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Research Article

Normality Criteria of Lahiri's Type and Their Applications

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We prove two normality criteria for families of some functions concerning Lahiri's type, the results generalize those given by Charak and Rieppo, Xu and Cao. As applications, we study a problem related to R. Brück's Conjecture and obtain a result that generalizes those given by Yang and Zhang, Lü, Xu and Chen.

1. Introduction and Main Results

Let $\mathbb{C}$ denote the complex plane, and let $f(z)$ be a nonconstant meromorphic function in $\mathbb{C}$. It is assumed that the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as the characteristic function $T(r,f)$, the proximity function $m(r,f)$, the counting function $N(r,f)$ (see, e.g., [1–4]), and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Let $a(z)$ and $b(z)$ be small functions of $f(z)$ and $g(z)$. $f(z) = a(z) \rightleftharpoons g(z) = b(z)$ means $f(z) - a(z)$ and $g(z) - b(z)$ have the same zeros (counting multiplicity) and $f(z) = \infty \rightleftharpoons g(z) = \infty$ means that f and g have the same poles (counting multiplicity). If $g(z) - b(z) = 0$ whenever $f(z) - a(z) = 0$, we write $f(z) = a(z) \Rightarrow g(z) = b(z)$. If $f(z) = a(z) \Rightarrow g(z) = b(z)$ and $g(z) = b(z) \Rightarrow f(z) = a(z)$, we write $f(z) = a(z) \Leftrightarrow g(z) = b(z)$. If $f(z) = a(z) \Leftrightarrow g(z) = a(z)$, then we say that f and g share a.

Set

$$P(f) = f^{n+n_1+\cdots+n_k},$$
$$M_1\left(f, f', \ldots, f^{(k)}\right) = f^n (f')^{n_1} \cdots \left(f^{(k)}\right)^{n_k},$$
$$M_2\left(f, f', \ldots, f^{(k)}\right) = f^m (f')^{m_1} \cdots \left(f^{(k)}\right)^{m_k}, \tag{1.1}$$
$$\gamma_{M_1} = n + n_1 + \cdots + n_k, \qquad \gamma_{M_2} = m + m_1 + \cdots + m_k,$$
$$\gamma^*_{M_1} = \sum_{j=1}^{k-1} n_j, \qquad \Gamma_{M_1} = \sum_{j=1}^{k} j n_j, \qquad \gamma^*_{M_2} = \sum_{j=1}^{k-1} m_j, \qquad \Gamma_{M_2} = \sum_{j=1}^{k} j m_j,$$

where $n, n_1, \ldots, n_k$, $m, m_1, \ldots, m_k$ are nonnegative integers. $M_i(f, f', \ldots, f^{(k)})$ is called the differential monomial of f and γ_{M_i} is called the degree of $M_i(f, f', \ldots, f^{(k)})$ $(i = 1,\ 2)$.

Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$, there exists a subsequence f_{n_j} such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

According to Bloch's principle, every condition which reduces a meromorphic function in $\mathbb{C}$ to a constant makes a family of meromorphic functions in a domain D normal. Although the principle is false in general, many authors proved normality criteria for families of meromorphic functions starting from Picard type theorems, for instance.

Theorem A (see [5]). *Let $n \geq 5$ be an integer, $a, b \in \mathbb{C}$ and $a \neq 0$. If, for a meromorphic function f, $f' + af^n \neq b$ for all $z \in \mathbb{C}$, then f must be a constant.*

Theorem B (see [6, 7]). *Let $n \geq 3$ be an integer, $a, b \in \mathbb{C}$, $a \neq 0$, and let $\mathcal{F}$, be a family of meromorphic functions in a domain D. If $f' + af^n \neq b$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

In 2005, Lahiri [8] got a normality criterion as follows.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Define*

$$E_f = \left\{ z \in D : f'(z) + \frac{a}{f(z)} = b \right\}. \tag{1.2}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

In 2009, Charak and Rieppo [9] generalized Theorem C and obtained two normality criteria of Lahiri's type.

