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Research Article

Stochastic Delay Lotka-Volterra Model

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This paper examines the asymptotic behaviour of the stochastic extension of a fundamentally important population process, namely the delay Lotka-Volterra model. The stochastic version of this process appears to have some intriguing properties such as pathwise estimation and asymptotic moment estimation. Indeed, their solutions will be stochastically ultimately bounded.

1. Introduction

As is well known, Lotka-Volterra Model is nonlinear and tractable models of predator-prey system. The predator-prey system is also studied in many papers. In the last few years, Mao et al. change the deterministic model in this field into the stochastic delay model. and give it more important properties [1–8].

Fluctuations play an important role for the self-organization of nonlinear systems; we will study their influence on a simple nonlinear model of interacting populations, that is, the Lotka-Volterra model. A simple analysis shows the result that the system allows extreme behaviour, leading to the extinction of both of their species or to the extinction of the predator and explosion of the prey. For example, in Mao et al. [1–8], we can see that once the population dynamics are corporate into the deterministic subclasses of the delay Lotka-Voterra model, the stochastic model will bear more attractive properties: the solutions will be be stochastically ultimately bounded, and their pathwise estimation and asymptotic moment estimation will be well done.

The most simple stochastic model is given in the form of a stochastic delay differential equation (also called a diffusion process); we call it a delay Lotka-Volterra model with diffusion. The model will be

$$dx(t) = \operatorname{diag}(x(t))\left[(b + Ax(t))dt + By(t)dt + Gdw(t)\right], \tag{1.1}$$

where $y(t) = x(t-\tau)$, $x(t) = (x_1(t), \dots, x_d(t))^T$ (where $(x_1(t), \dots, x_d(t))^T$ denotes the transpose of a vector or matrix $(x_1(t), \dots, x_d(t))$, $b = (b_1, b_2, \dots, b_d)^T$, $A = [a_{ij}] \in R^{d\times m}$, $B = [b_{ij}] \in R^{d\times m}$, $G = [\gamma_{ij}] \in R^{d\times m}$ and $w(t)$ is the m-dimensional Brownian motion, diag $(x(t))$ is the diag matrix.

This model of the stochastic delay Lotka-Volterra is different from Mao et al. [3–10], which paid more attention to the mathematical properties of the model than the real background of the model. However, our model has the following three characteristics. First, it is another stochastic delay subclass of the Lotka-Volterra model which is different from Mao et al. Then we can obtain more comprehensive properties in Theorem 2.1. Second, in this field no paper gives more attention to it so far, especially for the stochastic delay model which is the focus in our model. Third, this model has many real applications, for example, in economic growth model it is different from the old delay Lotka-Volterra model which only palys a role in predator-prey system, for example, the stochastic R&D model [9, 10] is the best application of this model. We hope our model can have new applications of the Lotka-Volterra model. Throughout this paper, we impose the condition

$$-a_{ii} > A_i = \sum_{j\neq i} a_{ij}^{+}, \quad (1 \le i \le d), \tag{1.2}$$

where $a_{ij}^{+} = a_{ij}$ if $a_{ij} > 0$.

Of course, it is important for us to point that the condition (1.2) may be not real in predator-prey interactions, but in the stochastic R&D model in economic growth model, it has a special meaning

$$-K_\theta = \max_i \left[a_{ii} + A_i - \frac{\theta^\theta A_i^{1+\theta}}{(1+\theta)^{1+\theta} |a_{ii}|^\theta} \right] + \sum_i \frac{\theta^\theta A_i^{1+\theta}}{(1+\theta)^{1+\theta} |a_{ii}|^\theta} < 0. \tag{1.3}$$

If $\theta = 1/2$ or 1, the inequality (1.3) can be deduced to

$$-K_{1/2} = \max_i \left[a_{ii} + A_i - \frac{2A_i^{3/2}}{\sqrt{27|a_{ii}|}} \right] + \sum_i \frac{2A_i^{3/2}}{\sqrt{27|a_{ii}|}} < 0, \tag{1.4}$$

$$-K_1 = \max_i \left[a_{ii} + A_i - \frac{A_i^2}{4|a_{ii}|} \right] + \sum_i \frac{A_i^2}{4|a_{ii}|} < 0. \tag{1.5}$$

If condition (1.2) is satisfied, then

$$\lim_{\theta\to\infty} \frac{\theta^\theta A_i^{1+\theta}}{(1+\theta)^{1+\theta} |a_{ii}|^\theta} = 0. \tag{1.6}$$

Therefore, if θ is big enough, condition (1.2) implies condition (1.3).

