

LOCAL SOLUTION OF CARRIER'S EQUATION IN A NONCYLINDRICAL DOMAIN

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We study Carrier's equation in a noncylindrical domain. We use the penalty method combined with Faedo-Galerkin and compactness arguments. We obtain results of the existence and uniqueness of the local solution.

1. Introduction

The wave equation of the type

$$\frac{\partial^2 u}{\partial t^2} - \left[P_0 + P_1 \int_{\alpha}^{\beta} \left(\frac{\partial u}{\partial x}(t) \right)^2 dx \right] \frac{\partial^2 u}{\partial x^2} = 0 \quad (1.1)$$

is a model for small vibrations of an elastic string fixed at α, β with P_0 and P_1 constants, and was investigated by Kirchhoff [4] and Carrier [1]. By approximations on the above equation, Carrier obtained the model

$$\frac{\partial^2 u}{\partial t^2} - \left[P_0 + P_1 \int_{\alpha}^{\beta} (u(t))^2 dx \right] \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which is known in the literature as Carrier's equation. Medeiros et al. [7, 8] studied the problem of small vibrations of an elastic string with moving boundary.

In this work, we study the following generalization of (1.2):

$$\frac{\partial^2 u}{\partial t^2} - \widehat{M} \left(x, t, \int_{\alpha(t)}^{\beta(t)} (u(t))^2 dx \right) \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } \widehat{Q}, \quad (1.3a)$$

$$u = 0 \quad \text{on } \widehat{\Sigma}, \quad (1.3b)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } (\alpha(0), \beta(0)), \quad (1.3c)$$

where

(i) $\widehat{Q} = \bigcup_{0 < t < T}]\alpha(t), \beta(t)[\times \{t\}$ is a noncylindrical domain,

(ii) $\widehat{\Sigma} = \bigcup_{0 < t < T} \{(\alpha(t), t), (\beta(t), t)\}$ is the lateral boundary of \widehat{Q} .

We use the penalty method as in Ebihara [2] combined with the Faedo-Galerkin method and compactness arguments. We obtain results of the existence and uniqueness of the local solution. This method was used by several authors, for example, Ebihara et al. [3], Pereira et al. [9]. We use some ideas contained in [7, 8] to obtain the necessary estimates.

2. Notations, assumptions, and main results

We use the notations contained in Lions and Magenes [6] and Lions [5] for the spaces $L^p(\Omega)$, $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$. In particular, for $p = 2$, $m = 1$, and $\Omega = (0, 1)$, we consider $L^2(0, 1)$ and $H_0^1(0, 1)$ with the scalar product and norms represented by (\cdot, \cdot) , $|\cdot|$, $((\cdot, \cdot))$, and $\|\cdot\|$, respectively, and given by

$$\begin{aligned} (u, v) &= \int_0^1 u(x)v(x)dx, & |u|^2 &= \int_0^1 (u(x))^2 dx, \\ ((u, v)) &= \int_0^1 \frac{du}{dx}(x) \frac{dv}{dx}(x)dx, & \|u\|^2 &= \int_0^1 \left| \frac{du}{dx}(x) \right|^2 dx. \end{aligned} \tag{2.1}$$

We consider the following assumptions about \widehat{M} :

- (M1) $\widehat{M} \in L_{loc}^\infty([0, \infty), W^{1,\infty}(\widehat{Q}))$ and for each $L > 0$, $\partial\widehat{M}/\partial\lambda \in L^\infty(\widehat{Q} \times (0, L))$;
- (M2) \widehat{M} , $\partial\widehat{M}/\partial x$, $\partial\widehat{M}/\partial\lambda$, $\partial\widehat{M}/\partial t$ are continuous with respect to λ , a.e. $(x, t) \in \widehat{Q}$;
- (M3) there exists a real number $m_0 > 0$ such that $\widehat{M}(x, t, \lambda) \geq m_0$ for all $(x, t) \in \widehat{Q}$ and $\lambda \geq 0$.

With respect to the noncylindrical domain \widehat{Q} we consider the following assumptions:

- (H1) $\alpha, \beta \in C^3([0, T])$, $\alpha(t) < \beta(t)$, for all $t \in [0, T]$;
- (H2) $\alpha'(t) < 0$, $\beta'(t) > 0$, for all $t \in (0, T]$, and $\alpha'(0) = 0 = \beta'(0)$;
- (H3) $\gamma'(t) \leq \sqrt{m_0}/2$, for all $t \in [0, T]$, where $\gamma(t) = \beta(t) - \alpha(t)$;
- (H4) $|\alpha''(t) + \gamma''(t)y| \leq |\alpha'(t) + \gamma'(t)y|^2/\gamma(t)$, for all $t \in [0, T]$, $y \in [0, 1]$.

We observe that when (x, t) varies in \widehat{Q} the point (y, t) of \mathbb{R}^2 , with $y = (x - \alpha(t))/\gamma(t)$, varies in $Q = (0, 1) \times (0, T)$, so we define the diffeomorphism $\tau : \widehat{Q} \rightarrow Q$ by

$$\tau(x, t) = (y, t), \quad \text{where } y = \frac{x - \alpha(t)}{\gamma(t)}. \tag{2.2}$$

We also consider the diffeomorphism $\widehat{\tau} : \widehat{Q} \times [0, \infty) \rightarrow Q \times [0, \infty)$ defined by $\widehat{\tau}(x, t, \lambda) = (y, t, \lambda)$ with y as above.