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be positive integers such that $m_1n_2 - m_2n_1 > 0$, $m_1 + m_2 \geq 1$, $n_1 + n_2 \geq 2$, and put*

$$E_f = \left\{ z \in D : (f(z))^{n_1}(f'(z))^{m_1} + \frac{a}{(f(z))^{n_2}(f'(z))^{m_2}} = b \right\}. \tag{1.3}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1n_2 = m_2n_1$, and put*

$$E_f = \left\{ z \in D : (f(z))^{n_1}(f'(z))^{m_1} + \frac{a}{(f(z))^{n_2}(f'(z))^{m_2}} = b \right\}. \tag{1.4}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Very recently, Xu and Cao [10] further extended Theorems D and E by replacing f' with $f^{(k)}$; they got

Theorem F. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D, all of whose zeros have multiplicity at least k. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1n_2 - m_2n_1 > 0$, $m_1 + m_2 \geq 1$, $n_1 + n_2 \geq 2$, (if $n_1 = n_2 = 1$, $k \geq 5$), and put*

$$E_f = \left\{ z \in D : (f(z))^{n_1}\left(f^{(k)}(z)\right)^{m_1} + \frac{a}{(f(z))^{n_2}(f^{(k)}(z))^{m_2}} = b \right\}. \tag{1.5}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem G. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D, all of whose zeros have multiplicity at least k. Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let $m_1 \geq 2$, m_2, n_1, n_2 be positive integers such that $m_1n_2 = m_2n_1$, and put*

$$E_f = \left\{ z \in D : (f(z))^{n_1}\left(f^{(k)}(z)\right)^{m_1} + \frac{a}{(f(z))^{n_2}(f^{(k)}(z))^{m_2}} = b \right\}. \tag{1.6}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

To prove Theorems D–G, the authors used a key lemma (Lemma 2.4 in this paper) besides Zalcman-Pang's Lemma. It's natural to ask whether such normality criteria of Lahiri's type still hold for the general differential monomial $M(f, f', \ldots, f^{(k)})$. We study this problem and obtain the following theorem.

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k. Let a, $b \in \mathbb{C}$ such that $a \neq 0$, let m, n, $k(\geq 1)$, m_j, n_j $(j = 1, 2, \ldots, k)$ be nonnegative integers such that*

$$\gamma_{M_2}\Gamma_{M_1} - \gamma_{M_1}\Gamma_{M_2} > 0, \qquad n_k + m_k > 0, \qquad m + n \geq 2. \tag{1.7}$$

Put

$$E_f = \left\{ z \in D : M_1\left(f, f', \ldots, f^{(k)}\right) + \frac{a}{M_2\left(f, f', \ldots, f^{(k)}\right)} = b \right\}. \tag{1.8}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k. Let a, $b \in \mathbb{C}$ such that $a \neq 0$, let m, n, $k(\geq 1)$, m_j, n_j $(j = 1, 2, \ldots, k)$ be nonnegative integers such that $mnm_kn_k\gamma^*_{M_1}\gamma^*_{M_2} > 0$, ($k \neq 2$ when $n = 1$ or $m = 1$), $m/n = m_j/n_j$ for all positive integers m_j and n_j, $(1 \leq j \leq k)$. Put*

$$E_f = \left\{ z \in D : M_1\left(f, f', \ldots, f^{(k)}\right) + \frac{a}{M_2\left(f, f', \ldots, f^{(k)}\right)} = b \right\}. \tag{1.9}$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

As an application of Theorem 1.1, we obtain the following theorem.

