ON THE EXISTENCE OF MINIMAL AND MAXIMAL SOLUTIONS OF DISCONTINUOUS FUNCTIONAL STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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We apply a fixed point result for multifunctions to derive existence results for boundary value problems of Sturm-Liouville differential equations with nonlinearities that may involve discontinuous and functional dependencies.

1. Introduction

The main goal of this paper is to study the solvability of the following Sturm-Liouville boundary value problem (BVP)

$$\begin{aligned} -\frac{d}{dt}(\mu(t)u'(t)) &= \lambda g(t,u,u(t),u'(t)) \quad \text{a.e. in } J=[t_0,t_1], \\ a_0u(t_0)-b_0u'(t_0) &= c_0, \qquad a_1u(t_1)+b_1u'(t_1)=c_1, \end{aligned} \tag{1.1}$$

where $g: J\times C(J)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. We are looking for solutions of (1.1) out of

$$Y=\{u\in C^1(J)\mid \mu u'\in AC(J)\}. \tag{1.2}$$

In Section 2, we give first an existence result for problems where the second, the functional argument u of g, is replaced in (1.1) by fixed functions $v\in C(J)$, and study the dependence of solution sets of these problems on v. The so obtained results and a fixed point result for multifunctions proved recently in [7] are then used in Section 3 to derive existence results for minimal and maximal solutions of (1.1). Also in nonfunctional case we get new existence results. Because of weaker hypotheses than those assumed, for example, in [1, 3, 4, 5, 8, 9, 10], the fixed point results for single-valued operators do not apply.

2. Hypotheses and preliminaries

2.1. Hypotheses. Throughout this paper we assume that

$$\lambda,a_j,b_j\in\mathbb{R}_+,\quad a_0a_1+a_0b_1+a_1b_0>0,\quad c_j\in\mathbb{R},\ j=0,1,\qquad \mu\in C(J,(0,\infty)), \tag{2.1}$$

and that $C(J)$ is ordered pointwise.

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The function $g : J \times C(J) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following hypotheses.

($g0$) $(t,x,y) \mapsto g(t,v,x,y)$ is a Carathéodory function, that is measurable in t and jointly continuous in (x,y), for each $v \in C(J)$.

($g1$) $|g(t,v,x,y)| \le p(t)\max\{|x|,|y|\} + m(t)$ for all $x,y \in \mathbb{R}$, for all $v \in C(J)$, and for a.e. $t \in J$, where $p,m \in L^1_+(J)$.

($g2$) $g(t,\cdot,x,y)$ is increasing for a.e. $t \in J$ and for all $x,y \in \mathbb{R}$.

($g3$) For each fixed $v \in C(J)$, $|g(t,v,x,y) - g(t,v,x,z)| \le p_v(t)\phi_v(|y-z|)$ for a.e. $t \in J$ and for all $x,y,z \in \mathbb{R}$, where $p_v \in L^1_+(J)$, $\phi_v : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $\int_{0+}^1 (dx/\phi_v(x)) = \infty$.

Notice that g can be discontinuous in its first and second arguments, and is monotone only with respect to its second, functional argument. It is also worth to notice that no lower or upper solutions are assumed to exist, and no Nagumo-type hypotheses are imposed on g.

We are going to show that if λ is small enough, then the BVP (1.1) has under the above hypotheses a minimal solution u_- and a maximal solution u^+ in the sense that if u is any solution of (1.1), then $u \le u_-$ implies $u = u_-$ and $u_+ \le u$ implies $u = u_+$.

2.2. Auxiliary results. For the sake of completeness we recall in this section several auxiliary results whose proofs can be found, for example, in [1, 5].

