NEW WEIGHTED POINCARÉ-TYPE INEQUALITIES FOR DIFFERENTIAL FORMS

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We first prove local weighted Poincaré-type inequalities for differential forms. Then, by using the local results, we prove global weighted Poincaré-type inequalities for differential forms in John domains, which can be considered as generalizations of the classical Poincaré-type inequality.

1. Introduction

Differential forms have wide applications in many fields, such as tensor analysis, potential theory, partial differential equations, and quasiregular mappings, see [1, 2, 3, 5, 6, 7, 8]. Different versions of the classical Poincaré inequality have been established in the study of the Sobolev space and differential forms, see [2, 7, 10]. Susan G. Staples proved the Poincaré inequality in L^s-averaging domains in [10]. Tadeusz Iwaniec and Adam Lutoborski proved a local Poincaré-type inequality in [7], which plays a crucial role in generalizing the theory of Sobolev functions to differential forms. In this paper, we prove local weighted Poincaré-type inequalities for differential forms in any kind of domains, and the global weighted Poincaré-type inequalities for differential forms in John domains.

A-harmonic tensors are the special differential forms which are solutions to the A-harmonic equation for differential forms

$$d^\star A(x,d\omega)=0, \tag{1.1}$$

where $A:\Omega\times\Lambda^l(\mathbb{R}^n)\to\Lambda^l(\mathbb{R}^n)$ is an operator satisfying some conditions, see [6, 7, 9]. Thus, all of the results on differential forms in this paper remain true for A-harmonic tensors. Therefore, our new results concerning differential forms are of interest in some fields, such as those mentioned above.

Throughout this paper, we always assume Ω is a connected open subset of $\mathbb{R}^n$. Let $e_1,e_2,\dots,e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l=0,1,\dots,n$, the linear space of vectors, spanned by the exterior products, corresponding to all ordered l-tuples $I=(i_1,i_2,\dots,i_l)$, $1\le i_1<i_2<\cdots<i_l\le n$, is denoted by $\Lambda^l=\Lambda^l(\mathbb{R}^n)$. The Grassmann algebra $\Lambda=\oplus\Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha=\sum\alpha^I e_I\in\Lambda$ and $\beta=\sum\beta^I e_I\in\Lambda$, the inner product in Λ is given by $\langle\alpha,\beta\rangle=\sum\alpha^I\beta^I$ with summation over all

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l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator $\star : \Lambda \to \Lambda$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle(\star 1)$ for all $\alpha, \beta \in \Lambda$. Hence, the norm of $\alpha \in \Lambda$ is given by the formula $|\alpha| = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on Λ with $\star : \Lambda^l \to \Lambda^{n-l}$ and $\star\star (-1)^{l(n-l)} : \Lambda^l \to \Lambda^l$. Letting $0 < p < \infty$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x)dx \right)^{1/p}. \tag{1.2}$$

As we know, a differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\Lambda^l(\mathbb{R}^n)$. In particular, for $l = 0$, ω is a real function or a distribution. We denote the space of differential l-forms by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum_I \omega_{i_1 i_2 \ldots i_l} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I. Thus, $L^p(\Omega, \Lambda^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.3}$$

Similarly, $W_p^1(\Omega, \Lambda^l)$ are those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbb{R})$. The notations $W_{p,\mathrm{loc}}^1(\Omega, \mathbb{R})$ and $W_{p,\mathrm{loc}}^1(\Omega, \Lambda^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \Lambda^{l+1})$, $l = 0, 1, \ldots, n$.

We write $\mathbb{R} = \mathbb{R}^1$. Balls are denoted by B, and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. The n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $w > 0$ a.e. Also, in general $d\mu = w dx$, where w is a weight. The following result appears in [7]: let $Q \subset \mathbb{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \Lambda^l) \to C^\infty(Q, \Lambda^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1 \cdots \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}) dt \tag{1.4}$$

and the decomposition

$$\omega = d(K_y \omega) + K_y(d\omega). \tag{1.5}$$

We define another linear operator $T_Q : C^\infty(Q, \Lambda^l) \to C^\infty(Q, \Lambda^{l-1})$ by averaging K_y over all points y in Q: $T_Q \omega = \int_Q \varphi(y) K_y \omega dy$, where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y) dy = 1$. We define the l-form $\omega_Q \in D'(\Omega, \Lambda^l)$ by $\omega_Q = |Q|^{-1} \int_Q \omega(y) dy$, $l = 0$, and $\omega_Q = d(T_Q \omega)$, $l = 1, 2, \ldots, n$, for all $\omega \in L^p(Q, \Lambda^l)$, $1 \le p < \infty$.

