ON STRONG UNIFORM DISTRIBUTION IV

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Received 24 January 2003

Let $a = (a_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let $\mathcal{A}$ be a space of Lebesgue measurable functions defined on $[0,1)$. Let $\{y\}$ denote the fractional part of the real number y. We say that a is an $\mathcal{A}^*$ sequence if for each $f \in \mathcal{A}$ we set $A_N(f,x) = (1/N)\sum_{i=1}^{N} f(\{a_i x\})$ $(N = 1,2,\ldots)$, then $\lim_{N\to\infty} A_N(f,x) = \int_0^1 f(t)dt$, almost everywhere with respect to Lebesgue measure. Let $V_q(f,x) = (\sum_{N=1}^{\infty} |A_{N+1}(f,x) - A_N(f,x)|^q)^{1/q}$ $(q \geq 1)$. In this paper, we show that if a is an $(L^p)^*$ for $p > 1$, then there exists $D_q > 0$ such that if $\|f\|_p$ denotes $(\int_0^1 |f(x)|^p dx)^{1/p}$, $\|V_q(f,\cdot)\|_q \leq D_q\|f\|_p$ $(q > 1)$. We also show that for any $(L^1)^*$ sequence a and any nonconstant integrable function f on the interval $[0,1)$, $V_1(f,x) = \infty$, almost everywhere with respect to Lebesgue measure.

1. Introduction

Let $a = (a_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let $\mathcal{A}$ be a space of Lebesgue measurable functions defined on $[0,1)$. Let $\{y\}$ denote the fractional part of the real number y. Following Marstrand [3] we say that a is an $\mathcal{A}^*$ *sequence* if for each $f \in \mathcal{A}$ we set

$$A_N(f,x) = \frac{1}{N}\sum_{i=1}^{N} f(\{a_i x\}) \quad (N = 1,2,\ldots), \tag{1.1}$$

then

$$\lim_{N\to\infty} A_N(f,x) = \int_0^1 f(t)dt, \tag{1.2}$$

almost everywhere with respect to Lebesgue measure. We know that any strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ is a C^* sequence where C denotes the space of continuous functions on $[0,1)$. This is because of Weyl's theorem [9] that for any strictly increasing sequence of integers (a_n), the fractional parts $(\{a_n x\})_{n=1}^{\infty}$ are uniformly distributed modulo one for almost all x with Lebesgue measure. On the other hand as shown in [3], the sequence $a_n = n$ $(n = 1,2,\ldots)$ is not an $(L^\infty)^*$. There are however examples of sequences of

Journal of Inequalities and Applications 2005:3 (2005) 319–327
DOI: 10.1155/JIA.2005.319

integers that are $(L^p)^*$ $p \geq 1$ and indeed $(L^1(\log L)^k)^*$. These are constructed by primarily ergodic means [3, 4, 5, 6, 8]. Here of course L^p denotes the space of functions f such that the norm $\|f\|_p = \int_0^1 |f(x)|^p dx$ is finite and $L^1(\log_+ L)^k$ denotes the space of L^1 functions such that $\int_0^1 |f|(\log_+ |f|)^{k-1}(x)dx$ is finite. As usual $\log_+ x$ denotes $\log\max(1,x)$. While it is possible to pose many of the questions considered in this subject and indeed this paper for many Banach spaces of measurable functions $\mathcal{A}$, they are perhaps primarily of interest in the context of L^p spaces and perhaps $L^1(\log_+ L)^k$. Note that

$$\mathrm{Span}\left(\cup_{p>1} L^p\right) \subseteq L\left(\log_+ L\right)^d \subseteq L^1, \tag{1.3}$$

where the inclusions are strict in both cases for each $d \geq 1$. Here Span(A) denotes the linear space spanned by the set A. A natural question which arises is whether if (1.2) is known for a particular sequence $a = (a_n)_{n=1}^\infty$ and a particular function f, anything can be said about the rate at which the averages $(A_N(f,x))_{N=1}^\infty$ converge to $\int_0^1 f(t)dt$ almost everywhere. Using [1, Theorem 1] and the Denjoy-Koksma inequality [2] it can be shown that if f is of bounded variation, for any strictly increasing sequence of integers $(a_n)_{N=1}^\infty$, then given $\epsilon > 0$,

$$A_N(f,x) = \int_0^1 f(t)dt + O\left(N^{-1/2}(\log N)^{3/2+\epsilon}\right), \tag{1.4}$$