Theorem 1.3. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k. Let $a, b(\neq 0)$ be two finite values and $n, k, n_1, \ldots, n_k$ be nonnegative integers with $n \geq 1$, $k \geq 1$, $n_k \geq 1$. For every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, if $P(f) = a \Leftrightarrow M_1(f, f', \ldots, f^{(k)}) = b$, then $\mathcal{F}$ is normal in D.*

Example 1.4. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m\}$. If $a = 0$, let $f_m := e^{mz}$. For each function $f \in \mathcal{F}$, $P(f)$ and $M_1(f, f', \ldots, f^{(k)})$ share 0 in D. However, it can be easily verified that $\mathcal{F}$ is not normal in D. Example 1.4 shows that the condition $b \neq 0$ in Theorem 1.3 is sharp.

Example 1.5. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m\}$. If $a \neq 0$, let $f_m := m(e^{\lambda z} - e^{-\lambda z})$, where λ is the root of $z^2 = b/a$. For each function $f \in \mathcal{F}$, $f'' = (b/a)f$, $f^{n+1} = a \Leftrightarrow f^n f'' = b$ in D. However, it can be easily verified that $\mathcal{F}$ is not normal in D. Example 1.5 shows that the multiplicity restriction on zeros of f in Theorem 1.3 is sharp (at least for $k = 2$).

2. Preliminary Lemmas

Lemma 2.1 (see [11]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, all of whose zeros have the multiplicity at least k, then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha < k$*

(a) *a number r, $0 < r < 1$,*

(b) *points z_n, $|z_n| < r$,*

(c) *functions $f_n \in \mathcal{F}$, and*

(d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0)$. Here, as usual, $g^{\#}(z) = |g'(z)|/(1+|g(z)|^2)$ is the spherical derivative.

Lemma 2.2 (see [1, page 158]). *Let $\mathcal{F} = \{f\}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. Then $\mathcal{F}$ is normal in D if and only if the spherical derivatives of functions $f \in \mathcal{F}$ are uniformly bounded on each compact subset of D.*

Lemma 2.3 (see [12]). *Let f be an entire function and M a positive integer. If $f^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, then f has the order at most one.*

Lemma 2.4 (see [13]). *Take nonnegative integers $n, n_1, \ldots, n_k$ with $n \geq 1$, $n_1 + n_2 + \cdots + n_k \geq 1$ and define $d = n+n_1+n_2+\cdots+n_k$. Let f be a transcendental meromorphic function with the deficiency $\delta(0,f) > 3/(3d+1)$. Then for any nonzero value c, the function $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} - c$ has infinitely many zeros. Moreover, if $n \geq 2$, the deficient condition can be omitted.*

The following two lemmas can be seen as supplements of Lemma 2.4.

Lemma 2.5. *Take nonnegative integers $n, n_1, \ldots, n_k$ with $n \geq 1$, $n_k \geq 1$ and define $d = n + n_1 + n_2 + \cdots + n_k$. Let f be a transcendental meromorphic function whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} - c$ has infinitely many zeros, provided that $n_1 + n_2 + \cdots + n_{k-1} \geq 1$ and $k \neq 2$ when $n = 1$. Specially, if f is transcendental entire, the function $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} - c$ has infinitely many zeros.*

Proof. If $n_1 + n_2 + \cdots + n_{k-1} = 0$, then $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} = f^n(f^{(k)})^{n_k}$, this case has been considered (see [5, 12–20]).

If $n_1 + n_2 + \cdots + n_{k-1} \geq 1$ and if $n \geq 2$, we immediately get the conclusion from Lemma 2.4. Next we consider the case $n = 1$.

Let $\Psi = f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}$. Using the proof of Lemma 2.4 (see [13, page 161–163]), we obtain

$$\begin{aligned}(3d-2)T(r,f) \leq{}& 3dN\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f}\right) + 4\overline{N}\left(r,\frac{1}{\Psi - c}\right)\\ &+ \overline{N}\left(r,\frac{\Psi - c}{\Psi'}\right) - 3N\left(r,\frac{\Psi - c}{\Psi'}\right) + S(r,f).\end{aligned} \tag{2.1}$$