The following generalized Hölder's inequality will be used repeatedly.

Lemma 1.1. *Let* $0 < \alpha < \infty$, $0 < \beta < \infty$, *and* $s^{-1} = \alpha^{-1} + \beta^{-1}$. *If* f *and* g *are measurable functions on* $\mathbb{R}^n$, *then* $\|fg\|_{s,\Omega} \le \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$ *for any* $\Omega \subset \mathbb{R}^n$.

Definition 1.2. The weight $w(x) > 0$ satisfies the A_r-condition, $r > 1$, and write $w \in A_r$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)}dx\right)^{r-1} < \infty \tag{1.6}$$

for any ball $B \subset \mathbb{R}^n$.

We also need the following lemma [4].

Lemma 1.3. *If* $w \in A_r$, *then there exist constants* $\beta > 1$ *and* C, *independent of* w, *such that* $\|w\|_{\beta,Q} \le C|Q|^{(1-\beta)/\beta}\|w\|_{1,Q}$ *for any cube or any ball* $Q \subset \mathbb{R}^n$.

2. Local weighted Poincaré-type inequalities

We need the following lemma, see [7].

Lemma 2.1. *Let* $u \in D'(Q,\Lambda^l)$ *and* $du \in L^p(Q,\Lambda^{l+1})$, $1 < p < n$. *Then,* $u - u_Q$ *is in* $L^{np/(n-p)}(Q,\Lambda^l)$ *and*

$$\left(\int_Q |u-u_Q|^{np/(n-p)}dx\right)^{(n-p)/np} \le C(n,p)\left(\int_Q |du|^p dx\right)^{1/p} \tag{2.1}$$

for Q *a cube or a ball in* $\mathbb{R}^n$ *and* $l = 0,1,\ldots,n$.

We now prove the following version of the local weighted Poincaré-type inequalities for differential forms.

Theorem 2.2. *Let* $u \in D'(Q,\Lambda^l)$ *and* $du \in L^p(Q,\Lambda^{l+1})$, *where* $1 < p < n$, *and* $l = 0,1,\ldots,n$. *If* $w \in A_{1+\lambda}$ *for any* $\lambda > 0$, *then there exist a constant* C, *independent of* u *and* du, *and* $\beta > 1$, *such that for any* α *with* $1 < \alpha < \beta$ *and* $np(\alpha - 1) > (n-p)\beta$, *it holds that*

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s w(x)dx\right)^{1/s} \le C|B|^{1/n}\left(\frac{1}{|B|}\int_B |du|^p w^{p/s}(x)dx\right)^{1/p} \tag{2.2}$$

for any ball or any cube $B \subset \mathbb{R}^n$, *here* $s = np(\alpha-1)/(n-p)\beta$.

Proof. Since $w \in A_{1+\lambda}$, by Lemma 1.3, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\|w\|_{\beta,B} \le C_1|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{2.3}$$

for any cube or any ball $B \subset \mathbb{R}^n$. Choose $t = s\beta/(\beta-1)$, then $1 < s < t$ and $\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/st$, by Hölder's inequality, Lemmas 1.3 and 2.1, we have

$$\begin{aligned}
\|u-u_B\|_{s,B,w} &= \left(\int_B \left(|u-u_B|\,w^{1/s}\right)^s dx\right)^{1/s} \\
&\le \left(\int_B |u-u_B|^t dx\right)^{1/t} \cdot \left(\int_B \left(w^{1/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\
&= \left(\int_B |u-u_B|^t dx\right)^{1/t} \cdot \|w\|_{\beta,B}^{1/s} \\
&\le C_2\left(\int_B |du|^{p'} dx\right)^{1/p'} \cdot C_3|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s} \\
&= C_4|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\left(\int_B |du|^{p'} dx\right)^{1/p'},
\end{aligned} \tag{2.4}$$

here,

$$p' = \frac{nt}{n+t} = \frac{ns\beta}{n(\beta-1)+s\beta}. \tag{2.5}$$

Using the assumption $np(\alpha-1) > (n-p)\beta$, it is easy to see that $p' < p$. By Hölder's inequality, again we have