Lemma 2.1. *If $q \in L^1(J)$, then the BVP*

$$
\begin{gathered}
-\frac{d}{dt}(\mu(t)u'(t)) = q(t) \quad \text{a.e. in } J, \\
a_0 u(t_0) - b_0 u'(t_0) = c_0, \qquad a_1 u(t_1) + b_1 u'(t_1) = c_1
\end{gathered}
\tag{2.2}
$$

has a unique solution u in Y, and it can be represented as

$$
u(t) = \frac{c_0 y_1(t) + c_1 y_0(t)}{D} + \int_{t_0}^{t_1} k(t,s)q(s)\,ds, \quad t \in J, \tag{2.3}
$$

where

$$
\begin{gathered}
y_0(t) = \int_{t_0}^{t} \frac{a_0}{\mu(s)}\,ds + \frac{b_0}{\mu(t_0)}, \qquad y_1(t) = \int_{t}^{t_1} \frac{a_1}{\mu(s)}\,ds + \frac{b_1}{\mu(t_1)}, \\
D = \int_{t_0}^{t_1} \frac{a_0 a_1}{\mu(s)}\,ds + \frac{a_0 b_1}{\mu(t_1)} + \frac{a_1 b_0}{\mu(t_0)}, \qquad k(t,s) = \begin{cases} \dfrac{y_1(t)y_0(s)}{D}, & t_0 \le s \le t, \\ \dfrac{y_0(t)y_1(s)}{D}, & t \le s \le t_1. \end{cases}
\end{gathered}
\tag{2.4}
$$

Denote

$$
z_0(t) = \max\left\{y_0(t), \frac{a_0}{\mu(t)}\right\}, \qquad z_1(t) = \max\left\{y_1(t), \frac{a_1}{\mu(t)}\right\}, \tag{2.5}
$$

and define an operator $A : C(J) \to C(J)$ by

$$Au(t) = \int_{t_0}^{t_1} \ell(t,s)p(s)u(s)ds, \quad \text{where } l(t,s) = \begin{cases} \dfrac{z_1(t)y_0(s)}{D}, & t_0 \le s \le t, \\ \dfrac{z_0(t)y_1(s)}{D}, & t \le s \le t_1. \end{cases} \tag{2.6}$$

Lemma 2.2. *If the hypothesis* $(g1)$ *holds, and if* $\lambda \in [0,\lambda_1)$*, where* λ_1 *is the least positive eigenvalue of* A*, then the integral equation*

$$b(t) = \frac{|c_0|z_1(t) + |c_1|z_0(t)}{D} + \lambda \int_{t_0}^{t_1} \ell(t,s)\big(p(s)b(s) + m(s)\big)ds \tag{2.7}$$

has a unique solution $b \in C(J)$.

Lemma 2.3. *If the hypotheses* $(g0)$ *and* $(g1)$ *hold, then each solution of the BVP (1.1) belongs to the set*

$$B = \{u \in C^1(J) \mid \max\{|u(t)|, |u'(t)|\} \le b(t),\ t \in J\}, \tag{2.8}$$

where b *is the unique solution of (2.7).*

2.3. An auxiliary problem. In this section, we study the BVP

$$\begin{aligned} -\frac{d}{dt}\big(\mu(t)u'(t)\big) &= \lambda g\big(t,v,u(t),u'(t)\big) \quad \text{for a.e. } t \in J, \\ a_0u(t_0) - b_0u'(t_0) = c_0, &\qquad a_1u(t_1) + b_1u'(t_1) = c_1 \end{aligned} \tag{2.9}$$

in the case when v is a fixed element of the set

$$P = \{v \in C(J) \mid |v(t)| \le b(t),\ t \in J\}, \tag{2.10}$$

where b is the solution of (2.7).

The following existence result is proved in [5, Proposition 4.1.1].

Proposition 2.4. *Let the hypotheses* $(g0)$ *and* $(g1)$ *hold, assume that* $\lambda \in [0,\lambda_1)$*, where* λ_1 *is the least positive eigenvalue of the operator* A*, defined by (2.6), and let* B *and* P *be defined by (2.8) and (2.10), where* b *is the solution of (2.7). Then for each* $v \in P$ *the BVP (2.9) has a solution in* B.