Suppose that z_0 is a zero of f of multiplicity $p(\geq k)$, then z_0 is a zero of Ψ of multiplicity $dp - \Sigma_{j=1}^{k} jn_j$, and thus is a pole of $(\Psi - c)/\Psi'$ of multiplicity $dp - \Sigma_{j=1}^{k} jn_j - 1$. Thereby, from (2.1) we get

$$\begin{aligned}(3d-2)T(r,f) &\leq \left(3\sum_{j=1}^{k} jn_j + 5\right)\overline{N}\left(r,\frac{1}{f}\right) + 4\overline{N}\left(r,\frac{1}{\Psi - c}\right) + S(r,f)\\ &\leq \frac{3\sum_{j=1}^{k} jn_j + 5}{k} N\left(r,\frac{1}{f}\right) + 4\overline{N}\left(r,\frac{1}{\Psi - c}\right) + S(r,f).\end{aligned} \tag{2.2}$$

Note that $n = 1$, we deduce from (2.2) that

$$\frac{k - 5 + 3\sum_{j=1}^{k-1}(k-j)n_j}{k} T(r,f) \leq 4\overline{N}\left(r,\frac{1}{\Psi - c}\right) + S(r,f). \tag{2.3}$$

If $k = 1$, then $\Psi = f^n (f')^{n_1}$; this case has been proved as mentioned above (see [13–16]).

If $k \geq 5$, then we have $k - 5 + 3\sum_{j=1}^{k-1}(k-j)n_j > 0$; the conclusion is evident.

If $3 \leq k \leq 4$, note that $n_1 + n_2 + \cdots + n_{k-1} \geq 1$ and we deduce that $k - 5 + 3\sum_{j=1}^{k-1}(k - j)\ nj > 0$, thus the conclusion holds.

If f is a transcendental entire function, we only need to consider the case $k \geq 2$. Note that (see Hu et al. [21, page 67])

$$dT(r,f) \leq dN\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\Psi - c}\right) - N\left(r,\frac{\Psi - c}{\Psi'}\right) + S(r,f). \tag{2.4}$$

With similar discussion as above, we obtain

$$\left(n + \frac{\sum_{j=1}^{k-1}(k-j)n_j - 1}{k}\right) T(r,f) \leq \overline{N}\left(r,\frac{1}{\Psi - c}\right) + S(r,f). \tag{2.5}$$

In view of $n \geq 1$ and $k \geq 2$, we get $n + (\sum_{j=1}^{k-1}(k-j)n_j - 1)/k > 0$, thus we immediately obtain the conclusion. This completes the proof of Lemma 2.5. □

Lemma 2.6. *Take nonnegative integers $n, n_1, \ldots, n_k$, k with $n \geq 1$, $n_k \geq 1$, $k \geq 1$ and define $d = n + n_1 + n_2 + \cdots + n_k$. Let f be a nonconstant rational function whose zeros have multiplicity at least k. Then for any nonzero value c, the function $f^n (f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has at least one finite zero.*

Proof. Since the case $k = 1$ has been proved by Charak and Rieppo [9], we only need to consider $k \geq 2$.

Suppose that $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ has no zero.

Case 1. If f is a nonconstant polynomial, since the zeros of f have multiplicity at least k, we know that $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}$ is also a nonconstant polynomial, so $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ has at least one zero, which contradicts our assumption.

Case 2. If f is a nonconstant rational function but not a polynomial. Set

$$f(z)=A\frac{(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_s)^{m_s}}{(z-b_1)^{l_1}(z-b_2)^{l_2}\cdots(z-b_t)^{l_t}}, \tag{2.6}$$

where A is a nonzero constant and $m_i\ge k$ $(i=1,2,\dots,s)$, $l_j\ge 1$ $(j=1,2,\dots,t)$.