$$\begin{aligned}
\left(\int_B |du|^{p'} dx\right)^{1/p'} &= \left(\int_B \left(|du|w^{1/s}w^{-1/s}\right)^{p'} dx\right)^{1/p'} \\
&\le \left(\int_B \left(|du|w^{1/s}\right)^p dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{pp'/s(p-p')} dx\right)^{(p-p')/pp'}.
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.4) yields

$$\begin{aligned}
&\|u-u_B\|_{s,B,w} \\
&\quad\le C_4|B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\left(\int_B |du|^p w^{p/s} dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{pp'/s(p-p')} dx\right)^{(p-p')/pp'}.
\end{aligned} \tag{2.7}$$

Choose $\lambda > 0$, such that $\lambda < 1 - \alpha/\beta$. Then, $1 + \lambda < 2 - \alpha/\beta = r$. Hence, $w \in A_{1+\lambda} \subset A_r$. By simple computation, we find that $s(p-p')/pp' = 1 - \alpha/\beta = r - 1$. Thus, we have

$$\begin{aligned}
&\|w\|_{1,B}^{1/s}\left(\int_B \left(\frac{1}{w}\right)^{pp'/s(p-p')} dx\right)^{(p-p')/pp'} \\
&\quad= \left(|B|^{1+s(p-p')/pp'}\right)^{1/s}\left[\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\int_B \left(\frac{1}{w}\right)^{pp'/s(p-p')} dx\right)^{s(p-p')/pp'}\right]^{1/s} \\
&\quad= |B|^{1/s+1/p'-1/p}\left[\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right]^{1/s} \\
&\quad\le C_5|B|^{1/s+1/p'-1/p}.
\end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7) implies

$$||u-u_B||_{s,B,w} \le C_6|B|^{(1-\beta)/\beta s}|B|^{1/s+1/p'-1/p}\left(\int_B |du|^p w^{p/s}dx\right)^{1/p}, \tag{2.9}$$

that is,

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s w(x)dx\right)^{1/s} \le C_6|B|^{(1-\beta)/\beta s+1/p'}\left(\frac{1}{|B|}\int_B |du|^p w^{p/s}dx\right)^{1/p}. \tag{2.10}$$

Theorem 2.2 follows because $(1-\beta)/\beta s + 1/p' = 1/n$. □

We now prove another version of the local weighted Poincaré-type inequality for differential forms.

Theorem 2.3. *Let $u \in D'(B,\wedge^l)$ and $du \in L^p(B,\wedge^{l+1})$, where $1 < p < n$ and $l = 0,1,\dots,n$. If $w \in A_r$ for some $r > 1$, then there exist a constant C, independent of u and du, and $\beta > 1$, such that for any τ with $0 < \tau < 1/r(1-1/p+1/n)$, it holds that*

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s w^\tau(x)dx\right)^{1/s} \le C|B|^{1/n}\left(\frac{1}{B}\int_B |du|^p w^{\tau p/s}(x)dx\right)^{1/p} \tag{2.11}$$

for any ball or any cube $B \subset \mathbb{R}^n$. Here, $s = np(1-\tau r)/(n-p)$.

Proof. Let $T = s/(1-\tau)$. Using Lemma 1.3 and Hölder's inequality, we have

$$\begin{aligned}\left(\int_B |u-u_B|^s w^\tau(x)dx\right)^{1/s} &= \left(\int_B \left(|u-u_B|w^{\tau/s}\right)^s dx\right)^{1/s}\\ &\le \left(\int_B |u-u_B|^T dx\right)^{1/T}\left(\int_B w(x)dx\right)^{\tau/s}\\ &= ||u-u_B||_{T,B}\|w\|_{1,B}^{\tau/s}.\end{aligned} \tag{2.12}$$

Since

$$T = \frac{s}{1-\tau} = \frac{np(1-\tau r)}{(n-p)(1-\tau)} = \frac{np'}{n-p'}, \tag{2.13}$$

where $p' = ns/(n(1-\tau)+s) < p$, then using Lemma 2.1, we have

$$||u-u_B||_{T,B} \le C_7\left(\int_B |du|^{p'}dx\right)^{1/p'}. \tag{2.14}$$

Substituting (2.14) into (2.12), we obtain

$$||u-u_B||_{s,B,w^\tau} \le C_7\|w\|_{1,B}^{\tau/s}\left(\int_B |du|^{p'}dx\right)^{1/p'}. \tag{2.15}$$

