WEIGHTED POINCARÉ-TYPE ESTIMATES FOR CONJUGATE A-HARMONIC TENSORS

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We prove Poincaré-type estimates involving the Hodge codifferential operator and Green's operator acting on conjugate A-harmonic tensors.

1. Preliminary

In a survey paper [1], Agarwal and Ding summarized the advances achieved in the study of A-harmonic equations. Some recent results about A-harmonic equations can also be found in [2, 3, 5, 6]. The purpose of this note is to establish some estimates about Green's operator and the Hodge codifferential operator d^*, which will enrich the existing literature in the field of A-harmonic equations.

Let Ω be a connected open subset of $\mathbb{R}^n$, $n \geq 2$, B a ball in $\mathbb{R}^n$ and ρB denote the ball with the same center as B and with $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$ and a weight $w(x)$, we denote the weighted L^p-norm of a measurable function f over E by $\|f\|_{p,E,w^\alpha} = (\int_E |f(x)|^p w^\alpha dx)^{1/p}$, where α is a real number. Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ be the linear space of all l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$, $l = 0, 1, \ldots, n$. Assume that $D'(\Omega, \Lambda^l)$ is the space of all differential l-forms and $L^p(\Omega, \Lambda^l)$ is the space of all L^p-integrable l-forms, which is a Banach space with norm $\|\omega\|_{p,\Omega} = (\int_\Omega |\omega(x)|^p dx)^{1/p} = (\int_\Omega (\sum_I |\omega_I(x)|^2)^{p/2} dx)^{1/p}$. We denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n-1$. Its formal adjoint operator $d^* : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $d^* = (-1)^{nl+1} * d *$ on $D'(\Omega, \Lambda^{l+1})$, $l = 0, 1, \ldots, n-1$, where $*$ is the Hodge star operator. We call u and v a pair of conjugate A-harmonic tensor in Ω if u and v satisfy the conjugate A-harmonic equation

$$A(x, du) = d^* v \tag{1.1}$$

in Ω, where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ satisfies conditions: $|A(x,\xi)| \leq a|\xi|^{p-1}$ and $\langle A(x,\xi), \xi \rangle \geq |\xi|^p$ for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a > 0$ is a constant. In this paper, we always assume that p is the fixed exponent associated with (1.1), $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$.

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The following weak reverse Hölder inequality about d^*v appears in [3].

LEMMA 1.1. *Let u and v be a pair of solutions of (1.1) in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of v, such that $\|d^*v\|_{s,B} \le C|B|^{(t-s)/st}\|d^*v\|_{t,\sigma B}$ for all balls B with $\sigma B \subset \Omega$.*

Setting the differential form $u = d^*v$ in [2, Corollary 2.6], we obtain the following Poincaré-type inequality for Green's operator.

$$\left\|G(d^*v) - (G(d^*v))_B\right\|_{p,B} \le C\left\|d^*v\right\|_{p,B}. \tag{1.2}$$

Definition 1.2. A weight $w(x)$ is called an A_r-weight for some $r > 1$ on a subset $E \subset \mathbb{R}^n$, write $w \in A_r(E)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{1.3}$$

for any ball $B \subset E$.

We also need the following well-known reverse Hölder inequality for A_r-weights.

LEMMA 1.3. *If $w \in A_r$, then there exist constants $\beta > 1$ and C, independent of w, such that $\|w\|_{\beta,B} \le C|B|^{(1-\beta)/\beta}\|w\|_{1,B}$ for all balls $B \subset \mathbb{R}^n$.*

The following generalized Hölder inequality will be used repeatedly in this paper.

LEMMA 1.4. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then $\|fg\|_{s,E} \le \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E}$ for any $E \subset \mathbb{R}^n$.*

The following lemma appears in [6].