It is obvious the conditions (1.3)–(1.5) are dependent on the matrix A, independent on G.

Condition (1.4) will be used in a further topic in the paper; the condition (1.4) is complicated, we can find many matrixes A that have a property like this. For example,

$$A = \operatorname{diag}(a_{11}, a_{22}, \dots a_{dd}) \quad (a_{ii} < 0 \text{ for } 1 \leq i \leq d) \tag{1.7}$$

satisfy the condition (1.4). Furthermore, if $i \neq j$, $a_{ij} \leq 0$, or a_{ij} are proper small enough positive numbers, condition (1.4) holds too. Particularly, if $d = 2$, the condition can be induced into

$$a_{11} + a_{12}^{+} < \frac{-2\left(a_{21}^{+}\right)^{3/2}}{\sqrt{27a_{22}}}, \qquad a_{22} + a_{12}^{+} < \frac{-2\left(a_{12}^{+}\right)^{3/2}}{\sqrt{27a_{11}}} \tag{1.8}$$

It is clear that the upper inequalities are the key conditions in the stochastic R&D model in economic growth model.

Let

$$I_{\theta}(x) = \sum_{ij} a_{ij} x_i^{\theta} x_j. \tag{1.9}$$

The homogeneous function $I_{\theta}(x)$ of degree $1+\theta$ has the following key property.

Lemma 1.1. *Suppose the matrix A satisfies condition* (1.2)*. Let*

$$S = \left\{ x \in R_{+}^{d} : \|x\|_{\infty} = 1 \right\}; \tag{1.10}$$

then

$$\sup_{x \in S} I_{\theta}(x) \leq -K_{\theta}, \quad \theta > 0, \tag{1.11}$$

where K_{θ} is given in condition (1.3)*.*

Proof. Fix $x \in S$, so $0 < x_j \leq \|x\|_{\infty} = 1$. We will show $I_{\theta}(x) \leq -K_{\theta}$. We have

$$\begin{aligned} I_{\theta}(x) &\leq \sum_{i} a_{ii} x_i^{1+\theta} + \sum_{i} \sum_{j \neq i} a_{ij}^{+} x_i^{\theta} x_j \\ &\leq \sum_{i} \left(a_{ii} x_i^{1+\theta} + A_i x_i^{\theta} \right) \\ &= \sum_{i} \varphi_i(x_i) \end{aligned} \tag{1.12}$$

with A_i satisfying condition (1.2), where $\varphi_i(x_i) = a_{ii}x_i^{1+\theta} + A_i x_i^{\theta}$. Since, from condition (1.2), $\varphi_i(0) = 0$, $\varphi_i(1) = a_{ii} + A_i < 0$, and

$$\varphi_i'(t) = 0 \Longrightarrow t = t_0 = -\frac{\theta A_i}{(1+\theta)a_{ii}} = \frac{\theta A_i}{(1+\theta)|a_{ii}|} \in [0,1), \tag{1.13}$$

then

$$\max_{0\le t\le 1} \varphi_i(t) = \varphi_i(t_o) = \frac{\theta^{\theta} A_i^{1+\theta}}{(1+\theta)^{1+\theta}|a_{ii}|^{\theta}} = M_i. \quad (1 \le i \le d). \tag{1.14}$$

Since $x \in S$, we have $0 < x_i \le 1$, $(1 \le i \le d)$, $x \in S$, and there exits at least $x_i = 1$, such that $\varphi_i(x_i) \le M_i$, $(1 \le i \le d)$ and at least $\varphi_i(x_i) = \varphi_i(1) = a_{ii} + A_i$ for some i. Thus

$$\begin{aligned}\sum \varphi_i(x_i) &\le \max_i\left(a_{ii} + A_i + \sum_{j\neq i} M_j\right) \\ &= \max_i(a_{ii} + A_i - M_i) + \sum_i M_i.\end{aligned} \tag{1.15}$$

Now, from condition (1.3), the right hand of the upper equation is just $-K_\theta$, so $I_\theta(x) \le -K_\theta$; Lemma 1.1 is proved. □

We use the ordinary result of the polynomial functions.