Since $p' < p$, using Hölder's inequality again, we have

$$\begin{aligned}\left(\int_B |du|^{p'}dx\right)^{1/p'} &= \left(\int_B \left(|du|w^{\tau/s}w^{-\tau/s}\right)^{p'}dx\right)^{1/p'}\\ &\leq \left(\int_B \left(|du|w^{\tau/s}\right)^{p}dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{\tau pp'/s(p-p')}dx\right)^{(p-p')/pp'}\\ &= \left(\int_B |du|^{p}w^{p\tau/s}dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\tau(r-1)/s}.\end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16) yields

$$\|u-u_B\|_{s,B,w^\tau} \leq C_7\|w\|_{1,B}^{\tau/s}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\tau(r-1)/s}\left(\int_B |du|^{p}w^{p\tau/s}dx\right)^{1/p}. \tag{2.17}$$

Since $w \in A_r$, we obtain

$$\begin{aligned}&\|w\|_{1,B}^{\tau/s}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\tau(r-1)/s}\\ &\quad= \left[\left(\int_B w(x)dx\right)\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right]^{\tau/s}\\ &\quad= |B|^{r\tau/s}\left[\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right]^{\tau/s}\\ &\quad\leq C_8|B|^{r\tau/s}.\end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.17) yields

$$\|u-u_B\|_{s,B,w^\tau} \leq C_9|B|^{r\tau/s}\left(\int_B |du|^{p}w^{p\tau/s}dx\right)^{1/p}. \tag{2.19}$$

Simple calculation shows that $r\tau/s = 1/n + 1/s - 1/p$, this clearly implies (2.11) and completes the proof of Theorem 2.3. □

3. Global weighted Poincaré-type inequalities

Definition 3.1. Call Ω, a proper subdomain of $\mathbb{R}^n$, a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined to any other point $x \in \Omega$ by a continuous curve $\nu \subset \Omega$ so that $d(\xi, \partial\Omega) \geq \delta|x-\xi|$ for each $\xi \in \nu$. Here, $d(\xi,\partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

As we know, John domains are bounded. Bounded quasiballs and bounded uniform domains are John domains. We also know that a δ-John doamin has the following properties [9].

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a δ-John domain. Then, there exists a covering ν of Ω consisting of open cubes such that the following hold:*

(1) $\sum_{Q\in\nu}\chi_{\sigma Q}(x) \le N\chi_{\Omega}(x)$, *$\sigma > 1$ and $x \in \mathbb{R}^n$,*

(2) *there is a distinguished cube $Q_0 \in \nu$ (called the central cube) which can be connected with every cube $Q \in \nu$ by a chain of cubes $Q_0, Q_1, \dots, Q_k = Q$ from ν such that for each $i = 0, 1, \dots, k-1$, $Q \subset NQ_i$, there is a cube $\mathbb{R}_i \subset \mathbb{R}^n$ (this cube does not need to be a member of ν) such that $\mathbb{R}_i \subset Q_i \cap Q_{i+1}$, and $Q_i \cup Q_{i+1} \subset N\mathbb{R}_i$.*

We also know that if $w \in A_r$, then the measure μ defined by $d\mu = w(x)dx$ is a doubling measure, that is, $\mu(2B) \le C\mu(B)$ for all balls B in $\mathbb{R}^n$, see [5, page 299]. Since the doubling property implies $\mu(B) \approx \mu(Q)$ whenever Q is an open cube with $B \subset Q \subset \sqrt{n}B$, we may use cubes in place of balls whenever it is convenient to us.

We now prove the following weighted global results in John domains.

Theorem 3.3. *Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^p(\Omega, \Lambda^{l+1})$, where $1 < p < n$ and $l = 0, 1, \dots, n$. If $w \in A_{1+\lambda}$ for any $\lambda > 0$, then there exists a constant C, independent of u and du, and $\beta > 1$, such that for any α with $1 < \alpha < \beta$ and $np(\alpha-1) > (n-p)\beta$, it holds that*

$$\left(\frac{1}{\mu(\Omega)}\int_{\Omega}|u-u_Q|^s w\,dx\right)^{1/s} \le C\mu(\Omega)^{1/n}\left(\frac{1}{\mu(\Omega)}\int_{\Omega}|du|^p w^{p/s}dx\right)^{1/p} \tag{3.1}$$

for any δ-John domain $\Omega \subset \mathbb{R}^n$. Here, Q is any cube in the covering ν of Ω appearing in Lemma 3.2 and $s = np(\alpha-1)/(n-p)\beta$.

