). \end{gathered} \tag{3.6}$$

Since $\sqrt{L\varepsilon + c^2} \le \sqrt{2L\varepsilon}$ when $c \le \sqrt{L\varepsilon}$, we have

$$\left| f\left(\frac{\pi}{2}\right) \pm c \right| \le \sqrt{2L\varepsilon} + \sqrt{L\varepsilon} = (1+\sqrt{2})\sqrt{L\varepsilon} \quad (\text{for } c \le \sqrt{L\varepsilon}). \tag{3.7}$$

Hence, it follows that

$$\left| f\left(\frac{\pi}{2}\right) - c \right| \le (1+\sqrt{2})\sqrt{L\varepsilon} \quad \left(\text{for } f\left(\frac{\pi}{2}\right) \ge 0\right) \tag{3.8}$$

and that

$$\left| f\left(\frac{\pi}{2}\right) + c \right| \le (1+\sqrt{2})\sqrt{L\varepsilon} \quad \left(\text{for } f\left(\frac{\pi}{2}\right) < 0\right). \tag{3.9}$$

Due to (2.2), we now get

$$\begin{aligned} |f(x) - c\sin x - \cos x| &\le \left| f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x \right| + \left| \left(f\left(\frac{\pi}{2}\right) - c \right)\sin x \right| \\ &\le \frac{2(1+M_f)}{|d|}\varepsilon + \left| f\left(\frac{\pi}{2}\right) - c \right| \end{aligned} \tag{3.10}$$

for all $x \in \mathbb{R}$ and

$$\begin{aligned} |f(x) + c\sin x - \cos x| &\le \left| f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x \right| + \left| \left(f\left(\frac{\pi}{2}\right) + c \right)\sin x \right| \\ &\le \frac{2(1+M_f)}{|d|}\varepsilon + \left| f\left(\frac{\pi}{2}\right) + c \right| \end{aligned} \tag{3.11}$$

for all $x \in \mathbb{R}$. Therefore, if $f(\pi/2) \ge 0$, then (3.10) and (3.8) imply

$$|f(x) - c\sin x - \cos x| \le \frac{2(1+M_f)}{|d|}\varepsilon + (1+\sqrt{2})\sqrt{L\varepsilon} \tag{3.12}$$

for all $x \in \mathbb{R}$. Similarly, if $f(\pi/2) < 0$, it then follows from (3.11) and (3.9) that

$$|f(x) + c\sin x - \cos x| \le \frac{2(1+M_f)}{|d|}\varepsilon + (1+\sqrt{2})\sqrt{L\varepsilon} \tag{3.13}$$

for all $x \in \mathbb{R}$.

By (2.1) and our hypothesis $0 < \varepsilon < |d|$, we have

$$|f(x)|\,|f(y)| \le |f(x+y)| + 2|d| \tag{3.14}$$

for all $x, y \in \mathbb{R}$, which implies that $M_f^2 \le 2|d| + M_f$, and hence

$$M_f \le \frac{1+\sqrt{1+8|d|}}{2}. \tag{3.15}$$

Subsequently, it follows from (3.15) that

$$L \leq \frac{3+\sqrt{1+8|d|}+|d|}{|d|}, \tag{3.16}$$

and hence

$$\begin{aligned} &\frac{2(1+M_f)}{|d|}\varepsilon+(1+\sqrt{2})\sqrt{L\varepsilon} \\ &\quad \leq \frac{3+\sqrt{1+8|d|}}{|d|}\varepsilon+(1+\sqrt{2})\sqrt{\frac{3+\sqrt{1+8|d|}+|d|}{|d|}}\sqrt{\varepsilon}. \end{aligned} \tag{3.17}$$

Note that if $d<-1$, then the Butler-Rassias functional equation (1.1) has exactly two solutions $\pm c\sin x+\cos x$ (see Theorem 1.2.) Thus, it follows from (3.12), (3.13), and (3.17) that

$$|f(x)-f_0(x)| \leq \frac{3+\sqrt{1+8|d|}}{|d|}\varepsilon+(1+\sqrt{2})\sqrt{\frac{3+\sqrt{1+8|d|}+|d|}{|d|}}\sqrt{\varepsilon} \tag{3.18}$$

for any $x\in\mathbb{R}$ and for some solution function f_0 of the Butler-Rassias functional equation. Putting

$$K=\max\left\{\frac{3+\sqrt{1+8|d|}}{|d|},(1+\sqrt{2})\sqrt{\frac{3+\sqrt{1+8|d|}+|d|}{|d|}}\right\} \tag{3.19}$$

in the last inequality, we conclude that our assertion is true. □

Remark 3.2. If we set $d=0$ in the Butler-Rassias functional equation (1.1), then the equation is called the exponential functional equation. Baker, Lawrence, and Zorzitto [2] have investigated the stability problem for the exponential equation (see also [1]).

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