Then by mathematical induction, we get

$$f^{(k)}(z)=A\frac{(z-a_1)^{m_1-k}(z-a_2)^{m_2-k}\cdots(z-a_s)^{m_s-k}g_k(z)}{(z-b_1)^{l_1+k}(z-b_2)^{l_2+k}\cdots(z-b_t)^{l_t+k}}, \tag{2.7}$$

where $g_k(z)=(M-N)(M-N-1)\cdots(M-N-k+1)z^{k(s+t-1)}+c_m z^{k(s+t-1)-1}+\cdots+c_0$, $c_m,\dots,c_0$ are constants and

$$m_1+m_2+\cdots+m_s=M\ge ks, \tag{2.8}$$

$$l_1+l_2+\cdots+l_t=N\ge t. \tag{2.9}$$

It is easily obtained that

$$\deg(g_k)\le k(s+t-1). \tag{2.10}$$

Combining (2.6) and (2.7) yields

$$f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}=A^d\frac{(z-a_1)^{dm_1-\sum_{j=1}^k jn_j}\cdots(z-a_s)^{dm_s-\sum_{j=1}^k jn_j}g(z)}{(z-b_1)^{dl_1+\sum_{j=1}^k jn_j}\cdots(z-b_t)^{dl_t+\sum_{j=1}^k jn_j}}, \tag{2.11}$$

where $g(z)=\prod_{j=1}^k g_j^{n_j}(z)$ with $\deg(g)\le\sum_{j=1}^k jn_j(s+t-1)$.

Moreover, $g(z)$ is not a constant, or else, we get g_j is a constant for $j=1,\dots,k$. The leading coefficient of g_j is $M-N-(j-1)(s+t)$.

If g_1 is a constant, then we get

$$M=N. \tag{2.12}$$

If g_k is a constant, then we get

$$(k-1)(s+t)=0, \tag{2.13}$$

which implies $k=1$, a contradiction with the assumption $k\ge 2$.

Then from (2.11), we obtain

$$\left(f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}\right)' = A^d\frac{(z-a_1)^{dm_1-\sum_{j=1}^k jn_j-1}\cdots(z-a_s)^{dm_s-\sum_{j=1}^k jn_j-1}h(z)}{(z-b_1)^{dl_1+\sum_{j=1}^k jn_j+1}\cdots(z-b_t)^{dl_t+\sum_{j=1}^k jn_j+1}}, \tag{2.14}$$

where $h(z)$ is a polynomial with $s+t-1 \le \deg(h) \le (\sum_{j=1}^k jn_j+1)\ (s+t-1)$.

Since $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c \neq 0$, we obtain from (2.11) that

$$f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k} = c+\frac{B}{(z-b_1)^{dl_1+\sum_{j=1}^k jn_j}\cdots(z-b_t)^{dl_t+\sum_{j=1}^k jn_j}}, \tag{2.15}$$

where B is a nonzero constant. Then

$$\left(f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}\right)' = \frac{B\cdot H(z)}{(z-b_1)^{dl_1+\sum_{j=1}^k jn_j+1}\cdots(z-b_t)^{dl_t+\sum_{j=1}^k jn_j+1}}, \tag{2.16}$$

where $H(z)$ is a polynomial with $\deg(H)=t-1$.

From (2.14) and (2.16), we deduce that

$$dM-\left(\sum_{j=1}^k jn_j+1\right)s+\deg(h)=\deg(H)=t-1, \tag{2.17}$$

in view of $\deg(h)\ge s+t+1$, together with (2.8), we have

$$dks \le \sum_{j=1}^k jn_j s, \tag{2.18}$$

namely

$$nks+\sum_{j=1}^k (k-j)n_j s \le 0. \tag{2.19}$$

which is a contradiction since $n\ge 1$.

Hence $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ has at least one finite zero.

This proves Lemma 2.6. □

Remark 2.7. Lemma 2.6 is a generalization of Lemma 2.2 in [10]. The proof of Lemma 2.6 is quite different from its proof. Actually, the proof of Lemma 2.2 in [10] is incorrect. The main problem appears at (2.2) in [10]. Since f has only zero with multiplicity at least k, then each zero of f^n is of multiplicity at least nk, but this does not mean that each zero of $f^n(f^{(k)})^m$ is of multiplicity at least nk because the zeros of $f^{(k)}$ may not be the zeros of f, and thus their multiplicity may less than nk. Therefore, the inequality of (2.2) in [10] is not valid, which implies that the proof of Lemma 2.2 in [10] is not correct.