Lemma 1.2. *Let $f_i (1 \le i \le n)$ be a homogeneous function of degree θ_i, $\theta > \theta_i \ge 0$, and $a > 0$; then the function as follows has an upper bound for some constant K.*

$$F(x) = \sum_{i=1}^{n} f_i - a\sum_{i=1}^{d} x_i^{\theta} \le K. \tag{1.16}$$

2. Positive and Global Solutions

Let $(\Omega, F, \{F_t\}_{t\ge 0}, P)$ be a complete probability space with filtration $\{F_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is increasing and right continuous while F_0 contains all P-null sets [8]. Moreover, let $w(t)$ be an m-dimensional Brownian motion defined on the filtered space and $R_+^d = \{x \in R_+^d : x_i > 0 \text{ for all } 1 \le i \le d\}$. Finally, denote the trace norm of a matrix A by $|A| = \sqrt{\operatorname{trace}(A^T A)}$ (where A^T denotes the transpose of a vector or matrix A) and its operator norm by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R_+^d)$ the family of continuous functions from $[-\tau, 0]$ to R_+^d.

The coefficients of (1.1) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of (1.1) may explode at a finite time.

let us emphasize the important feature of this theorem. It is well known that a deterministic equation may explode to infinity at a finite time for some system parameters $b \in R^d$ and $A \in R^{d\times m}$. However, the explosion will no longer happen as long as conditions

(1.2) and (1.3) hold. In other words, this result reveals the important property that conditions (1.2) and (1.3) suppress the explosion for the equation. The following theorem shows that this solution is positive and global.

Theorem 2.1. *Let us assume that* $K_{1/2}$ *satisfy*

$$3K_{1/2} > d\left(\beta_i + 2\beta_i'\right), \quad \beta_i = \sum_j b_{ij}^+, \quad \beta_j' = \sum_i b_{ij}^+. \tag{2.1}$$

Then for any given initial data $\{x_t : -\tau \leq t \leq 0\} \in C([-\tau, 0], R_+^d)$, *there exists a unique global solution* $x = x(t)$ *to* (1.1) *on* $t \geq -\tau$. *Moreover, this solution remains in* R_+^d *with probability 1, namely,* $x_t \in R_+^d$ *for all* $t \geq -\tau$ *almost surely.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x_t : -\tau \leq t \leq 0\} \in C([-\tau, 0], R_+^d)$, there is a unique maximal local solution $x(t)$ on $t \in [0, \rho)$, where ρ is the explosion time [3–10]. To show this solution is global, we need to show that $\rho = \infty$ a.s. Let k_0 be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau \leq t \leq 0} |x(t)| \leq \max_{-\tau \leq t \leq 0} |x(t)| \leq k_0. \tag{2.2}$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in \left[0, \rho\right) : x_i(t) \notin \left(k^{-1}, k\right), \text{ for some } i = 1, \ldots, d\right\}, \tag{2.3}$$

where throughout this paper we set $\inf \phi = \infty$ (as usual ϕ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k \to \infty} \tau_k$, whence $\tau_\infty \leq \rho$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\rho = \infty$ a.s. and $x(t) \in R_+^d$ a.s. for all $t \geq -\tau$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. Or for all $t > 0$, we have $P(\tau_k \leq T) \to 0$, $(k \to \infty)$. To show this statement, let us define a C^2-functions $V : R_+^d - R_+$ by

$$u(t) = t - \ln(t), \quad V(t) = \sum u(\sqrt{x_i}) \quad \left(x \in R_+^d\right). \tag{2.4}$$

The nonnegativity of this function can be seen from

$$u(t) = t - \ln(t) > 0 \quad on\ t > 0. \tag{2.5}$$

Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq T \wedge \tau_k$, we apply the Itô formula to $V(x)$ to obtain that

$$\begin{aligned} LV(x) &= \frac{1}{2}\sum_i (\sqrt{x_i}-1)\left[b_i + \sum_j a_{ij}x_j + b_{ij}y_j\right] + \frac{1}{8}\sum_{ij}(2-\sqrt{x_i})r_{ij}^2 \\ &= \frac{1}{2}\sum_i b_i(\sqrt{x_i}-1) + \frac{1}{8}\sum_{ij}\left[-4a_{ij}x_j + r_{ij}^2(2-\sqrt{x_i})\right] \\ &\quad + \frac{1}{2}\sum_{ij} b_{ij}(\sqrt{x_i}-1) + \frac{1}{2}\sum_{ij} a_{ij}\sqrt{x_i}x_j \\ &= \phi(x) + \frac{1}{2}\sum_{ij} b_{ij}\sqrt{x_i}x_j - \frac{1}{2}\sum_{ij} a_{ij}y_j + \frac{1}{2}I(x), \end{aligned} \tag{2.6}$$