almost everywhere with respect to Lebesgue measure. As standard, for two sequences, $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$, by $f_n = O(g_n)$ we mean there exists a constant $C > 0$ such that $|f_n| \leq C|g_n|$ for all $n \geq 1$. The class of functions of bounded variation is however quite restrictive and if we look at a broader class of functions, problems arise. For instance, it can be shown that there exist sequences of integers $a = (a_n)_{n=1}^\infty$ for which (1.2) is true for all elements f of some L^q class, but for which for any null sequence $(b_n)_{n=1}^\infty$,

$$A_N(f,x) = \int_0^1 f(t)dt + O(b_N), \tag{1.5}$$

almost everywhere with respect to Lebesgue measure fails to be true for some f in L^∞ [7]. This means that assuming (1.2) to get more information about the sequence $(A_N(f,x))_{N=1}^\infty$ as N tends to infinity, we will have to consider something other than pointwise convergence. We could, for instance, consider norm convergence, that is, ask if it were true that

$$\lim_{N\to\infty} \left\| A_N(f,x) - \int_0^1 f(t)dt \right\|_p = 0. \tag{1.6}$$

Using Lemma 2.2 below and the dominated convergence theorem, (1.6) follows immediately from (1.2) if $a = (a_n)_{n=1}^\infty$ is an $(L^p)^*$ sequence and hence is not of much additional interest. However (1.6) implies that

$$\lim_{N\to\infty} \left\| A_{N+1}(f) - A_N(f) \right\|_p = 0. \tag{1.7}$$

It is (1.7) which admits a nontrivial refinement. One can prove that for a particular $a = (a_n)_{N=1}^{\infty}$ and a particular $p > 1$ if a is $(L^p)^*$, then (1.7) follows from (1.2) without recourse to the rather sophisticated Lemma 2.2. To see this argue as follows. First note that, in light of the bounded convergence theorem if g is in L^{∞}, then (1.2) implies that

$$\lim_{N\to\infty} \left\| A_N(g) - \int_0^1 g(t)dt \right\|_p = 0. \tag{1.8}$$

Now if we are given $\epsilon > 0$, there exists a natural number $n = n(\epsilon, g)$ such that if $N > n$ and k is a positive integer, then

$$\lim_{N\to\infty} \|A_{N+k}(g) - A_N(g)\|_p = 0. \tag{1.9}$$

Now consider a general function f in L^p. Notice that for each $N \geq 1$,

$$\|A_N(f)\|_p \leq \|f\|_p. \tag{1.10}$$

Suppose we are given $\epsilon > 0$ and g is an L^{∞} function with $\|f - g\|_p \leq \epsilon/3$. Then

$$\begin{aligned} &\|A_{N+1}(f) - A_N(f)\|_p \\ &\quad \leq \|A_N(f) - A_N(g)\|_p + \|A_{N+1}(f) - A_{N+1}(g)\|_p + \|A_{N+1}(g) - A_N(g)\|_p \end{aligned} \tag{1.11}$$

which is less than ϵ if $N > n(\epsilon, g)$. Thus (1.7) is proved.

Let

$$V_q(f,x) = \left(\sum_{N=1}^{\infty} |A_{N+1}(f,x) - A_N(f,x)|^q \right)^{1/q} \quad (q \geq 1). \tag{1.12}$$

Our refinement of (1.7) is the following theorem.

Theorem 1.1. *Suppose* $a = (a_n)_{n=1}^{\infty}$ *is an* $(L^p)^*$ *sequence for each* $p > 1$, *then if* $q > 1$, *then there exists a constant* $D_q > 0$ *such that*

$$\|V_q(f,\cdot)\| \leq D_q \|f\|_p \quad (q > 1). \tag{1.13}$$

When $q = 1$, this seems to break down.

Theorem 1.2. *For any* $(L^1)^*$ *sequence* $a = (a_n)_{N=1}^{\infty}$ *and any nonconstant integrable function* f *defined on* $[0,1)$,

$$V_1(f,x) = \infty, \tag{1.14}$$

almost everywhere with respect to Lebesgue measure.