Lemma 2.8. *Let $a, b \in \mathbb{C}$ such that $a \neq 0$. Let $m, n, k(\geq 1), m_j, n_j$ $(j = 1, 2, \ldots, k)$ be nonnegative integers such that $mnm_k n_k \gamma^*_{M_1}\gamma^*_{M_2} > 0$, ($k \neq 2$ when $n = 1$ or $m = 1$), $m/n = m_j/n_j$ for all positive integers m_j and n_j, $(1 \leq j \leq k)$. Let f be a meromorphic function in $\mathbb{C}$; all zeros of f have multiplicity at least k. Define*

$$\Phi(z) = M_1\left(f, f', \ldots, f^{(k)}\right) + \frac{a}{M_2\left(f, f', \ldots, f^{(k)}\right)} - b. \tag{2.20}$$

Then $\Phi(z)$ has a finite zero.

Proof. The algebraic complex equation

$$x + \frac{a}{x^{m/n}} = b \tag{2.21}$$

has always a nonzero solution, say $x_0 \in \mathbb{C}$. By Lemmas 2.5 and 2.6, the differential monomial $M_1(f, f', \ldots, f^{(k)})$ cannot avoid it and thus there exists $z_0 \in \mathbb{C}$ such that $M_1(f(z_0), f'(z_0), \ldots, f^{(k)}(z_0)) = x_0$.

Under the assumptions, for all positive integers m, n, m_j, n_j, we have

$$m = n\frac{m}{n}, \qquad m_j = n_j\frac{m}{n}. \tag{2.22}$$

Thus

$$\Phi(z_0) = M_1\left(f(z_0), f'(z_0), \ldots, f^{(k)}(z_0)\right) + \frac{a}{M_1^{m/n}\left(f(z_0), f'(z_0), \ldots, f^{(k)}(z_0)\right)} - b = 0. \tag{2.23}$$

This proves Lemma 2.8. □

Lemma 2.9 (see [2, page 51]). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty}\frac{\log \nu(r, f)}{\log r}, \tag{2.24}$$

where $\nu(r, f)$ denotes the central-index of $f(z)$.

Lemma 2.10 (see [22, page 187–199] or [2, page 51]). *If g is a transcendental entire function, let $0 < \delta < 1/4$ and z be such that $|z| = r$ and that $|g(z)| = M(r, g)\nu(r, g)^{-(1/4)+\delta}$ holds. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure, that is, $\int_F dt/t < +\infty$ such that*

$$\frac{g^{(m)}(z)}{g(z)} = \left(\frac{\nu(r, g)}{z}\right)^m (1 + o(1)) \tag{2.25}$$

holds for all $m \geq 0$ and all $r \notin F$.

3. Proof of Theorem 1.1

Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 2.1, for $0 \le \alpha < k$, there exist $r < 1$, $z_j \in \Delta$ such that $z_j \to z_0$, $f_j \in \mathcal{F}$ and $\rho_j \to 0$ such that $g_j(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k. For simplicity, we denote $f_j(z_j + \rho_j \xi)$ by f_j. By Lemmas 2.4 and 2.6, there exists $\xi_0 \in \mathbb{C}$ such that

$$g(\xi_0)^n (g'(\xi_0))^{n_1} \cdots \left(g^{(k)}(\xi_0)\right)^{n_k} + \frac{a}{g(\xi_0)^m (g'(\xi_0))^{m_1} \cdots (g^{(k)}(\xi_0))^{m_k}} = 0. \tag{3.1}$$