Hint to the proof. Obviously, B is closed and convex subset of $C^1(J)$ with respect to the norm of $C^1(J)$ defined by

$$\|u\| = \max\{|u(t)|, |u'(t)| \mid t \in J\}. \tag{2.11}$$

Let $v \in P$ be given. It can be shown (cf. [5, Proposition 4.1.1]) that relation

$$F_vu(t) = \frac{c_0y_1(t) + c_1y_0(t)}{D} + \lambda \int_{t_0}^{t_1} k(t,s)g\big(s,v,u(s),u'(s)\big)ds, \quad t \in J \tag{2.12}$$

defines a compact mapping $F_v : B \to B$. Thus F_v has by Schauder's fixed point theorem a fixed point u in B. It then follows from Lemma 2.1 that u is a solution of (2.9) in B. □

Remark 2.5. The BVP

$$-u''(t) = \lambda u(t), \quad t \in J = [0,\pi], \qquad u(0) = 0, \qquad u(\pi) = 1 \tag{2.13}$$

does not have any solution when $\lambda = 1$. Thus the result of Proposition 2.4 is not valid in general if condition $\lambda \in [0,\lambda_1)$ is dropped.

The auxiliary problem (2.9) does not, in general, have a unique solution. However, the results of the next Proposition show that the solutions of (2.9), or equivalently, the fixed points of F_v defined by (2.12) have properties which enable us to apply a fixed point result for multivalued functions that has been proved recently in [7].

Proposition 2.6. *Let the hypotheses $(g0)$–$(g3)$ hold, and assume that $v_1, v_2 \in P$, $v_1 \leq v_2$, where P is given by (2.10). Then the fixed points of the operators F_{v_i}, defined by (2.12), have the following properties.*

(a) *If $u \in P$ and $u = F_{v_1}u$, there exists a $w \in P$ such that $u \leq w = F_{v_2}w$.*

(b) *If $w \in P$ and $w = F_{v_2}w$, there exists a $u \in P$ such that $u = F_{v_1}u \leq w$.*

Proof. (a) Let $u \in P$, $u = F_{v_1}u$, be given. Define a mapping $f : J \times C(J) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,v,x,y) = \begin{cases} g(t,v,u(t),y), & \text{if } u(t) > x, \\ g(t,v,x,y), & \text{if } u(t) \leq x. \end{cases} \tag{2.14}$$

The so-defined mapping f satisfies the hypotheses $(g0)$–$(g3)$ given for g, with m replaced by $m + pu$ in $(g1)$. Thus the BVP

$$\begin{gathered} -\frac{d}{dt}\big(\mu(t)w'(t)\big) = \lambda f\big(t,v_2,w(t),w'(t)\big) \quad \text{for a.e. } t \in J, \\ a_0 w(t_0) - b_0 w'(t_0) = c_0, \qquad a_1 w(t_1) + b_1 w'(t_1) = c_1 \end{gathered} \tag{2.15}$$

has a solution $w \in Y$. Moreover, u is a solution of the BVP

$$\begin{gathered} -\frac{d}{dt}\big(\mu(t)u'(t)\big) = \lambda g\big(t,v_1,u(t),u'(t)\big) \quad \text{for a.e. } t \in J, \\ a_0 u(t_0) - b_0 u'(t_0) = c_0, \qquad a_1 u(t_1) + b_1 u'(t_1) = c_1. \end{gathered} \tag{2.16}$$

To prove that $u \leq w$, we will show that if $u \not\leq w$, then $u(t) - w(t) \equiv c > 0$, which yields $a_0 = a_1 = 0$, and hence $a_0a_1 + a_0b_1 + a_1b_0 = 0$, contradicting with (2.1). Consider first the case when $u - w$ attains a positive maximum c at $t_2 \in (t_0,t_1)$.

The proof that $u(t) - w(t) \equiv c$ is divided into two steps.

(i) Let t_3 be the greatest number on $(t_2,t_1]$ such that $u(t) \geq w(t)$ for each $t \in [t_2,t_3]$.