Denoting

$$\begin{aligned} v(y, t) &= (u \circ \tau^{-1})(y, t), \\ M(y, t, \lambda) &= (\widehat{M} \circ (\widehat{\tau})^{-1})(y, t, \lambda), \\ g(y, t) &= (f \circ \tau^{-1})(y, t), \end{aligned} \tag{2.3}$$

$v_0(y) = u_0(\gamma_0 y + \alpha_0)$, and $v_1 = (\gamma_0 y + \alpha_0)$, where $\gamma_0 = \gamma(0)$, $\alpha_0 = \alpha(0)$, and defining

$$\begin{aligned}
 b(y, t) &= -2 \frac{\alpha'(t) + \gamma'(t)y}{\gamma(t)}, \\
 c(y, t) &= - \left[\frac{\alpha''(t) + \gamma''(t)y}{\gamma(t)} - 2\gamma'(t) \frac{\alpha'(t) + \gamma'(t)y}{\gamma^2(t)} \right],
 \end{aligned}
 \tag{2.4}$$

problem (1.3) is transformed into the following cylindrical problem:

$$v''(y, t) - \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v(t)|^2) - \frac{b^2(y, t)}{4} \right] \frac{\partial^2 v}{\partial y^2}(y, t)
 \tag{2.5a}$$

$$+ b(y, t) \frac{\partial v'}{\partial y}(y, t) + c(y, t) \frac{\partial v}{\partial y}(y, t) = g(y, t), \quad \text{in } Q,$$

$$v = 0 \quad \text{on } \Sigma,
 \tag{2.5b}$$

$$v(y, 0) = v_0(y), \quad v'(y, 0) = v_1(y), \quad \text{in } (0, 1),
 \tag{2.5c}$$

where ' denotes the derivative with respect to t .

If $\Omega_t = (\alpha(t), \beta(t))$ and $\Omega_0 = (\alpha_0, \beta_0)$, where $\alpha_0 = \alpha(0)$ and $\beta_0 = \beta(0)$, the main result of this work is given by the following theorem.

THEOREM 2.1. *Let \widehat{M} satisfy (M1)–(M3) and let $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy (H1)–(H4). Given $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, and $f \in L^2(0, T, H_0^1(\Omega_t))$ with $f' \in L^2(0, T, L^2(\Omega_t))$, there exist $T_0 > 0$ and a unique u satisfying*

$$\begin{aligned}
 u &\in L^\infty(0, T_0, H_0^1(\Omega_t) \cap H^2(\Omega_t)), \\
 u' &\in L^\infty(0, T_0, H^1(\Omega_t)), \\
 u'' &\in L^2(0, T_0, L^2(\Omega_t)),
 \end{aligned}
 \tag{2.6}$$

and u is the solution of (1.3).

First of all we prove the equivalent result in a cylindrical domain.

THEOREM 2.2. *Let $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ be as in (H1)–(H4) and let M satisfy*

(i) $M \in L_{\text{loc}}^\infty([0, \infty), W^{1,\infty}(Q))$ and for each $L > 0$, $\partial M / \partial \lambda \in L^\infty(Q \times (0, L))$,

(ii) $M, \partial M / \partial \lambda, \partial M / \partial t$ are continuous with respect to λ , a.e. $(y, t) \in Q$.

There exists a real number $m_0 > 0$ such that

$$M(y, t, \lambda) \geq m_0, \quad \forall (y, t) \in Q, \lambda \geq 0.
 \tag{2.7}$$

Given $v_0 \in H_0^1(0, 1) \cap H^2(0, 1)$, $v_1 \in H_0^1(0, 1)$, and $g \in L^2([0, T], H_0^1(0, 1))$ with $g' \in L^2([0, T], L^2(0, 1))$, there exist $T_0 > 0$ and a unique v satisfying

$$v \in L^\infty(0, T_0, H_0^1(0, 1) \cap H^2(0, 1)),
 \tag{2.8a}$$

$$v' \in L^\infty(0, T_0, H_0^1(0, 1)),
 \tag{2.8b}$$

$$v'' \in L^\infty(0, T_0, L^2(0, 1)),
 \tag{2.8c}$$

and v is the solution of (2.5).

3. Proof of results

Let $(w_\nu)_{\nu \in \mathbb{N}}$ be the orthonormal complete set of $L^2(0, 1)$ given by the eigenvectors of the operator $-d^2/dx^2$, that is,

$$-\frac{d^2}{dx^2} w_\nu = \lambda_\nu w_\nu, \quad w_\nu = 0 \quad \text{on } \Gamma. \tag{3.1}$$

We represent by $V_m = [w_1, \dots, w_m]$ the subspace of $H_0^1(0, 1) \cap H^2(0, 1)$ generated by the m first vectors w_ν .

We consider $F : (0, \infty) \rightarrow \mathbb{R}$ a function satisfying

$$F \in C^1(0, \infty), \quad F'(\xi) < 0, \quad \forall \xi > 0, \quad F(\xi) = 1, \quad \forall \xi \geq 1. \tag{3.2}$$

$$\text{There exist } \mu_0, \eta_0, \delta > 0 \text{ such that } F(\xi) \geq \frac{\mu_0}{\xi^{\eta_0}}, \quad \forall \xi \in (0, \delta]. \tag{3.3}$$

Let $K > 0$ such that

$$|v_1|^2 < K. \tag{3.4}$$

For each $\varepsilon > 0$, we consider the following penalized problem: for each $m \in \mathbb{N}$, let $v_{\varepsilon m}(t) \in V_m$ satisfy

$$\begin{aligned} & \left(v''_{\varepsilon m}(t) - \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\varepsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t), w \right) \\ & + \left(c(y, t) \frac{\partial v_{\varepsilon m}}{\partial y}(t) + b(y, t) \frac{\partial v'_{\varepsilon m}}{\partial y}(t) + \varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) v'_{\varepsilon m}(t), w \right) = (g(t), w), \end{aligned} \tag{3.5}$$

for all $w \in V_m$, with the initial conditions

$$v_{\varepsilon m}(0) = v_{0m} \longrightarrow v_0 \quad \text{in } H_0^1(0, 1) \cap H^2(0, 1), \tag{3.6a}$$

$$v'_{\varepsilon m}(0) = v_{1m} \longrightarrow v_1 \quad \text{in } H_0^1(0, 1). \tag{3.6b}$$