where $\phi(x) = (1/2)\sum_i b_i(\sqrt{x_i}-1) + (1/8)\sum_{ij}[-4a_{ij}x_j + r_{ij}^2(2-\sqrt{x_i})]$ is a homogeneous function of a degree not above 1, $G = [\gamma_{ij}] \in R^{d\times m}$, and by (1.9), $I(x) = I_{1/2}(x)$, and let $z = x/\|x\|_\infty$, for all $x \in R_+^d$; then $\|z\|_\infty = 1$. By Lemma 1.1, we obtain

$$\begin{aligned} I(x) &= I(z\|x\|_\infty) = I(z)\|x\|_\infty^{3/2} \\ &\leq -k_{1/2}\|x\|_\infty^{3/2} \leq -d^{-3/2}k_{1/2}|x|^{3/2}, \end{aligned} \tag{2.7}$$

where we use the fact $V_{3/2}(x) = \sum_{i=1}^d x_i^{3/2}$ and $V_{3/2}(x) \leq d\|x\|_\infty^{3/2}$, $K_{1/2} > 0$, and

$$\begin{aligned} \sum_{ij} b_{ij}\sqrt{x_i}y_j &\leq \sum_{ij} b_{ij}^+\left(\frac{x_i^{3/2}}{3} + \frac{2y_j^{3/2}}{3}\right) \\ &= \frac{1}{3}\sum_i\sum_j b_{ij}^+ x_i^{3/2} + \frac{2}{3}\sum_j\sum_i b_{ij}^+ y_j^{3/2} \\ &= \frac{1}{3}\sum_i \beta_i x_i^{3/2} + \frac{2}{3}\sum_j \beta_j' y_j^{3/2} \\ -\sum_{ij} b_{ij}y_j &\leq -\sum_{ij} b_{ij}^- y_j = -\sum_j \rho_j y_j, \end{aligned} \tag{2.8}$$

where $b_{ij}^- = -b_{ij}$, if $b_{ij} < 0$, and $\rho_j = \sum_i b_{ij}^-$.

Thus

$$LV(x) \leq \phi(x) + \frac{1}{6}\sum_i \beta_i x_i^{3/2} - \frac{1}{2d}K_{1/2}V_{3/2}(x) + \sum_i\left(\frac{1}{3}\beta_i' y_i^{3/2} + \frac{1}{2}\rho_i y_i\right). \tag{2.9}$$

Put

$$W(t,x(t)) = V(x) + \int_{t-\tau}^{t} \sum_i \left[\frac{1}{3}\beta_i' x_i^{3/2}(s) + \frac{1}{2}\rho_i x_i(s)\right] ds. \tag{2.10}$$

Then, if $t \le \tau_k$, by Lemma 1.2, we obtain

$$\begin{aligned} LW(t,x(t)) &= LV(x) + \sum_i \left[\frac{1}{3}\beta_i' x_i^{3/2}(t) - y_i^{3/2}(t) + \frac{1}{2}\rho_i\left(x_i(t) - y_i(t)\right)\right] \\ &\le \phi(x) + \frac{1}{2}\sum_i \rho_i x_i(t) - \frac{1}{6d}\sum_i \left[3k_{1/2} - d\left(\beta_i + 2\beta_i'\right)\right] x_i^{3/2}(t) \\ &\le K \end{aligned} \tag{2.11}$$

with a constant K.

Consequently,

$$\begin{aligned} EW(x(\tau_k \wedge T)) &\le EW(\tau_k \wedge T, x(\tau_k \wedge T)) \\ &= W(0 \cdot x(0)) + E\int_0^{\tau_k \wedge T} LW(t,x(t))dt \\ &\le W(0 \cdot x(0)) + KT. \end{aligned} \tag{2.12}$$