To prove that $w'(t) \leq u'(t)$ for each $t \in [t_2,t_3]$, assume on the contrary: there is a subinterval $[a,b]$ of $[t_2,t_3]$ such that

$$0 < w'(t) - u'(t), \quad t \in (a,b], \qquad w'(a) - u'(a) = 0. \tag{2.17}$$

Denoting $K = \max(1/\mu)$, we have

$$x(t) := K\big(\mu(t)w'(t) - \mu(t)u'(t)\big) \geq w'(t) - u'(t) \quad \forall t \in [a,b]. \tag{2.18}$$

Since $u, w \in Y$, then $x \in AC(J)$ by (1.2). Applying (2.14), (2.15), (2.16), (2.18), ($g2$), and ($g3$) we get

$$\begin{aligned} x'(t) &= K\frac{d}{dt}(\mu(t)w'(t)) - K\frac{d}{dt}(\mu(t)u'(t)) \\ &= K\lambda\big(g(t,v_1,u(t),u'(t)) - f(t,v_2,w(t),w'(t))\big) \\ &\leq K\lambda\big(g(t,v_2,u(t),u'(t)) - g(t,v_2,u(t),w'(t))\big) \\ &\leq K\lambda\,|g(t,v_2,u(t),u'(t)) - g(t,v_2,u(t),w'(t))| \\ &\leq K\lambda p_{v_2}(t)\phi_{v_2}(|u'(t) - w'(t)|) = K\lambda p_{v_2}(t)\phi_{v_2}(w'(t) - u'(t)) \\ &\leq K\lambda p_{v_2}(t)\phi_{v_2}(K(\mu(t)w'(t) - \mu(t)u'(t))) = K\lambda p_{v_2}(t)\phi_{v_2}(x(t)) \end{aligned} \tag{2.19}$$

for a.e. $t \in (a,b]$. Thus we have $x'(t) \leq K\lambda p_{v_2}(t)\phi_{v_2}(x(t))$ a.e. in $(a,b]$, $x(a) = 0$, whence $x(t) \equiv 0$ on $[a,b]$ by [5, Lemma B.6.1]. This contradicts (2.17). Consequently, $w'(t) \leq u'(t)$ on $[t_2,t_3]$, whence

$$\begin{aligned} u(t) - w(t) &= u(t_2) - w(t_2) + \int_{t_2}^{t} (u'(s) - w'(s))\,ds \\ &\geq u(t_2) - w(t_2), \quad t \in [t_2,t_3]. \end{aligned} \tag{2.20}$$

Because t_2 was the maximum point of $u(t) - w(t)$, then $u(t) - w(t) \equiv c$ on $[t_2,t_3]$. This result and the choice of t_3 imply that $t_3 = t_1$. Thus $u(t) - w(t) \equiv c$ on $[t_2,t_1]$.

(ii) Choose next t_4 to be the least number on $[t_0,t_2)$ such that $u(t) \geq w(t)$, for each $t \in [t_4,t_2]$.

To prove that $u'(t) \leq w'(t)$ for each $t \in [t_4,t_2]$, assume on the contrary: there is a subinterval $[a,b]$ of $[t_4,t_2]$ such that

$$0 < u'(t) - w'(t), \quad t \in [a,b), \qquad u'(b) = w'(b). \tag{2.21}$$

Thus

$$x(t) := K\big(\mu(t)u'(t) - \mu(t)w'(t)\big) \geq u'(t) - w'(t) \quad \forall t \in [a,b]. \tag{2.22}$$

In view of (2.14), (2.15), (2.16), (2.22), ($g2$) and ($g3$) we obtain

$$\begin{aligned} -x'(t) &= K\frac{d}{dt}(\mu(t)w'(t)) - K\frac{d}{dt}(\mu(t)u'(t)) \\ &= K\lambda\big(g(t,v_1,u(t),u'(t)) - f(t,v_2,w(t),w'(t))\big) \\ &\leq K\lambda\big(g(t,v_2,u(t),u'(t)) - g(t,v_2,u(t),w'(t))\big) \\ &\leq K\lambda\,|g(t,v_2,u(t),u'(t)) - g(t,v_2,u(t),w'(t))| \\ &\leq K\lambda p_{v_2}(t)\phi_{v_2}(|u'(t) - w'(t)|) = K\lambda p_{v_2}(t)\phi_{v_2}(u'(t) - w'(t)) \\ &\leq K\lambda p_{v_2}(t)\phi_{v_2}(K(\mu(t)u'(t) - \mu(t)w'(t))) = K\lambda p_{v_2}(t)\phi_{v_2}(x(t)) \end{aligned} \tag{2.23}$$