As in [9] we prove that we can apply Carathéodory’s theorem to (3.5)–(3.6b) obtaining the existence of $T_{\varepsilon m} > 0$ and $v_{\varepsilon m} : [0, T_{\varepsilon m}] \rightarrow V_m$ as a solution of (3.5) satisfying

$$v_{\varepsilon m} \in C^2([0, T_{\varepsilon m}]), \quad |v'_{\varepsilon m}(t)|^2 < K, \quad \forall m \geq m_1, \quad t \in [0, T_{\varepsilon m}], \quad \text{for each } \varepsilon > 0. \tag{3.7}$$

LEMMA 3.1. *There exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$|v''_{\varepsilon m}(0)| \leq C_0, \tag{3.8}$$

for all $m \geq m_1, 0 < \varepsilon < \varepsilon_0$.

Proof. From (3.6) the existence of a constant $N > 0$ follows, such that

$$\|v'_{\varepsilon m}(0)\| \leq N, \quad \|v_{\varepsilon m}(0)\| \leq N, \quad |v_{\varepsilon m}(0)| \leq N, \quad |\Delta v_{\varepsilon m}(0)| \leq N, \quad \forall m \geq m_1. \tag{3.9}$$

From (3.4) there exists $\rho > 0$ such that

$$|v_1|^2 < \rho < K, \tag{3.10}$$

so

$$|v_{1m}|^2 < \rho < K, \quad \forall m \geq m_1. \tag{3.11}$$

Then, for each $0 < \varepsilon < \min\{1, K - \rho\} = \varepsilon_0$, we have

$$\frac{K - |v_{1m}|^2}{\varepsilon} > 1, \quad \forall m \geq m_1, \tag{3.12}$$

thus, from (3.2),

$$F\left(\frac{K - |v_{1m}|^2}{\varepsilon}\right) = 1, \quad \forall m \geq m_1. \tag{3.13}$$

Let

$$M_0 = \max_{[0,1] \times [0, \gamma_0 N^2]} |M(y, 0, \lambda)|. \tag{3.14}$$

We supposed that $\alpha'(0) = \beta'(0) = 0$, then it follows from (H4) that $b(y, 0) = c(y, 0) = 0$, so by considering $t = 0$ and $w = v''_{\varepsilon m}(0)$ in (3.5), we obtain, from (3.13),

$$|v''_{\varepsilon m}(0)| \leq \frac{1}{\gamma_0^2} [M_0 + \gamma_0^2] N + |g(0)|, \tag{3.15}$$

and the proof of the lemma is complete. □

LEMMA 3.2. Fix $\varepsilon > 0$ and $m \geq m_1$. Then

(i) there exists $\lim_{t \rightarrow T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2$ such that $\lim_{t \rightarrow T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 < K$.

Proof. Taking the derivate with respect to t of (3.5) we obtain

$$\begin{aligned} & \left(v'''_{\varepsilon m}(t) + \left[\frac{2\gamma'(t)}{\gamma^3(t)} M(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) - \frac{1}{\gamma^2(t)} M'(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right. \right. \\ & \quad \left. \left. - \frac{\gamma'(t)}{\gamma^2(t)} \frac{\partial M}{\partial \lambda}(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) | v_{\varepsilon m}(t) |^2 - \frac{2}{\gamma(t)} \frac{\partial M}{\partial \lambda}(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right. \right. \\ & \quad \left. \left. \times (v_{\varepsilon m}(t), v'_{\varepsilon m}(t)) + \frac{1}{4} (b^2(y, t))' \right] \frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t) \right. \\ & \quad \left. + \left[\frac{b^2(y, t)}{4} - \frac{1}{\gamma^2(t)} M(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right] \frac{\partial^2 v'_{\varepsilon m}}{\partial y^2}(t) + c'(y, t) \frac{\partial v_{\varepsilon m}}{\partial y}(t) \right. \\ & \quad \left. + [c(y, t) + (b(y, t))'] \frac{\partial v'_{\varepsilon m}}{\partial y}(t) + b(y, t) \frac{\partial v''_{\varepsilon m}}{\partial y}(t) - 2F' \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) \right. \\ & \quad \left. \times (v''_{\varepsilon m}(t), v'_{\varepsilon m}(t)) v'_{\varepsilon m}(t) + \varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) v''_{\varepsilon m}(t), w \right) = (g'(t), w). \end{aligned} \tag{3.16}$$

From (3.7) we obtain that

$$|v_{\varepsilon m}(t)|^2 < K_{\varepsilon m}, \quad \forall t \in [0, T_{\varepsilon m}). \tag{3.17}$$

Let

$$C_{1\varepsilon m} = \sup_{[0,1] \times [0,T] \times [0,\gamma(T)K_{\varepsilon m}]} \text{ess} \left\{ |M(y, t, \lambda)|, |M'(y, t, \lambda)|, \left| \frac{\partial M}{\partial \lambda}(y, t, \lambda) \right| \right\}. \tag{3.18}$$