On the other hand, if $\tau_k \le T$, then $x_i(\tau_k) \notin (k^{-1}, k)$ for some i; therefore,

$$\begin{aligned} V(x(\tau_k)) &\ge u\left(\frac{1}{\sqrt{k}}\right) \wedge u\left(\sqrt{k}\right) \longrightarrow \infty, \\ EV(x(\tau_k \wedge T)) &\ge P(\tau_k \le T)\left(u\left(\frac{1}{\sqrt{k}}\right) \wedge u\left(\sqrt{k}\right)\right) \end{aligned} \tag{2.13}$$

so $\lim_{k\to\infty} P(\tau_k \le T) = 0$; Theorem 2.1 is proved. □

3. Stochastically Ultimate Boundedness

Theorem 2.1 shows that under simple hypothesis conditions (1.2), (1.3), and (2.1), the solutions of (1.1) will remain in the positive cone R_+^d. This nice positive property provides us with a great opportunity to construct other types of Lyapunov functions to discuss how the solutions vary in R_+^d in more detail.

As mentioned in Section 2, the nonexplosion property in a population dynamical system is often not good enough but the property of ultimate boundedness is more desired. Let us now give the definition of stochastically ultimate boundedness.

Theorem 3.1. *Suppose* (2.1) *and the following condition:*

$$\min_i(-a_{ii} - A_i) > \max_i d\beta_i \tag{3.1}$$

hold. Then for all $\theta > 0$ and any initial data $\{x_t : -\tau \le t \le 0\} \in C([-\tau, 0], R_+^d)$, there is a positive constant K, which is independent of the initial data, such that the solution $x(t)$ of (1.1) *has the property that*

$$\limsup_{t\to\infty} E|x(t)|^\theta \le K. \tag{3.2}$$

Proof. If condition (1.2) is satisfied, then

$$\lim_{\theta\to\infty} \frac{\theta^\theta A_i^{1+\theta}}{(1+\theta)^{1+\theta}|a_{ii}|^\theta} = 0. \tag{3.3}$$

By Liapunov inequality,

$$\left(E|x|^r\right)^{1/r} \le \left(E|x|^\theta\right)^{1/\theta}, \quad \text{if } 0 < r < \theta < \infty. \tag{3.4}$$

So in the proof, we suppose θ is big enough, and these hypotheses will not effect the conclusion of the theorem.

Define the Lyapunov functions.

$$V(x(t)) = V_\theta(x(t)) = \sum_{i=1}^d x_i^\theta, \quad \left(x \in R_+^d\right). \tag{3.5}$$

It is sufficient to prove

$$\limsup_{t\to\infty} E|V(x(t))| \le K_0, \tag{3.6}$$

with a constant K_0, independent of initial data $\{x_t : -\tau \le t \le 0\} \in C([-\tau, 0], R_+^d)$.

We have

$$\begin{aligned} LV_\theta(x,y) &= \sum_i \theta x_i^\theta \left[b_i + \sum_j (a_{ij}x_j + b_{ij}y_j)\right] + \frac{\theta(\theta-1)}{2}\sum_{ij}\gamma_{ij}^2 x_i^\theta \\ &= \sum_i \theta x_i^\theta \left(b_i + \frac{\theta-1}{2}\sum_j \gamma_{ij}^2\right) + \theta I_\theta(x) + \theta\sum_{ij} b_{ij}x_i^\theta y_j \\ &\le cV_\theta(x) + \theta I_\theta(x) + \theta\sum_{ij} b_{ij}x_i^\theta y_j, \end{aligned} \tag{3.7}$$

where $c = \max_i \theta(b_i + ((\theta-1)/2)\sum_j \gamma_{ij}^2)$ is constant and $I_\theta(x)$ is given in (1.9). Let $z = x/\|x\|_\infty$, for all $x \in R_+^d$; by Lemma 1.1, we have

$$I_\theta(x) \; = I_\theta(z\|x\|_\infty) = I_\theta(z)\|x\|_\infty^{1+\theta} \le -K_\theta d^{-1-\theta}|x|^{1+\theta}. \tag{3.8}$$

Then,

$$\begin{gathered} I_\theta(x) \le -K_\theta \|x\|_\infty^{1+\theta} \le -K_\theta d^{-1} V_{\theta+1}(x), \\ \sum_{ij} b_{ij} x_i^\theta y_j \le \sum_{ij} b_{ij}^+ \frac{1}{1+\theta}\left(\theta x_i^{1+\theta} + y_j^{1+\theta}\right). \end{gathered} \tag{3.9}$$