LEMMA 1.5. *Let u and v be a pair of solutions of (1.1) in a domain Ω. Then, there exists a constant C, independent of u and v, such that*

$$\|du\|^p_{p,D,w^\alpha} \le \left\|d^*v\right\|^q_{q,D,w^\alpha} \le C\|du\|^p_{p,D,w^\alpha} \tag{1.4}$$

for any subset $D \subset \Omega$. Here w is any weight and $\alpha > 0$ is any constant.

2. Main results and proofs

Now, we prove the following A_r-weighted Poincaré-type inequality for Green's operator G acting on solutions of (1.1).

THEOREM 2.1. *Let u and v be a pair of solutions of (1.1) in Ω, and assume that $\omega \in A_r(\Omega)$ for some $r > 1$, $\sigma > 1$, $0 < \alpha \le 1$, and $1 + \alpha(r-1) < q < \infty$. Then, there exists a constant C, independent of u and v, such that*

$$\left\|G(d^*v) - (G(d^*v))_B\right\|^q_{q,B,w^\alpha} \le C\|du\|^p_{p,\sigma B,w^\alpha} \tag{2.1}$$

for all balls B with $\sigma B \subset \Omega$.

Proof. First, we assume that $0 < \alpha < 1$. Let $s = q/(1-\alpha)$. Using Hölder inequality we get

$$\begin{aligned}
&\left(\int_B \left|G(d^*v) - (G(d^*v))_B\right|^q w^\alpha dx\right)^{1/q} \\
&\quad\le \left(\int_B \left(\left|G(d^*v) - (G(d^*v))_B\right| w^{\alpha/q}\right)^q dx\right)^{1/q} \\
&\quad\le \left(\int_B \left|G(d^*v) - (G(d^*v))_B\right|^s dx\right)^{1/s}\left(\int_B w^{\alpha s/(s-q)} dx\right)^{(s-q)/qs} \\
&\quad= \left\|G(d^*v) - (G(d^*v))_B\right\|_{s,B}\left(\int_B w\,dx\right)^{\alpha/q}.
\end{aligned} \tag{2.2}$$

Select $t = q/(\alpha(r-1)+1)$, then $t < q$. Using Lemma 1.1 and (1.2), we find that

$$\left\|G(d^*v) - (G(d^*v))_B\right\|_{s,B} \le C_1\|d^*v\|_{s,B} \le C_2|B|^{(t-s)/ts}\|d^*v\|_{t,\sigma B} \tag{2.3}$$

for all balls B with $\sigma B \subset \Omega$. Since $1/t = 1/q + (q-t)/qt$, by Hölder inequality again, we have

$$\begin{aligned}
\|d^*v\|_{t,\sigma B} &= \left(\int_{\sigma B} \left(|d^*v| w^{\alpha/q} w^{-\alpha/q}\right)^t dx\right)^{1/t} \\
&\le \left(\int_{\sigma B} |d^*v|^q w^\alpha dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{\alpha t/(q-t)} dx\right)^{(q-t)/qt} \\
&= \left(\int_{\sigma B} |d^*v|^q w^\alpha dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/q}.
\end{aligned} \tag{2.4}$$

Combining (2.2), (2.3), and (2.4) yields

$$\begin{aligned}
&\left(\int_B \left|G(d^*) - (G(d^*v))_B\right|^q w^\alpha dx\right)^{1/q} \\
&\quad\le C_2|B|^{(t-s)/ts}\left(\int_B w\,dx\right)^{\alpha/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/q}\left(\int_{\sigma B} |d^*v|^q w^\alpha dx\right)^{1/q}.
\end{aligned} \tag{2.5}$$

Noting that $w \in A_r$, we have

$$\begin{aligned}
&\left(\int_B w\,dx\right)^{\alpha/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/q} \\
&\quad\le \left(|\sigma B|^r\left(\frac{1}{|\sigma B|}\int_{\sigma B} w\,dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)}\right)^{\alpha/q} \le C_3|B|^{\alpha r/q}.
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5) with $(t-s)/ts + \alpha r/q = 0$, it follows that

$$\left(\int_B \left|G(d^*v) - (G(d^*v))_B\right|^q w^\alpha dx\right)^{1/q} \le C_4\left(\int_{\sigma B} |d^*v|^q w^\alpha dx\right)^{1/q}. \tag{2.7}$$