We remember that

$$\begin{aligned} & |\alpha'(t) + \gamma'(t)y| \leq \gamma'(t), \\ & |\alpha''(t) + \gamma''(t)y| \leq \frac{|\alpha'(t) + \gamma'(t)y|^2}{\gamma(t)}, \quad t \in [0, T], y \in [0, 1], \end{aligned} \tag{3.19}$$

so we obtain from (H3)

$$\begin{aligned}
 \left| \frac{1}{4}(b^2(y, t))' \right| &\leq 4 \left(\frac{y'(t)}{y(t)} \right)^3 \leq \frac{m_0 \sqrt{2m_0}}{\gamma_0^3}, \\
 \left| 2 \frac{y'(t)}{\gamma^3(t)} M(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right| &\leq \frac{\sqrt{2m_0}}{\gamma_0^3} C_{1\varepsilon m}, \\
 \left| \frac{1}{\gamma^2(t)} M(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right| &\leq \frac{1}{\gamma_0^2} C_{1\varepsilon m}, \\
 \left| \frac{1}{\gamma^2(t)} \frac{\partial M}{\partial \lambda}(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) [\gamma'(t) | v_{\varepsilon m}(t) |^2 + 2\gamma(t)(v_{\varepsilon m}(t), v'_{\varepsilon m}(t))] \right| \\
 &\leq K_{\varepsilon m} C_{1\varepsilon m} \left[\frac{1}{\gamma_0^2} \sqrt{\frac{m_0}{2}} + \frac{2}{\gamma_0} K \right], \\
 \left| \frac{b^2(y, t)}{4} - \frac{1}{\gamma^2(t)} M(y, t, \gamma(t) | v_{\varepsilon m}(t) |^2) \right| &\leq \left(\frac{y'(t)}{y(t)} \right)^2 + \frac{1}{\gamma^2(t)} C_{1\varepsilon m} \leq \frac{1}{\gamma_0^2} \left[\frac{m_0}{2} + C_{1\varepsilon m} \right], \\
 |c(y, t) + b'(y, t)| &\leq \frac{7m_0}{2\gamma_0^2}.
 \end{aligned} \tag{3.20}$$

Taking $w = v''_{\varepsilon m}(t)$ in (3.16) it follows from (3.20) that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} |v''_{\varepsilon m}(t)|^2 + \varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) |v''_{\varepsilon m}(t)|^2 - 2F' \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) (v''_{\varepsilon m}(t), v'_{\varepsilon m}(t))^2 \\
 \leq |v''_{\varepsilon m}(t)| \left\{ \left[C_{1\varepsilon m} \left(\frac{\sqrt{2m_0}}{\gamma_0^3} + \frac{1}{\gamma_0^2} + \frac{1}{\gamma_0^2} \sqrt{\frac{m_0}{2}} K_{\varepsilon m} + \frac{2}{\gamma_0} K_{\varepsilon m} K \right) + \frac{m_0 \sqrt{2m_0}}{\gamma_0^3} \right] \left| \frac{\partial^2 v_{\varepsilon m}(t)}{\partial y^2} \right| \right. \\
 \left. + \left[\frac{m_0}{2\gamma_0^2} + \frac{1}{\gamma_0^2} C_{1\varepsilon m} \right] \left| \frac{\partial^2 v'_{\varepsilon m}(t)}{\partial y^2} \right| + C_2 \|v_{\varepsilon m}(t)\| + \left(\frac{3m_0}{2\gamma_0^2} + \frac{2m_0}{\gamma_0^2} \right) \|v'_{\varepsilon m}(t)\| \right\} \\
 + |g'(t)| + \frac{\sqrt{2m_0}}{\gamma_0} |v''_{\varepsilon m}(t)| \|v'_{\varepsilon m}(t)\|,
 \end{aligned} \tag{3.21}$$

where $C_2 = \max_{[0,1] \times [0,T]} |c'(y, t)|$. But in V_m the norms are equivalent, so by using (3.7) and (3.17) we obtain that there exist constants $C_{2\varepsilon m} > 0$ and $C_{3\varepsilon m} > 0$ depending on ε and m such that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} |v''_{\varepsilon m}(t)|^2 + \varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) |v''_{\varepsilon m}(t)|^2 \\
 - 2F' \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) (v''_{\varepsilon m}(t), v'_{\varepsilon m}(t))^2 \leq C_{2\varepsilon m} [|v''_{\varepsilon m}(t)|^2 + |g'(t)|^2] + C_{3\varepsilon m}.
 \end{aligned} \tag{3.22}$$

Using the hypothesis on F and F' , Lemma 3.1, and Gronwall's inequality in (3.22) it follows that

$$|v''_{\varepsilon m}(t)|^2 \leq C_{4\varepsilon m}, \tag{3.23}$$

where $C_{4\varepsilon m} > 0$ is a constant depending on $\varepsilon > 0$ and m .

It follows from (3.7) and (3.23) that $\lim_{t \rightarrow T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2$ exists and

$$\lim_{t \rightarrow T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 \leq K. \tag{3.24}$$

Suppose that

$$\lim_{t \rightarrow T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 = K. \tag{3.25}$$

Taking $w = v'_{\varepsilon m}(t)$ in (3.5), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |v'_{\varepsilon m}(t)|^2 + \varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) |v'_{\varepsilon m}(t)|^2 \\ & + \int_0^1 \left[\frac{b^2(y,t)}{4} - \frac{1}{\gamma^2(t)} M(y,t, \gamma(t) |v_{\varepsilon m}(t)|^2) \right] \frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t) v'_{\varepsilon m}(y,t) dy \\ & + \int_0^1 c(y,t) \frac{\partial v_{\varepsilon m}}{\partial y}(y,t) v'_{\varepsilon m}(y,t) dy + \int_0^1 b(y,t) \frac{\partial v'_{\varepsilon m}}{\partial y}(y,t) v'_{\varepsilon m}(y,t) dy \\ & = \int_0^1 g(y,t) v'_{\varepsilon m}(y,t) dy, \end{aligned} \tag{3.26}$$