for a.e. $t \in [a,b)$. Because $x(b) = 0$, we have

$$-x'(t) \leq K\lambda p_{\nu_2}(t)\phi_{\nu_2}(x(t)) \quad \text{a.e. in } [a,b), \quad x(b) = 0, \tag{2.24}$$

which implies a contradiction:

$$\infty = \int_{0+}^{x(a)} \frac{dx}{\phi_{\nu_2}(x)} = \int_{b-}^{a} \frac{x'(t)dt}{\phi_{\nu_2}(x(t))} = \int_{a}^{b-} \frac{-x'(t)dt}{\phi_{\nu_2}(x(t))} \leq \int_a^b K\lambda p_{\nu_2}(t)dt < \infty. \tag{2.25}$$

Thus $u'(t) \leq w'(t)$ on $[t_4,t_2]$, whence

$$u(t_2) - w(t_2) = u(t) - w(t) + \int_t^{t_2} (u'(s) - w'(s))ds \leq u(t) - w(t), \quad t \in [t_4,t_2]. \tag{2.26}$$

Because t_2 was the maximum point of $u(t) - w(t)$, then $u(t) - w(t) \equiv c$ in $[t_4,t_2]$. This result and the choice of t_4 imply that $t_4 = t_0$. Thus $u(t) - w(t) \equiv c$ on $[t_0,t_2]$.

The results of (i) and (ii) imply that $u(t) - w(t) \equiv c > 0$ in the considered case. Applying this result and the boundary conditions

$$\begin{aligned} a_0u(t_0) - b_0u'(t_0) &= a_0w(t_0) - b_0w'(t_0) = c_0, \\ a_1u(t_1) + b_1u'(t_1) &= a_1w(t_1) + b_1w'(t_1) = c_1, \end{aligned} \tag{2.27}$$

one can show that $a_0 = a_1 = 0$ by the reasoning used in the proof of [5, Lemma 3.4.2]. This proof covers also cases when $u(t) - w(t)$ attains its positive maximum at t_0 or at t_1.

The above proof shows that if $u \nleq w$, then $a_0 = b_0 = 0$. But then $a_0a_1 + a_0b_1 + a_1b_0 = 0$, which contradicts with the assumption (2.1). Consequently, the maximum of $u(t) - w(t)$ is not positive, whence $u(t) \leq w(t)$ for all $t \in J$. This result, (2.14) and (2.15) imply that w is a solution of the BVP

$$\begin{aligned} &-\frac{d}{dt}(\mu(t)w'(t)) = \lambda g(t,\nu_2,w(t),w'(t)) \quad \text{for a.e. } t \in J, \\ &a_0w(t_0) - b_0w'(t_0) = c_0, \qquad a_1w(t_1) + b_1w'(t_1) = c_1, \end{aligned} \tag{2.28}$$

or equivalently, $w = F\nu_2 w$. This proves (a) because $u \leq w$.

The proof of case (b) is similar. □

3. Main results

In the proof of our main existence theorem we need the following special case of a fixed point result proved recently in [7] as a slight modification to [6, Theorem 2.1].

Lemma 3.1. *Let X be an ordered normed space, and let $G : P \subset X \to 2^P \setminus \emptyset$ satisfy the following hypotheses.*

(G1) *The set $P_0 = \{u \in P \mid u \leq v$ for some $v \in G(u)\}$ is nonempty.*

(G2) *If $u_n \leq v_n \in G(u_n)$, $n \in \mathbb{N}$, and if (v_n) is increasing, then $v_n \to v \in P_0$.*

Then G has a maximal fixed point u_+, that is $u_+ \in G(u_+)$, and if $u \in G(u)$ and $u_+ \leq u$, then $u = u_+$.