Applying Lemma 1.5 and (2.7), we conclude that

$$\left\|G(d^*v)-(G(d^*v))_B\right\|_{q,B,w^{\alpha}}^q \le C_5\|d^*v\|_{q,\sigma B,w^{\alpha}}^q \le C_6\|du\|_{p,\sigma B,w^{\alpha}}^p. \tag{2.8}$$

We have proved that (2.1) is true if $0<\alpha<1$.

Next, we show that (2.1) is also true for $\alpha=1$. By Lemma 1.3, there exist constants $\beta>1$ and $C_7>0$, such that

$$\|w\|_{\beta,B} \le C_7|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{2.9}$$

for any ball $B\subset\mathbb{R}^n$. Choose $s=q\beta/(\beta-1)$, then $1<q<s$ and $\beta=s/(s-q)$. Since $1/q=1/s+(s-q)/qs$, using Lemma 1.4 and (2.9), we obtain

$$\begin{aligned}
&\left(\int_B\left|G(d^*v)-(G(d^*v))_B\right|^q w\,dx\right)^{1/q}\\
&\quad\le\left(\int_B\left|G(d^*v)-(G(d^*v))_B\right|^s dx\right)^{1/s}\left(\int_B\left(w^{1/q}\right)^{qs/(s-q)}dx\right)^{(s-q)/sq}\\
&\quad=\left\|G(d^*v)-(G(d^*v))_B\right\|_{s,B}\cdot\|w\|_{\beta,B}^{1/q}\\
&\quad\le C_8\left\|G(d^*v)-(G(d^*v))_B\right\|_{s,B}\cdot|B|^{(1-\beta)/\beta q}\|w\|_{1,B}^{1/q}.
\end{aligned}\tag{2.10}$$

Now, choose $t=q/r$, then $t<q$. From Lemma 1.1 and (1.2), we have

$$\left\|G(d^*v)-(G(d^*v))_B\right\|_{s,B}\le C_9\|d^*v\|_{s,B}\le C_{10}|B|^{(t-s)/st}\|d^*v\|_{t,\sigma B}. \tag{2.11}$$

Using Hölder inequality again, we find that

$$\begin{aligned}
\|d^*v\|_{t,\sigma B}&=\left(\int_{\sigma B}\left(|d^*v|w^{1/q}w^{-1/q}\right)^t dx\right)^{1/t}\\
&\le\left(\int_{\sigma B}|d^*v|^q w\,dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{t/(q-t)}dx\right)^{(q-t)/qt}\\
&=\left(\int_{\sigma B}|d^*v|^q w\,dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{(r-1)/q}.
\end{aligned}\tag{2.12}$$

Combining (2.11) and (2.12) yields

$$\begin{aligned}
&\left\|G(d^*v)-(G(d^*v))_B\right\|_{s,B}\\
&\quad\le C_{11}|B|^{(t-s)/st}\left(\int_{\sigma B}|d^*v|^q w\,dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{(r-1)/q}.
\end{aligned}\tag{2.13}$$

Since $w\in A_r$, we obtain

$$\left(\int_B w\,dx\right)^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{(r-1)/q}\le C_{12}|B|^{r/q}. \tag{2.14}$$

Substituting (2.13) into (2.10) and using (2.14), we find that

$$\begin{aligned}
&\left\|G(d^*v)-(G(d^*v))_B\right\|_{q,B,w}\\
&\quad\le C_{13}|B|^{(1-\beta)/\beta q}|B|^{(t-s)/st}\left\|d^*v\right\|_{q,\sigma B,w}\|w\|_{1,B}^{1/q}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{(r-1)/q}\\
&\quad\le C_{14}|B|^{(1-\beta)/\beta q}|B|^{(t-s)/st}|B|^{r/q}\left\|d^*v\right\|_{q,\sigma B,w}\le C_{15}\left\|d^*v\right\|_{q,\sigma B,w}.
\end{aligned}\tag{2.15}$$