but

$$\begin{aligned} \left| \frac{b^2(y,t)}{4} - \frac{1}{\gamma^2(t)} M(y,t, \gamma(t) |v_{\varepsilon m}(t)|^2) \right| & \leq \frac{1}{\gamma_0^2} \left[\frac{m_0}{2} + C_{1\varepsilon m} \right], \\ |c(y,t)| & \leq \frac{3}{\gamma_0^2} \frac{m_0}{2}, \\ |b(y,t)| & \leq \frac{\sqrt{2m_0}}{\gamma_0}, \end{aligned} \tag{3.27}$$

so we have

$$\begin{aligned} & \varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) |v'_{\varepsilon m}(t)|^2 \\ & \leq |v''_{\varepsilon m}(t)| |v'_{\varepsilon m}(t)| + \frac{1}{\gamma_0^2} \left[\frac{m_0}{2} + C_{1\varepsilon m} \right] \left| \frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t) \right| |v'_{\varepsilon m}(y,t)| \\ & + \frac{3}{\gamma_0^2} \frac{m_0}{2} |v_{\varepsilon m}(t)| |v'_{\varepsilon m}(t)| + \frac{\sqrt{2m_0}}{\gamma_0} |v'_{\varepsilon m}(t)| |v'_{\varepsilon m}(t)| + |g(t)| |v'_{\varepsilon m}(t)|, \end{aligned} \tag{3.28}$$

then from (3.23), (3.17), and (3.7) and the equivalence of the norms in V_m there exists a constant $C_{5\epsilon m} > 0$ such that

$$\epsilon F \left(\frac{K - |v'_{\epsilon m}(t)|^2}{\epsilon} \right) |v'_{\epsilon m}(t)|^2 \leq C_{5\epsilon m}. \tag{3.29}$$

We conclude from (3.25) and (3.29) that there exists $\delta_1 > 0$ such that

$$\epsilon F \left(\frac{K - |v'_{\epsilon m}(t)|^2}{\epsilon} \right) \leq C_{5\epsilon m} \sqrt{\frac{2}{K}}, \quad \text{for } t \in (T_{\epsilon m} - \delta_1, T_{\epsilon m}). \tag{3.30}$$

But by (3.25) and the hypothesis on F we have

$$\lim_{t \rightarrow T_{\epsilon m}} \epsilon F \left(\frac{K - |v'_{\epsilon m}(t)|^2}{\epsilon} \right) = \infty, \tag{3.31}$$

which is a contradiction to (3.30). Then the proof is completed. □

From (3.7) and Lemma 3.2(i) we can extend the solution to $[0, T]$ and

$$|v_{\epsilon m}(t)|^2 \leq K_1 \text{ e } |v'_{\epsilon m}(t)|^2 \leq K, \quad \forall t \in [0, T], \tag{3.32}$$

where K and K_1 are independent of $\epsilon > 0$ and $m \geq m_1$.

LEMMA 3.3. *There exists a constant $K_2 > 0$ independent of $\epsilon > 0$ and $m \geq m_1$ such that*

$$\|v'_{\epsilon m}(t)\|^2 + \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2 \leq K_2, \quad \forall t \in [0, T]. \tag{3.33}$$

Proof. We observe that

$$\begin{aligned} \int_0^1 c(y,t) \frac{\partial v_{\epsilon m}(y,t)}{\partial y} \frac{\partial^2 v'_{\epsilon m}(y,t)}{\partial y^2} dy &= c(1,t) \frac{\partial v_{\epsilon m}}{\partial y}(1,t) \frac{\partial v'_{\epsilon m}}{\partial y}(1,t) \\ &\quad - c(0,t) \frac{\partial v_{\epsilon m}}{\partial y}(0,t) \frac{\partial v'_{\epsilon m}}{\partial y}(0,t) - \int_0^1 c(y,t) \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y,t) \frac{\partial v'_{\epsilon m}}{\partial y}(y,t) dy \\ &\quad - \int_0^1 \frac{\partial c}{\partial y}(y,t) \frac{\partial v_{\epsilon m}}{\partial y}(y,t) \frac{\partial v'_{\epsilon m}}{\partial y}(y,t) dy, \\ \int_0^1 b(y,t) \frac{\partial v'_{\epsilon m}}{\partial y}(y,t) \frac{\partial^2 v'_{\epsilon m}}{\partial y^2}(y,t) dy &= \frac{1}{2} \int_0^1 b(y,t) \frac{\partial}{\partial y} \left(\frac{\partial v'_{\epsilon m}}{\partial y}(y,t) \right)^2 dy \\ &= \frac{1}{2} b(1,t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(1,t) \right)^2 - \frac{1}{2} b(0,t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(0,t) \right)^2 \\ &\quad - \frac{1}{2} \int_0^1 \frac{\partial b}{\partial y}(y,t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(y,t) \right)^2 dy, \end{aligned} \tag{3.34}$$

so if we take $w = -(\partial^2 v'_{\epsilon m} / \partial y^2)(y, t)$ in (3.5) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ |v'_{\epsilon m}(t)|^2 + \int_0^1 \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \left(\frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \right)^2 dy \right\} \\ & + \frac{1}{2} \int_0^1 \frac{d}{dt} \left[\frac{b^2(y, t)}{4} - \frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) \right] \left(\frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \right)^2 dy \\ & + c(0, t) \frac{\partial v_{\epsilon m}}{\partial y}(0, t) \frac{\partial v'_{\epsilon m}}{\partial y}(0, t) - c(1, t) \frac{\partial v_{\epsilon m}}{\partial y}(1, t) \frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \\ & + \int_0^1 c(y, t) \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \frac{\partial v'_{\epsilon m}}{\partial y}(y, t) dy + \int_0^1 \frac{\partial c}{\partial y}(y, t) \frac{\partial v_{\epsilon m}}{\partial y}(y, t) \frac{\partial v'_{\epsilon m}}{\partial y}(y, t) dy \\ & + \frac{1}{2} b(0, t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(0, t) \right)^2 - \frac{1}{2} b(1, t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \right)^2 + \frac{1}{2} \int_0^1 \frac{\partial b}{\partial y}(y, t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(y, t) \right)^2 dy \\ & \leq \|g(t)\| \|v'_{\epsilon m}(t)\|. \end{aligned} \tag{3.35}$$