Proof. Supposing $\sigma > 1$, by Theorem 2.2 and Lemma 3.2(1), we have

$$\begin{aligned}\int_{\Omega}|u-u_Q|^s w\,dx &\le \sum_{Q\in\nu}\int_Q |u-u_Q|^s w\,dx \\ &\le C_{10}\sum_{Q\in\nu}|Q|^{s(1/n+1/s-1/p)}\left(\int_Q |du|^p w^{p/s}dx\right)^{s/p}.\end{aligned} \tag{3.2}$$

Since $s = np(\alpha-1)/(n-p)\beta$, then $1/s + 1/n - 1/p > 0$. Therefore,

$$\begin{aligned}\int_{\Omega}|u-u_Q|^s w\,dx &\le C_{10}\sum_{Q\in\nu}|\Omega|^{s(1/n+1/s-1/p)}\left(\int_Q |du|^p w^{p/s}dx\right)^{s/p} \\ &\le C_{10}|\Omega|^{s(1/n+1/s-1/p)}\sum_{Q\in\nu}\left(\int_{\sigma Q}|du|^p w^{p/s}\right)^{s/p} \\ &\le C_{10}N|\Omega|^{s(1/n+1/s-1/p)}\left(\int_{\Omega}|du|^p w^{p/s}dx\right)^{s/p} \\ &= C_{11}|\Omega|^{s(1/n+1/s-1/p)}\left(\int_{\Omega}|du|^p w^{p/s}dx\right)^{s/p}.\end{aligned} \tag{3.3}$$

This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let $u \in D'(\Omega, \Lambda^{l+1})$ and $du \in L^p(\Omega, \Lambda^{l+1})$, where $1 < p < n$ and $l = 0, 1, \ldots, n$. If $w \in A_r$ for some $r > 1$, then there exists a constant C, independent of u and du, and $\beta > 1$, such that for any τ with $0 < \tau < 1/r(1 - 1/p + 1/n)$, it holds that*

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u - u_Q|^s w^\tau(x)dx\right)^{1/s} \le C|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_\Omega |du|^p w^{\tau p/s}dx\right)^{1/p} \tag{3.4}$$

for any δ-John domain $\Omega \subset \mathbb{R}^n$. Here, Q is any cube in the covering ν of Ω appearing in Lemma 3.2 and $s = np(1 - \tau r)/(n - p)$.

Proof. Supposing $\sigma > 1$, by Theorem 2.3 and the Lemma 3.2(1), we have

$$\begin{aligned}\int_\Omega |u - u_Q|^s w^\tau dx &\le \sum_{Q\in\nu}\int_Q |u - u_Q|^s w^\tau dx \\ &\le C_{12}\sum_{Q\in\nu}|Q|^{s(1/n+1/s-1/p)}\left(\int_Q |du|^p w^{p\tau/s}dx\right)^{s/p}.\end{aligned} \tag{3.5}$$

Since $s = np(1 - \tau r)/(n - p) < p < np/(n - p)$, then $1/n + 1/s - 1/p > 0$. Therefore,

$$\begin{aligned}\int_\Omega |u - u_Q|^s w^\tau dx &\le C_{12}\sum_{Q\in\nu}|\Omega|^{s(1/n+1/s-1/p)}\left(\int_Q |du|^p w^{p\tau/s}dx\right)^{s/p} \\ &\le C_{12}|\Omega|^{s(1/n+1/s-1/p)}\sum_{Q\in\nu}\left(\int_{\sigma Q} |du|^p w^{p\tau/s}dx\right)^{s/p} \\ &\le C_{12}N|\Omega|^{s(1/n+1/s-1/p)}\left(\int_\Omega |du|^p\, w^{p\tau/s}dx\right)^{s/p} \\ &\le C_{13}|\Omega|^{s(1/n+1/s-1/p)}\left(\int_\Omega |du|^p w^{p\tau/s}dx\right)^{s/p}.\end{aligned} \tag{3.6}$$

This completes the proof of Theorem 3.4. □

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