Combining Lemma 1.5 and (2.15), we conclude that

$$\left\|G(d^*v)-(G(d^*v))_B\right\|_{q,B,w}^q\le C_{16}\left\|d^*v\right\|_{q,\sigma B,w}^q\le C_{17}\|du\|_{p,\sigma B,w}^p.\tag{2.16}$$

This ends the proof of Theorem 2.1. □

For any weight w, we define the weighted norm of $\omega\in W^{1,p}(\Omega,\Lambda^l,w^\alpha)$ in Ω by

$$\|\omega\|_{W^{1,p}(\Omega),w^\alpha}=\operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega,w^\alpha}+\|\nabla\omega\|_{p,\Omega,w^\alpha},\quad 0<p<\infty.\tag{2.17}$$

Now we can give the following Sobolev norm estimates for Green operator in terms of Hodge codifferential operator.

Theorem 2.2. *Let u and v be a pair of solutions of (1.1) in Ω, and assume that $\omega\in A_r(\Omega)$ for some $r>1$, $\sigma>1$, $0<\alpha\le 1$, and $r<p<\infty$. Then, there exists a constant C, independent of u and v, such that*

$$\left\|G(u)-(G(u))_B\right\|_{W^{1,p}(B),w^\alpha}^p\le C\left\|d^*v\right\|_{q,\sigma B,w^\alpha}^q\tag{2.18}$$

for all balls B with $\sigma B\subset\Omega$. Here α is any constant with $0<\alpha\le 1$.

Proof. We know that Green's operator commutes with d in [4], that is, for any smooth differential form u, we have $dG(u)=Gd(u)$. Since $|\nabla\omega|=|d\omega|$ for any differential form ω, we have $\|\nabla G(u)\|_{p,B}=\|dG(u)\|_{p,B}=\|G(du)\|_{p,B}\le C_1\|du\|_{p,B}$ from [2, Lemma 2.1]. Using the same method as we did above, we can also have the following A_r-weighted inequalities

$$\begin{aligned}
\left\|G(u)-(G(u))_B\right\|_{p,B,w^\alpha}&\le C_2\operatorname{diam}(B)\|du\|_{p,\sigma B,w^\alpha},\\
\left\|\nabla\left(G(u)-(G(u))_B\right)\right\|_{p,B,w^\alpha}&\le C_3\|du\|_{p,\sigma B,w^\alpha}.
\end{aligned}\tag{2.19}$$

Combining (2.17) and (2.19), it follows that

$$\begin{aligned}
&\left\|G(u)-(G(u))_B\right\|_{W^{1,p}(B),w^\alpha}\\
&\quad=\operatorname{diam}(B)^{-1}\left\|G(u)-(G(u))_B\right\|_{p,B,w^\alpha}+\left\|\nabla\left(G(u)-(G(u))_B\right)\right\|_{p,B,w^\alpha}\\
&\quad\le\operatorname{diam}(B)^{-1}\cdot C_2\operatorname{diam}(B)\|du\|_{p,\sigma_1 B,w^\alpha}+C_3\|du\|_{p,\sigma_2 B,w^\alpha}\\
&\quad\le C_4\|du\|_{p,\sigma B,w^\alpha},
\end{aligned}\tag{2.20}$$

here $\sigma = \max(\sigma_1, \sigma_2)$ with $\sigma B \subset M$. Applying Lemma 1.5 and (2.20), we conclude that

$$||G(u) - (G(u))_B||^p_{W^{1,p}(B),w^\alpha} \le C_5 \|du\|^p_{p,\sigma B,w^\alpha} \le C_5 ||d^* v||^q_{q,\sigma B,w^\alpha}. \tag{2.21}$$

Therefore, we have completed the proof of Theorem 2.2. □

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