It follows from (H3) and (H4) that

$$\begin{aligned} & \left| c(1, t) \frac{\partial v_{\epsilon m}}{\partial y}(1, t) \frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \right| \\ & \leq \left[2 \frac{\gamma'(t) \beta'(t)}{\gamma^2(t)} + \left(\frac{\beta'(t)}{\gamma(t)} \right)^2 \right] \left| \frac{\partial v_{\epsilon m}}{\partial y}(1, t) \right| \left| \frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \right| \\ & \leq \frac{\beta'(t)}{\gamma(t)} \left| \frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \right|^2 + \frac{4(\gamma'(t))^2 \beta'(t) + (\beta'(t))^3}{2\gamma^3(t)} \left| \frac{\partial v_{\epsilon m}}{\partial y}(1, t) \right|^2 \\ & \leq \frac{\beta'(t)}{\gamma(t)} \left| \frac{\partial v'_{\epsilon m}}{\partial y}(1, t) \right|^2 + \frac{5}{2\gamma_0^3} \left(\frac{m_0}{2} \right)^{3/2} \left(\frac{\pi + 1}{\pi} \right)^2 \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2, \\ & \left| c(0, t) \frac{\partial v_{\epsilon m}}{\partial y}(0, t) \frac{\partial v'_{\epsilon m}}{\partial y}(0, t) \right| \\ & \leq \frac{-\alpha'(t)}{\gamma(t)} \left| \frac{\partial v'_{\epsilon m}}{\partial y}(0, t) \right|^2 + \frac{5}{2\gamma_0^3} \left(\frac{m_0}{2} \right)^{3/2} \left(\frac{\pi + 1}{\pi} \right)^2 \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2, \\ & \int_0^1 \frac{d}{dt} \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \right|^2 dy \\ & = \int_0^1 \left\{ - \frac{2\gamma'(t)}{\gamma^3(t)} M(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) + \frac{1}{\gamma^2(t)} \frac{\partial M}{\partial t}(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) \right. \\ & \quad + \frac{1}{\gamma^2(t)} \frac{\partial M}{\partial \lambda}(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) [\gamma'(t) |v_{\epsilon m}(t)|^2 + 2\gamma(t)(v_{\epsilon m}(t), v'_{\epsilon m}(t))] \\ & \quad + \frac{2}{\gamma^3(t)} (\alpha'(t) + \gamma'(t)y) [(\alpha''(t) + \gamma''(t)y)\gamma(t) - (\alpha'(t) + \gamma'(t)y)\gamma'(t)] \\ & \quad \left. \times \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \right|^2 dy \right\} \end{aligned}$$

$$\begin{aligned}
 &\leq \left\{ M_1 \left[\frac{2}{\gamma_0^3} \left(\frac{m_0}{2} \right)^{1/2} + \frac{1}{\gamma_0^2} \left(\frac{m_0}{2} \right)^{1/2} K_1 + \frac{2}{\gamma_0} K K_1 \right] + \frac{4}{\gamma_0^3} \left(\frac{m_0}{2} \right)^{3/2} \right\} \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2, \\
 &\int_0^1 \frac{\partial c}{\partial y}(y, t) \frac{\partial v_{\epsilon m}}{\partial y}(y, t) \frac{\partial v'_{\epsilon m}}{\partial y}(y, t) dy \leq \frac{2}{\gamma_0^2} m_0 \|v_{\epsilon m}(t)\| \|v'_{\epsilon m}(t)\|, \\
 &\int_0^1 \frac{\partial b}{\partial y}(y, t) \left(\frac{\partial v'_{\epsilon m}}{\partial y}(y, t) \right)^2 dy \leq \frac{2}{\gamma_0} \left(\frac{m_0}{2} \right)^{1/2} \|v'_{\epsilon m}(t)\|^2, \\
 &\int_0^1 c(y, t) \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \frac{\partial v'_{\epsilon m}}{\partial y}(y, t) dy \leq \frac{3}{\gamma_0^2} \frac{m_0}{2} \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right| \|v'_{\epsilon m}(t)\|.
 \end{aligned}
 \tag{3.36}$$

By substituting the above inequalities in (3.35) we obtain

$$\begin{aligned}
 &\frac{1}{2} \left\{ \frac{d}{dt} \|v'_{\epsilon m}(t)\|^2 + \int_0^1 \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\epsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(y, t) \right|^2 dy \right\} \\
 &\leq \left[M_1 \left(\frac{\sqrt{2m_0}}{\gamma_0^3} + \frac{1}{\gamma_0^2} + \frac{1}{\gamma_0^2} \sqrt{\frac{m_0}{2}} K_1 + \frac{2}{\gamma_0} \sqrt{K_1 K} \right) + \frac{m_0 \sqrt{2m_0}}{\gamma_0^3} \right] \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2 \\
 &\quad + \frac{3}{\gamma_0^2} \frac{m_0}{2} \|v'_{\epsilon m}(t)\| \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right| + \frac{2m_0}{\gamma_0^2} \|v_{\epsilon m}(t)\| \|v'_{\epsilon m}(t)\| + \frac{2}{\gamma_0} \sqrt{\frac{m_0}{2}} \|v'_{\epsilon m}(t)\|^2 \\
 &\quad + \frac{5}{\gamma_0^3} \left(\frac{m_0}{2} \right)^{3/2} \left(\frac{\pi + 1}{\pi} \right)^2 \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2 + \|g(t)\| \|v'_{\epsilon m}(t)\| \\
 &\leq C_3 \left[\|v'_{\epsilon m}(t)\|^2 + \left| \frac{\partial^2 v_{\epsilon m}}{\partial y^2}(t) \right|^2 \right] + \frac{1}{2} \|g(t)\|^2,
 \end{aligned}
 \tag{3.37}$$

where

$$M_1 = \max_{[0, T] \times [0, 1] \times [0, \gamma_0 K_1]} \sup \text{ess} \left\{ M(y, t, \lambda), |M'(t, y, \lambda)|, \left| \frac{\partial}{\partial y} M(y, t, \lambda) \right|, \left| \frac{\partial}{\partial \lambda} M(y, t, \lambda) \right| \right\}.
 \tag{3.38}$$