Now we are ready to prove our main existence theorem.

THEOREM 3.2. *Assume that g satisfies the hypotheses (g0)–(g3), and that* $\lambda \in [0,\lambda_1)$, *where* λ_1 *is the least positive eigenvalue of the operator A, defined by (2.6). Then the BVP (1.1) has a minimal solution* u_- *and a maximal solution* u_+ *in Y, given by (1.2).*

Proof. Choose $X = C(J)$, equipped with the sup-norm and pointwise ordering. Let $b \in X$ be the solution of (2.7), and let P be defined by (2.10) and F_v, $v \in P$ by (2.12). In view of Proposition 2.4 the relation

$$G(v) = \{u \in X \mid u = F_v u\}, \quad v \in P, \tag{3.1}$$

defines a mapping $G : P \to 2^P \setminus \emptyset$. We will show that the hypotheses $(G1)$ and $(G2)$ of Lemma 3.1 hold. $(G1)$ holds because $-b \in P_0$. To prove $(G2)$, assume that $u_n \le v_n \in G(u_n)$, and that (v_n) is increasing. It follows from [5, (4.1.16)] that

$$|F_v u(t)| \le b(t), \quad \left|\frac{d}{dt} F_v(u(t))\right| \le b(t) \quad \forall t \in J \text{ and } v \in P. \tag{3.2}$$

Thus $G[P] = \cup\{G(v) \mid v \in P\} = \cup\{u \mid u = F_v u,\ v \in P\}$ is a bounded and equicontinuous subset of P. This implies that $v = \lim_n v_n$ exists in X. Because P is closed, then $v \in P$. Since $u_n \le v_n \le v$ for each $n \in \mathbb{N}$, there exists by Proposition 2.6(a) a $w_n \in G(v)$ such that $v_n \le w_n$. Since the operator F_v is compact with respect to the norm of $C^1(J)$ defined by (2.11), then sequence $(w_n) = (F_v w_n)$ has a subsequence, say (w_k) which has a limit w in $C^1(J)$ in the sense that $w_k \to w$ and $w_k' \to w'$ uniformly on J. Denoting $k_0 = \max\{k(t,s) \mid t,s \in J\}$, it follows from (2.12) that

$$|F_v w_k(t) - F_v w(t)| \le k_0 \lambda \int_{t_0}^{t_1} |g(s,v,w_k(s),w_k'(s)) - g(s,v,w(s),w'(s))| \, ds, \quad t \in J. \tag{3.3}$$

Because $(x,y) \mapsto g(s,v,x,y)$ is continuous by $(g0)$, the above inequality and the dominated convergence theorem imply that $F_v w_k \to F_v w$ in X as $n \to \infty$. Thus it follows from $w_k = F_v w_k$ as $k \to \infty$ that $w = F_v w$, so that $w \in G(v)$. Since $v_k \le w_k$ for each k we get, as $k \to \infty$, that $v \le w \in G(v)$. Thus $v \in P_0$, so that $(G2)$ holds. Thus all the hypotheses of Lemma 3.1 hold, which implies that G has a maximal fixed point u. In particular, $u \in G(u)$, whence $u = F_u u$ by (3.1). In view of (2.12) u is a solution of the integral equation

$$u(t) = \frac{c_0 y_1(t) + c_1 y_0(t)}{D} + \lambda \int_{t_0}^{t_1} k(t,s) g(s,v,u(s),u'(s)) \, ds, \quad t \in J. \tag{3.4}$$

As a maximal fixed point of G, u is a maximal solution of (3.4) in P, and hence by Lemma 2.1 a maximal solution of the BVP (1.1) in P. Since all the solutions of (1.1) are contained in $B \subset P$ by Lemma 2.3, then u is a maximal solution of (1.1) in Y.