Let $M = (M_t)_{t=1}^{\infty}$ denote a strictly increasing sequence of integers and let

$$V_q(f,M,x) = \left(\sum_{N=1}^{\infty} |A_{M_{t+1}}(f,x) - A_{M_t}(f,x)|^q \right)^{1/q} \quad (q \geq 1). \tag{1.15}$$

It would be interesting to know if Theorem 1.1 can be generalised to show that for each M and $q > 1$ there exists $D'_p > 0$ such that

$$\left|\left|V_q(f,M,\cdot)\right|\right|_q \leq D'_p \| f \|_p. \tag{1.16}$$

By a modification of the proof of Theorem 1.1, the author has verified the special case of (1.16) where $M_t \approx t^\rho$ for $\rho \geq 1$. For two sequences $(a_t)_{t=1}^\infty$ and $(b_t)_{t=1}^\infty$, $a_t \approx b_t$ means $a_t = O(b_t)$ and $b_t = O(a_t)$ as t tends to infinity. Henceforth in this paper C refers to a constant, not necessarily the same on each occurrence.

2. Proof of Theorem 1.1

From the definition of $A_N(f,x)$ we have

$$\left(A_{N+1}(f,x) - A_N(f,x)\right) = \frac{1}{N+1}\left(f\left(\langle a_N x\rangle\right) - A_N(f,x)\right). \tag{2.1}$$

So using the $l^q(\mathbf{Z})$ triangle inequality,

$$\begin{aligned} V_q(f,x) \leq {} & \left(\sum_{N\geq 1}\left(\frac{f(\{a_N x\})}{N+1}\right)^q\right)^{1/q} + \left(\sum_{N\geq 1}\left(\frac{A_N(f,x)}{N+1}\right)^q\right)^{1/q} \\ & \times G_1(f,x) + G_2(f,x). \end{aligned} \tag{2.2}$$

For a subset A of $[0,1)$, we use $|A|$ to denote its Lebesgue measure. We use the following lemma [6].

Lemma 2.1. *Suppose $a = (a_n)_{n=1}^\infty$ is an $(L^p)^*$ sequence, then there exists $C > 0$ such that if f is in L^p, then if*

$$M_a f(x) = \sup_{N\geq 1}\left|\frac{1}{N}\sum_{k=1}^N f(\{a_k x\})\right|, \tag{2.3}$$

$$\left|\{x \in [0,1) : Mf(x) :> \alpha\}\right| \leq \frac{C}{\alpha^p}\| f \|_p. \tag{2.4}$$

Before we proceed we need another lemma. Recall that

$$\| f \|_\infty = \inf\{M : \left|\{x : |f(x)| > M\}\right| = 0\}. \tag{2.5}$$

Lemma 2.2. *Suppose f is in $L^p([0,1))$ and that (2.4) holds with $p > 1$ and $p' > p$, then there exists C such that*

$$\left|\left|M_a f\right|\right|_{p'} \leq C\| f \|_{p'}. \tag{2.6}$$

Proof. First notice that by the way $\| \cdot \|_\infty$ norm is defined there exists C such that

$$\left|\left|M_a f\right|\right|_\infty \leq C\| f \|_\infty. \tag{2.7}$$

Lemma 2.2 now follows in light of the Marcinkiewiez interpolation theorem [10, page 111]. □

Notice that there exists $C > 0$ such that

$$G_2 f(x) \le CM(f)(x). \tag{2.8}$$

This means that G_2 inherits the estimates of Mf so

$$\|G_2 f\|_p \le C\|f\|_p \quad (p > 1). \tag{2.9}$$

We now show that for $p > 1$

$$\|G_1 f\|_p \le C\|f\|_p. \tag{2.10}$$

Set

$$f(\{a_k x\}) = e_k(x) + f_k(x), \tag{2.11}$$

where

$$\begin{aligned} e_k(x) &= f(\{a_k x\}) I_{[f(\{a_k x\}) \le (k+1)]}, \\ f_k(x) &= f(\{a_k x\}) I_{[f(\{a_k x\}) > (k+1)]} \end{aligned} \tag{2.12}$$

with I_A denoting the indicator function of the set A. This means by Minkowski's inequality that

$$G_1 f(x) \le B_1 f(x) + B_2 f(x), \tag{2.13}$$

where

$$B_1 f(x) = \left(\sum_{n \ge 0} \left(\frac{e_n(x)}{n+1} \right)^q \right)^{1/q}, \qquad B_2 f(x) = \left(\sum_{n \ge 0} \left(\frac{f_n(x)}{n+1} \right)^q \right)^{1/q}. \tag{2.14}$$

We therefore know that

$$\|G_1 f\|_p \le \|B_1 f\|_p + \|B_2 f\|_p, \tag{2.15}$$

hence our result is proved if we show that there exists $C_p > 0$ such that

$$\|B_i f\|_p \le C_p \|f\|_p \tag{2.16}$$

for each $i = 1, 2$. We prove something slightly stronger. That is, we show that

$$|\{x \in X : B_i f(x) \ge \lambda\}| \le C_p \frac{\int_0^1 |f| dx}{\lambda}. \tag{2.17}$$