Integrating inequality (3.37) from 0 to t , the result follows from the convergence of the initial data, Gronwall's lemma, (H6), and (2.7). □

LEMMA 3.4. *There exists a constant $K_3 > 0$ independent of $\epsilon > 0$ and $m \geq m_1$ such that*

$$|v''_{\epsilon m}(t)| \leq K_3, \quad \forall t \in [0, T].
 \tag{3.39}$$

Proof. Considering $w = v''_{\epsilon m}(t)$ in (3.16), using Lemmas 3.2 and 3.3, and by the same arguments used in the proof of Lemma 3.2, it follows that there exists a constant $C_4 > 0$

independent of $\varepsilon > 0$ and $m \geq m_1$ such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |v''_{\varepsilon m}(t)|^2 + \varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) |v''_{\varepsilon m}(t)|^2 - 2F' \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) (v''_{\varepsilon m}(t), v'_{\varepsilon m}(t))^2 \\ & + \frac{1}{2} \frac{d}{dt} \int_0^1 \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\varepsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \left| \frac{\partial v'_{\varepsilon m}}{\partial y}(y, t) \right|^2 dy \\ & \leq C_4 + \frac{1}{2} |v''_{\varepsilon m}(t)|^2 + \frac{1}{2} |g'(t)|^2. \end{aligned} \tag{3.40}$$

The result follows from this inequality, the convergence of the initial data, and from Gronwall's lemma. \square

3.1. Proof of Theorem 2.2. We observe that

(i) if $|v'_{\varepsilon m}(t)|^2 \leq K/2$, then $(K - |v'_{\varepsilon m}(t)|^2)/\varepsilon \geq K/2\varepsilon$, so from the hypothesis on F we have

$$F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) = 1, \quad \text{for } \varepsilon > 0 \text{ such that } 2\varepsilon < K; \tag{3.41}$$

(ii) if $|v'_{\varepsilon m}(t)|^2 > K/2$, taking $w = v'_{\varepsilon m}(t)$ in (3.5), we have

$$\begin{aligned} & \varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) \\ & \leq \frac{2}{K} \left\{ |v''_{\varepsilon m}(t)| |v'_{\varepsilon m}(t)| + \int_0^1 \left[\frac{1}{\gamma^2(t)} M(y, t, \gamma(t) |v_{\varepsilon m}(t)|^2) - \frac{b^2(y, t)}{4} \right] \left| \frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y, t) \right| \right. \\ & \quad \times |v'_{\varepsilon m}(t)| dy + \int_0^1 \left| c(y, t) \frac{\partial v_{\varepsilon m}}{\partial y}(y, t) v'_{\varepsilon m}(y, t) \right| dy \\ & \quad \left. + \int_0^1 \left| b(y, t) \frac{\partial v'_{\varepsilon m}}{\partial y}(y, t) v'_{\varepsilon m}(y, t) \right| dy + \int_0^1 |g(y, t) v'_{\varepsilon m}(y, t)| dy \right\}. \end{aligned} \tag{3.42}$$

Using the estimates obtained in the above inequality, there exists a constant $K_4 > 0$ such that

$$\varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) \leq K_4, \quad \text{for } \varepsilon > 0. \tag{3.43}$$

From the above inequalities, there exists a constant $K_5 > 0$ independent of $\varepsilon > 0$ and $m \geq m_1$ such that

$$\varepsilon F \left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \right) \leq K_4, \quad \forall t \in [0, T], \quad m \geq m_1, \quad 0 < \varepsilon < \frac{K}{2}. \tag{3.44}$$

From the lemmas and (3.44) it follows that there exists a subsequence of $(v_{\varepsilon m})$, still denoted by $(v_{\varepsilon m})$, such that

$$v_{\varepsilon m} \rightharpoonup v, \quad \text{weakly* in } L^\infty(0, T, H_0^1(0, 1) \cap H^2(0, 1)), \tag{3.45a}$$

$$v'_{\varepsilon m} \rightharpoonup v', \quad \text{weakly* in } L^\infty(0, T, H_0^1(0, 1)), \tag{3.45b}$$

$$v''_{\varepsilon m} \rightharpoonup v'', \quad \text{weakly* in } L^\infty(0, T, L^2(0, 1)), \tag{3.45c}$$

$$\varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) \rightharpoonup \chi, \quad \text{weakly* in } L^\infty(0, T). \tag{3.45d}$$

From (3.45a)–(3.45c) we conclude that $(v_{\varepsilon m}(t))$ and $(v'_{\varepsilon m}(t))$ are equicontinuous with respect to the norms on $H_0^1(0, 1)$ and $L^2(0, 1)$, respectively. So there exists a subsequence of $(v_{\varepsilon m})$, still denoted by $(v_{\varepsilon m})$, such that

$$v_{\varepsilon m} \rightarrow v, \quad \text{strongly in } H_0^1(0, 1), \text{ uniformly in } [0, T], \tag{3.46}$$

$$v'_{\varepsilon m} \rightarrow v', \quad \text{strongly in } L^2(0, 1), \text{ uniformly in } [0, T]. \tag{3.47}$$