The proof of the existence of a minimal solution of (1.1) can be reduced to the above proof, replacing the order relation $\le$ of $C(J)$ by its dual relation $\preceq$, defined by $u \preceq v$ if and only if $v \le u$, using Proposition 2.6(b), and replacing $-b$ by b in the proof of $(G1)$. $\square$

The hypotheses $(g2)$ and $(g3)$ are needed only in the proof of Proposition 2.6. If g is constant with respect to its second argument v, the results of Proposition 2.6 are trivially valid. As a consequence of this remark and Theorem 3.2 we obtain the following result.

Proposition 3.3. *Assume that $q : J \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function satisfying*

(qa) $|q(t,x,y)| \le p(t)\max\{|x|,|y|\} + m(t)$ *for all $x, y \in \mathbb{R}$ and for a.e. $t \in J$, where $p, m \in L^1_+(J)$ and $\|A^n\|^{1/n} < 1$ for some $n \ge 1$ with $A : C(J) \to C(J)$ given by (2.6).*

Then the BVP

$$\begin{gathered} -\frac{d}{dt}(\mu(t)u'(t)) = q(t,u(t),u'(t)) \quad \text{for a.e. } t \in J, \\ a_0u(t_0) - b_0u'(t_0) = c_0, \qquad a_1u(t_1) + b_1u'(t_1) = c_1 \end{gathered} \tag{3.5}$$

has minimal and maximal solutions in Y.

In particular, the following result holds.

Corollary 3.4. *If $q : J \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-bounded Carathéodory function, then the BVP (3.5) has minimal and maximal solutions for each choices of $c_j \in \mathbb{R}$ and $a_j, b_j \in \mathbb{R}_+$, $j = 0, 1$, satisfying $a_0a_1 + a_0b_1 + a_1b_0 > 0$.*

The next result is also a direct consequence of Theorem 3.2.

Proposition 3.5. *Assume that $q : J \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function satisfying (qa) and the following hypothesis.*

(qb) *There exists a $p_1 \in L^1_+(J)$ and an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_{0+}^1 (dz/\phi(z)) = \infty$ such that $|q(t,x,z) - q(t,x,y)| \le p_1(t)\phi(|z-y|)$ for a.e. $t \in J$ and all $x, y, z \in \mathbb{R}$.*

If $h : J \times \mathbb{R} \to \mathbb{R}$ is an L^1-bounded and sup-measurable function, and if $h(t,\cdot)$ is increasing for a.e. $t \in J$, then the BVP

$$\begin{gathered} -\frac{d}{dt}(\mu(t)u'(t)) = q(t,u(t),u'(t)) + h(t,u(t)) \quad \text{for a.e. } t \in J, \\ a_0u(t_0) - b_0u'(t_0) = c_0, \qquad a_1u(t_1) + b_1u'(t_1) = c_1 \end{gathered} \tag{3.6}$$

has minimal and maximal solutions in Y.

Proof. The function $g(t,v,x,y) = q(t,x,y) + h(t,v(t))$ satisfies the hypotheses $(g0)$–$(g3)$. $\square$

The next result is a special case to Theorem 3.2.

Proposition 3.6. *Assume that the function $g : J \times C(J) \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses.*

(ga) *$g(\cdot,v,x)$ is measurable, $g(t,\cdot,x)$ is increasing and $g(t,v,\cdot)$ is continuous for a.e. $t \in J$ and for all $v \in C(J)$ and $x \in \mathbb{R}$.*

(gb) *$|g(t,v,x)| \le p(t)|x| + m(t)$ for all $x \in \mathbb{R}$, for all $v \in C(J)$, and for a.e. $t \in J$, where $p, m \in L^1_+(J)$.*