The Marcinkiewiez interpolation gives (2.16). The bound (2.10) follows from (2.16). We first prove (2.16) with $i = 1$,

$$\mu\left(\left\{x \in X : B_1 f(x) > \frac{\lambda}{2}\right\}\right) \le \frac{C}{\lambda^q} \int_0^1 \sum_{n=0} \left(\frac{e_n(x)}{n+1} \right)^q dx = C\lambda^{-q} \sum_{n \ge 0} \left(\frac{1}{n+1} \right)^q \int_0^1 e_n(x)^q dx \tag{2.18}$$

which, as

$$\int_0^1 e_n^q(x)dx \le C\int_0^\infty y^{q-1}\,|\,\{x\in X : e_n(x) > y\}\,|\,dy, \tag{2.19}$$

is

$$\frac{C}{\lambda^q}\sum_{n\ge 0}\left(\frac{1}{n+1}\right)^q\int_0^\infty y^{q-1}\,|\,\{x\in X : e_n(x) > y\}\,|\,dy. \tag{2.20}$$

The map $x \to \{a_n x\}$ preserves, Lebesgue measure on $[0,1)$, that is, for any Lebesgue measurable set A in $[0,1)$,

$$|A| = |\,\{x : \{a_n x\} \in A\}\,|. \tag{2.21}$$

From this it follows that $\int_0^1 f(\{a_n x\})dx = \int_0^1 f(x)dx$ for any L^1 function f. The identity is evident where $f = I_A$, for some Lebesgue measurable A and for simple f by taking linear combinations. The case for general integrable f follows by approximating f by a sequence of simple functions in L^1 norm. This and the definition of e_n tells us that (2.20) is less than or equal to

$$\frac{C}{\lambda^q}\sum_{n\ge 0}\left(\frac{1}{n+1}\right)^q\int_0^{\lambda(n+1)} y^{q-1}\,|\,\{x\in X : f(x) > y\}\,|\,dy \tag{2.22}$$

which is less than or equal to

$$\frac{C}{\lambda^q}\int_0^\infty \sum_{n\ge [y/\lambda]}\left(\frac{1}{n+1}\right)^q y^{q-1}\,|\,\{x\in X : f(x) > y\}\,|\,dy. \tag{2.23}$$

This is less than or equal

$$\frac{C}{\lambda^q}\int_0^\infty y^{q-1}\left(\frac{\lambda}{y}\right)^{1-q}|\,\{x\in X : f(x) > y\}\,|\,dy, \tag{2.24}$$

and is equal to

$$\frac{C}{\lambda}\int_0^\infty |\,\{x : f(x) > y\}\,|\,dy \tag{2.25}$$

which is equal to

$$C\int_0^1 |f|(y)dy. \tag{2.26}$$

Because $q > 1$, this is finite and we have shown (2.16). We now show (2.16), $i = 2$. Here

$$\mu(\{B_2 f(x) > 0\}) \le \sum_{n\ge 0} |\{x : e_n(x) > 0\}| \tag{2.27}$$

which using the fact $x \to \{a_n x\}$ is Lebesgue measure preserving is less than or equal to

$$\begin{aligned}
&\sum_{n\ge 0} |\{x : f(x) > \lambda(n+1)\}| \\
&\qquad \le \int_0^\infty |\{x : f(x) > y\}|\, dy \\
&\qquad \le \frac{1}{\lambda}\int_0^1 |f|(y)\,dy.
\end{aligned} \tag{2.28}$$

This completes the proof of Theorem 1.1.

The proof of Theorem 1.1 crucially uses the fact that $G_2(f,x) \le CM_a f(x)$. It is natural to ask if

$$V_q(f,x) \le CM_a f(x). \tag{2.29}$$

It turns out this is not true in general. To see this argue as follows. We consider the sequence $a_k = 2^k$ $(k = 1,2,\dots)$. For a natural number k and a set contained in $[0,1)$ let

$$kB = \{\{kx\} : x \in B\}. \tag{2.30}$$

For a large natural number L let C denote the interval $[(2^L - 2)/2^L, (2^L - 1)/2^L]$. Note that

$$C, 2^1 C, \dots, 2^{(L-1)} C \tag{2.31}$$

are pairwise disjoint,

$$g_l(x) = \begin{cases} 2^l & \text{if } x \in 2^{(2^l-1)}C,\ 1 \le 2^l - 1 < L, \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Note that

$$\begin{aligned}
M_a f(x) &= \sup_{l\ge 1}\left|\frac{1}{l}\sum_{k=0}^{l} f(\{2^k x\})\right| = \sup_{\substack{l\ge 1\\ 2^l<N+1}} \frac{1}{2^l}\sum_{k=1}^{l} f(\{2^{2^k-1}x\}) \\
&= \sup_{\substack{l\ge 1\\ 2^l<L+1}} \frac{1}{2^l}\sum_{k=1}^{l} 2^k = \frac{2^{l+1}}{2^l} = 2.
\end{aligned} \tag{2.33}$$