Thus we obtain

$$LV_\theta(x,y) \le cV_\theta(x) - \frac{\theta}{d(1+\theta)}\sum_i \left[(1+\theta)K_\theta x_i^{1+\theta} - d\beta_i \theta x_i^{1+\theta} - d\beta_i' y_i^{1+\theta}\right], \tag{3.10}$$

and from (1.3)

$$\lim_{\theta\to\infty} K_\theta = \min_i(-a_{ii} - A_i), \tag{3.11}$$

if θ is big enough, then

$$(1+\theta)K_\theta > d\left(\theta\beta_i + e^\tau \beta_i'\right). \tag{3.12}$$

By Lemma 1.2 and inequality (3.12),

$$\begin{aligned} &e^s E V_\theta(x(s))|_0^t \\ &\quad = E\int_0^t e^s [V_\theta(x(s)) + LV_\theta(x(s))]ds \\ &\quad \le \int_0^t e^s \left[c_1 V_\theta(x(s)) - \frac{\theta}{d(1+\theta)}\sum_i ((1+\theta)K_\theta - d\beta_i\theta) x_i^{1+\theta}(s)\right] ds \\ &\qquad + E\int_0^t e^s \sum_i \frac{\beta_i'\theta}{1+\theta} x_i^{1+\theta}(s-\tau)ds \quad (c_1 = c+1) \\ &\quad \le E\int_0^t e^s \left[c_1 V_\theta(x(s)) - \frac{\theta}{d(1+\theta)}\sum_i ((1+\theta)K_\theta - d\beta_i\theta - de^\tau\beta_i') x_i^{1+\theta}(s)\right] ds \\ &\qquad + Ee^\tau \int_{-\tau}^0 e^s \sum_i \frac{\beta_i'\theta}{1+\theta} x_i^{1+\theta}(s)ds \end{aligned}$$

$$\begin{aligned}
&\leq E\int_0^t e^s[c_1V_\theta(x(s))-c_2V_{1+\theta}(x(s))]ds+Ee^\tau\int_{-\tau}^0 e^s\sum_i\frac{\beta_i'\theta}{1+\theta}x_i^{1+\theta}(s)ds\\
&\leq\int_0^t K_0e^sds+Ee^\tau\int_{-\tau}^0 e^s\sum_i\frac{\beta_i'\theta}{1+\theta}x_i^{1+\theta}(s)ds\\
&\leq K_0e^t-K_0+Ee^\tau\int_{-\tau}^0 e^s\sum_i\frac{\beta_i'\theta}{1+\theta}x_i^{1+\theta}(s)ds,
\end{aligned}\tag{3.13}$$

where $c_2=\inf_i(\theta/d(1+\theta))[(1+\theta)K_\theta-d\beta_i\theta-de^\tau\beta_i']>0$ is a constant. Then (3.2) follows from the above inequality and Theorem 3.1 is proved. □

4. Asymptotic Pathwise Estimation

In the previous sections, we have discussed how the solutions vary in R_+^d in probability or in moment. In this section, we will discuss the solutions pathwisely.

Theorem 4.1. *Suppose* (2.1) *holds and the following condition:*

$$K_1>d^{3/2}e^\tau\|B\|\tag{4.1}$$

is satisfied, where K_1 is given by (1.5) *and $\|B\|=\sup_{\|x\|=1}|Bx|$. Then for any initial data $\{x_t:-\tau\leq t\leq 0\}\in C([-\tau,0],R_+^d)$, the solution $x(t)$ of* (1.1) *has the property that*

$$\limsup_{t\to\infty}\frac{1}{t}\left[\ln|x(t)|+\left(K_1d^{-3/2}-\|B\|\right)\int_0^t|x(s)|ds\right]\leq|b|-\frac{\lambda}{2d}\quad a.s.,\tag{4.2}$$

where $\lambda=\lambda_{\min}GG^T$.

Proof. Define the Lyapunov functions

$$V(x)=V_1(x),\quad\text{for } x\in R_+^d.\tag{4.3}$$

By Itô's formula, we have

$$\begin{aligned}
\frac{LV(x,y)}{V(x)}&=\frac{b^Tx+I_1(x)+x^TBy}{V(x)}\\
&\leq\frac{|b||x|-K_1d^{-1}|x|^2+|x|\|B\|\left|y\right|}{V(x)}\\
&\leq|b|-K_1d^{-3/2}|x|+\|B\|\left|y\right|.
\end{aligned}\tag{4.4}$$