But $M \in L^\infty_{\text{loc}}([0, \infty), W^{1,\infty}(Q))$ and for each $L > 0$, $(\partial M/\partial \lambda) \in L^\infty(Q \times (0, L))$, so

$$M\left(t, \gamma(t) |v_{\varepsilon m}(t)|^2\right) \rightarrow M\left(t, \gamma(t) |v(t)|^2\right) \quad \text{strongly in } L^2(0, 1), \text{ uniformly in } [0, T]. \tag{3.48}$$

Passing to the limit, when $m \rightarrow \infty$ and $\varepsilon \rightarrow 0$, in (3.5), we obtain

$$\begin{aligned} v''(t) + \left[-\frac{1}{\gamma^2(t)} M\left(y, t, \gamma(t) |v(t)|^2\right) + \frac{b^2(y, t)}{4} \right] \frac{\partial^2 v}{\partial y^2}(t) + c(y, t) \frac{\partial v}{\partial y}(t) \\ + b(y, t) \frac{\partial v'}{\partial y}(t) + \chi(t) v'(t) = g(t) \quad \text{in } L^2(0, T, L^2(0, 1)), \end{aligned} \tag{3.49}$$

and $v(0) = v_0$ and $v'(0) = v_1$.

We have $v' \in C^0([0, T], L^2(0, 1))$ and $|v'(0)|^2 = |v_1|^2 < K$, so there exists $T_0 > 0$ such that

$$|v'(t)|^2 < K, \quad \forall t \in [0, T_0]. \tag{3.50}$$

By considering $\beta_0 = \max_{[0, T_0]} |v'(t)|^2$ and $\rho_0 = K - \beta_0 > 0$, we have

$$|v'(t)|^2 \leq \beta_0 = K - \rho_0 < K - \frac{\rho_0}{2}, \quad \forall t \in [0, T_0]. \tag{3.51}$$

From (3.47) there exists $m_2 \in \mathbb{N}$ such that

$$|v'_{\varepsilon m}(t)|^2 < \frac{\rho_0}{4} + |v'(t)|^2 < K - \frac{\rho_0}{4}, \quad \forall t \in [0, T_0]. \tag{3.52}$$

Then, for $0 < \varepsilon < (\rho_0/4)$, we have

$$\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon} \geq 1, \quad m \geq m_2, t \in [0, T_0], \tag{3.53}$$

so

$$\varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) = \varepsilon, \quad t \in [0, T_0]. \tag{3.54}$$

Thus by the uniqueness of the limit it follows that $\chi = 0$ in $[0, T_0]$.

For the uniqueness we consider v and \hat{v} solutions in $[0, T_0]$, and $w = v - \hat{v}$. So we have

$$\begin{aligned} w''(t) + \left[-\frac{1}{y^2(t)}M(y, t, \gamma(t) |v(t)|^2) + \frac{b^2(y, t)}{4} \right] \frac{\partial^2 w}{\partial y^2}(t) \\ + \left[\frac{1}{y^2(t)}M(y, t, \gamma(t) |\hat{v}(t)|^2) - \frac{1}{y^2(t)}M(y, t, \gamma(t) |v(t)|^2) \right] \frac{\partial^2 \hat{v}}{\partial y^2}(t) \\ + c(y, t) \frac{\partial w}{\partial y}(t) + b(y, t) \frac{\partial w'}{\partial y}(t) = 0 \quad \text{in } L^2(0, T, L^2(0, 1)). \end{aligned} \tag{3.55}$$

By defining

$$\Psi(t) = \frac{1}{2} |w'(t)|^2 + \frac{1}{2} \int_0^1 \left[\frac{1}{y^2(t)}M(y, t, \gamma(t) |v(t)|^2) - \frac{b^2(y, t)}{4} \right] \left(\frac{\partial w}{\partial y}(t) \right)^2 dy \tag{3.56}$$

and multiplying the equation in (3.55) by w' , we obtain

$$\Psi'(t) - C\Psi(t) \leq 0, \tag{3.57}$$

for some constant $C > 0$, and since $\Psi(0) = 0$ we conclude that $w = 0$ in $[0, T_0]$.

3.2. Proof of Theorem 2.1. In this section, we study the noncylindrical problem (1.3). We represent by Ω_t and Ω_0 the intervals $(\alpha(t), \beta(t))$ and (α_0, β_0) , respectively, and we denote by $|\cdot|_0, \|\cdot\|_0, |\cdot|_t, \|\cdot\|_t$ the norms in $L^2(\Omega_0), H_0^1(\Omega_0), L^2(\Omega_t), H_0^1(\Omega_t)$, respectively.

We consider the change of variables $x = \alpha(t) + \gamma(t)y$, and we define

$$\begin{aligned} M(y, t, \lambda) &= \widehat{M}(\alpha(t) + \gamma(t)y, t, \lambda), \\ v_0(y) &= u_0(\alpha_0 + \gamma_0 y) \text{ e } v_1(y) = u_1(\alpha_0 + \gamma_0 y), \\ g(y, t) &= f(\alpha(t) + \gamma(t)y, t). \end{aligned} \tag{3.58}$$

We can see that we are in the conditions of Theorem 2.2. It follows that there exist $T_0 > 0$ and a unique solution v of problem (2.5) satisfying conditions (2.8). By considering $u(x, t) = v(y, t)$ with $x = \alpha(t) + \gamma(t)y$ we verify that u is the solution of Theorem 2.1.

The regularity and uniqueness of u claimed in Theorem 2.1 are obtained by the regularity and uniqueness of the solution v of Theorem 2.2.

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