On the other hand if $2^m \leq N < 2^{m+1}$, for x in C,

$$\begin{aligned} V_q(f,x) &= \left(\sum_{N=0}^{\infty} \left| A_{N+1}(f,x) - A_N(f,x) \right|^q \right)^{1/q} \\ &= \left(\sum_{N=0}^{\infty} \left| g_{N+1}(x) - g_N(x) \right|^q \right)^{1/q} \\ &\geq \left(\sum_{N=0}^{2m} \left| g_{N+1}(x) - g_N(x) \right|^q \right)^{1/q} \\ &\geq \left(\sum_{N=0}^{m} \left| g_{2^{N+1}}(x) - g_{2^N}(x) \right|^q \right)^{1/q} \\ &\geq \left(\sum_{N=0}^{m} \left| \frac{2^{N+1}}{2^N} - \frac{2^N}{2^N} \right|^q \right)^{1/q} \\ &= m^{1/q}. \end{aligned} \tag{2.34}$$

This tells us that (2.29) is not true in general.

3. Proof of Theorem 1.2

Let

$$E(\delta) = \left\{ x \in X : \left| f(x) - \int_0^1 f(x)dx \right| > \delta \right\}, \tag{3.1}$$

and note that

$$\left| A_{N+1}f(x) - A_N f(x) \right| = \frac{1}{N+1} \left| A_{N+1}f(x) - f(\{a_n x\}) \right|. \tag{3.2}$$

Because a is $(L^1)^*$, there exists $N_0(x)$ such that if $N > N_0(x)$, for almost all x we have

$$\left| A_N f(x) - \int_0^1 f(x)dx \right| < \frac{\delta}{2}. \tag{3.3}$$

Thus

$$\left| A_{N+1}f(x) - A_N f(x) \right| \geq \frac{1}{N+1} \left| A_N f(x) - f(\{a_n x\}) \right| - \frac{\delta}{2(N+1)}. \tag{3.4}$$

So if $\{a_n x\}$ is in $E(\delta)$, we have

$$\left| A_{N+1}(f,x) - A_N(f,x) \right| \geq \frac{\delta}{2(N+1)}. \tag{3.5}$$

This means that

$$V_1(f,x) \geq \sum_{N \geq N_0(x)} \frac{\delta}{2(N+1)} \chi_{E(\delta)}(\{a_n x\}) \times \frac{\delta}{2}\left(\sum_{l \geq N_0(x)} \frac{1}{l+2}\left(\frac{1}{l+1}\sum_{n=N_0(x)}^{l} \chi_{E(\delta)}(\{a_n x\})\right)\right) \tag{3.6}$$

which for suitably large $N_0(x)$ is greater than or equal to

$$\frac{\delta}{2}\left(\frac{\mu(E(\delta))}{2}\right)\sum_{l \geq N_0(x)} \frac{1}{N+1} = \infty, \tag{3.7}$$

as required.

References

[1] H. Kamarul Haili and R. Nair, *The discrepancy of some real sequences*, Math. Scand. **93** (2003), no. 2, 268–274.

[2] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1974.

[3] J. M. Marstrand, *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc. (3) **21** (1970), 540–556.

[4] R. Nair, *On strong uniform distribution*, Acta Arith. **56** (1990), no. 3, 183–193.

[5] ______, *On strong uniform distribution. II*, Monatsh. Math. **132** (2001), no. 4, 341–348.

[6] ______, *On strong uniform distribution. III*, Indag. Math. (N.S.) **14** (2003), no. 2, 233–240.

[7] ______, *On strong uniform distribution. V*, unpublished manuscript.

[8] F. Riesz, *Sur la théorie ergodique*, Comment. Math. Helv. **17** (1945), 221–239 (French).

[9] H. Weyl, *Über de Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (German).

[10] A. Zygmund, *Trigonometric Series. Vols. I, II*, 2nd ed., Cambridge University Press, New York, 1959.

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