Therefore,

$$
\begin{aligned}
\ln V(x)|_0^t &= \int_0^t \left[\frac{LV(s)}{V(s)} - \frac{Z(s)^2}{2} \right] ds + M(t) \\
&\le \int_0^t \left[|b| - K_1 d^{-3/2} |x(s)| + \|B\| \left| y(s) \right| - \frac{Z(s)^2}{2} \right] ds + M(t),
\end{aligned}
\tag{4.5}
$$

where

$$
M(t) = \int_0^t Z(s) dw(s), \tag{4.6}
$$

where $Z = x^T G / V(x)$ is a real-valued continuous local martingale vanishing at $t = 0$ and its quadratic form is given by

$$
\langle M(t), M(t) \rangle = \int_0^t Z(s)^2 ds, \tag{4.7}
$$

and then

$$
|Z|^2 = V^{-2}(x) x^T G G^T x \ge \frac{\lambda}{d}. \tag{4.8}
$$

Now, let $\delta \in (0,1)$ be arbitrary. By the exponential martingale inequality [3–10], we can show that for every integer $n \ge 1$,

$$
P\left\{ \sup_{0 \le t \le n} \left[M(t) - \frac{\delta}{2} \int_0^t |Z|^2 ds \right] \ge \frac{2 lnn}{\delta} \right\} < \frac{1}{n^2}. \tag{4.9}
$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, the well-known Borel-Cantelli lemma yields that there is $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ there exists a random integer $n_0(\omega)$ such that for all $n \ge n_0(\omega)$,

$$
\sup_{0 \le t \le n} \left[M(t) - \frac{\delta}{2} \int_0^t |Z|^2 ds \right] \le \frac{2 lnn}{\delta} \tag{4.10}
$$

which implies

$$
M(t) \le \frac{\delta}{2} \int_0^t |Z(s)|^2 ds + \frac{2}{\delta} \ln(t+1) \quad on\ 0 \le t \le n \quad \text{a.s.} \tag{4.11}
$$

Substituting this into (4.6) and making use of the upper inequality, we derive that

$$\begin{aligned}
&\ln V(x(t)) - \ln V(x_0) - \frac{2}{\delta}\ln(t+1) \\
&\leq \int_0^t \left[|b| - \frac{K_1}{d^{3/2}}|x(s)| + \|B\|\left|y(s)\right| - \frac{\lambda(1-\delta)}{2d}\right] ds \\
&\leq t\left[|b| - \frac{\lambda(1-\delta)}{2d}\right] - \left(\frac{K_1}{d^{3/2}} - \|B\|\right)\int_0^t |x(s)|ds + \int_{-\tau}^0 \|B\||x(s)|ds.
\end{aligned} \tag{4.12}$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\left[\ln|x(t)| + \left(K_1 d^{-3/2} - \|B\|\right)\int_0^t |x(s)|ds\right] \leq |b| - \frac{\lambda(1-\delta)}{2d}, \quad \text{a.s.} \tag{4.13}$$

Puting $\delta \to 0$, we can get inequality(4.2). □

5. Further Topic

In this section, we introduce an economic model named stochastic R&D model in economic growth [10]; for the notion of the model, see details in reference [9, 10]. The equation is

$$d\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\left\{\left(\begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} -\theta & \xi \\ \alpha & -\beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)dt + \begin{bmatrix} a\theta & b\eta \\ a\alpha & b\alpha \end{bmatrix}\begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}\right\}, \tag{5.1}$$

where we put the delay $\tau = 0$; it is clear that the property of the model can be done by the example of condition (1.4). So we have the following theorem.

Theorem 5.1. *Let the following conditions be satisfied*

$$\xi - \theta < \frac{-2\alpha^{3/2}}{\sqrt{27\beta}}, \qquad \alpha - \beta < \frac{-2\xi^{3/2}}{\sqrt{27\theta}}. \tag{5.2}$$

Then for any given initial data $(x_0, y_0) \in R_+^d$, there exists a unique global solution to (5.1) *on* $t \geq 0$. *Moreover, this solution remains in R_+^d with probability 1.*

Remark 5.2. The explanations in population dynamic of the conditions (1.2), (1.3), and (2.1) for (1.1) are worth pointing out. Each species x_i has a special ability to inhibit the fast growth; the relationship of the species is the role of either species competition ($a_{ij} < 0, i \neq j$), or a low level of cooperation ($a_{ij} > 0, i \neq j$, but they are small enough).

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