Then the BVP

$$-\frac{d}{dt}(\mu(t)u'(t)) = \lambda g(t,u,u(t)) \quad \textit{for a.e. } t \in J,$$
$$a_0 u(t_0) - b_0 u'(t_0) = c_0, \qquad a_1 u(t_1) + b_1 u'(t_1) = c_1 \tag{3.7}$$

has minimal and maximal solutions in Y whenever $\lambda \in [0,\lambda_0)$, where λ_0 is the least positive eigenvalue of the operator $T : C(J) \to C(J)$, defined by

$$Tu(t) = \int_{t_0}^{t_1} k(t,s)p(s)u(s)ds, \quad t \in J. \tag{3.8}$$

Example 3.7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $q : J \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\varphi(x-y+j/k)}{2^{|j|+k}}, \quad \text{where } \varphi(z) = \begin{cases} z\cos\dfrac{1}{z}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

$$q(t,x,y) = \frac{f(x,y)}{1+|f(x,y)|} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2+\sin\left(1/(1+[k^{1/j}t]-k^{1/j}t)\right)}{(kj)^2}, \quad t \in J,\ x,y \in \mathbb{R}. \tag{3.9}$$

The function q satisfies the hypotheses of Corollary 3.4. Thus the BVP (3.5) has for each choices of $a_j, b_j \in \mathbb{R}_+$, $j = 0,1$, satisfying $a_0a_1 + a_0b_1 + a_1b_0 > 0$, and $c_0, c_1 \in \mathbb{R}$ minimal and maximal solutions. By Proposition 3.5 this result holds also for the BVP (3.6) if $q : J \times \mathbb{R}^2 \to \mathbb{R}$ and $h : J \times \mathbb{R} \to \mathbb{R}$ are defined by

$$q(t,x,y) = \frac{\sin(y)\varphi(x)}{1+|\varphi(x)|} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2+\sin\left(1/(1+[k^{1/j}t]-k^{1/j}t)\right)}{(kj)^2}, \quad t \in J,\ x,y \in \mathbb{R},$$
$$h(t,x) = \arctan\left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{[2+[k^{1/j}t]-k^{1/j}t]+[k^{1/j}x]}{(kj)^2}\right), \quad t \in J,\ x \in \mathbb{R}, \tag{3.10}$$

where $[z]$ denotes the greatest integer $\leq z$.

Remark 3.8. The hypothesis $(g2)$ of our main existence result, Theorem 3.2, is weaker than that of [1, Theorem 2.1] and [5, Theorem 4.1.1], because $g(t,v,\cdot,y)$ is not assumed to be decreasing. Thus we cannot use fixed point results for single-valued operators in the proof of Theorem 3.2. This is the reason that instead of least and greatest solutions we can prove the existence of minimal and maximal solutions of the BVP (1.1). The hypothesis $(g3)$ is also somewhat weaker than the corresponding hypothesis in [1, 5]. In fact, the proof of Proposition 2.6 could be carried out also when $(g3)$ is replaced by the following hypothesis.

- $(g3')$ For each fixed $v \in C(J)$ there exists an $r_v > 0$ and a $p_v \in L^1_+(J)$ such that $|g(t,v,x,y) - g(t,v,x,z)| \leq p_v(t)\phi_v(|y-z|)$ for a.e. $t \in J$ and for all $x,y,z \in \mathbb{R}$, $|y-z| \leq r_v$, where $\phi_v : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $\int_{0+}^{r_v}(dx/\phi_v(x)) = \infty$.

For instance, ϕ_v can be any one of the functions: $\phi_0(z) = z$, $z \in \mathbb{R}_+$, and

$$\phi_n(z) = \begin{cases} z \ln \dfrac{1}{z} \cdots \ln_n \dfrac{1}{z}, & 0 < z \le r_n = \dfrac{1}{\exp_n(1)}, \\ 0, & z = 0,\ n = 1, 2, \dots, \end{cases} \tag{3.11}$$

where $\ln_n$ and $\exp_n$ denote the n-fold iterated logarithm and exponential functions, respectively.

Implicit discontinuous Sturm-Liouville BVP's are studied, for example, in [1, 3, 4, 5, 8]. The special case when $\mu(t) \equiv 1$ is considered in [3, 4, 8, 10]. The existence of positive solutions for the Sturm-Liouville BVP's is studied in [2], and uniqueness results in [1, 5, 8, 11].

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