Obviously, $g(\xi_0) \ne 0, \infty$, so in some neighborhood of ξ_0, $g_j(\xi)$ converges uniformly to $g(\xi)$. We have

$$\begin{aligned}
&g_j(\xi)^n \left(g_j'(\xi)\right)^{n_1} \cdots \left(g_j^{(k)}(\xi)\right)^{n_k} + \frac{a}{g_j(\xi)^m \left(g_j'(\xi)\right)^{m_1} \cdots \left(g_j^{(k)}(\xi)\right)^{m_k}} - \rho_j^{\alpha\gamma_{M_2} - \Gamma_{M_2}} b \\
&= \rho_j^{-\alpha\gamma_{M_1} + \Gamma_{M_1}} f_j^n \left(f_j'\right)^{n_1} \cdots \left(f_j^{(k)}\right)^{n_k} + \frac{a}{\rho_j^{-\alpha\gamma_{M_2} + \Gamma_{M_2}} f_j^m \left(f_j'\right)^{m_1} \cdots \left(f_j^{(k)}\right)^{m_k}} - \rho_j^{\alpha\gamma_{M_2} - \Gamma_{M_2}} b \\
&= \rho_j^{\alpha\gamma_{M_2} - \Gamma_{M_2}} \left[\rho_j^{-\alpha(\gamma_{M_1} + \gamma_{M_2}) + \Gamma_{M_1} + \Gamma_{M_2}} f_j^n \left(f_j'\right)^{n_1} \cdots \left(f_j^{(k)}\right)^{n_k} + \frac{a}{f_j^m \left(f_j'\right)^{m_1} \cdots \left(f_j^{(k)}\right)^{m_k}} - b\right].
\end{aligned} \tag{3.2}$$

Let $\alpha = (\Gamma_{M_1} + \Gamma_{M_2})/(\gamma_{M_1} + \gamma_{M_2}) < k$, and under the assumption $\gamma_{M_2}\Gamma_{M_1} - \gamma_{M_1}\Gamma_{M_2} > 0$, we obtain

$$g^n (g')^{n_1} \cdots \left(g^{(k)}\right)^{n_k} + \frac{a}{g^m (g')^{m_1} \cdots (g^{(k)})^{m_k}} \tag{3.3}$$

is the uniform limit of

$$\rho_j^{(\gamma_{M_2}\Gamma_{M_1} - \gamma_{M_1}\Gamma_{M_2})/(\gamma_{M_1} + \gamma_{M_2})} \left[f_j^n \left(f_j'\right)^{n_1} \cdots \left(f_j^{(k)}\right)^{n_k} + \frac{a}{f_j^m \left(f_j'\right)^{m_1} \cdots \left(f_j^{(k)}\right)^{m_k}} - b\right] \tag{3.4}$$

in some neighborhood of ξ_0.

By (3.1) and Hurwitz's theorem, there exists a sequence $\xi_j \to \xi_0$ such that for all large values of j and $\zeta_j = z_j + \rho_j \xi_j$,

$$(f_j(\zeta_j))^n \left(f_j'(\zeta_j)\right)^{n_1} \cdots \left(f_j^{(k)}(\zeta_j)\right)^{n_k} + \frac{a}{(f_j(\zeta_j))^m \left(f_j'(\zeta_j)\right)^{m_1} \cdots \left(f_j^{(k)}(\zeta_j)\right)^{m_k}} = b. \tag{3.5}$$

Hence for all large values of j, $\zeta_j = z_j + \rho_j \xi_j \in E_f$, it follows from the condition that

$$\left|g_j(\xi_j)\right| = \frac{\left|f_j(\zeta_j)\right|}{\rho_j^{\alpha}} \geq \frac{M}{\rho_j^{\alpha}}. \tag{3.6}$$

Since ξ_0 is not a pole of g, there exists a positive number K such that in some neighborhood of ξ_0 we get $|g(\xi)| \leq K$.

Since $g_j(\xi)$ converges uniformly to $g(\xi)$ in some neighborhood of ξ_0, we get for all large values of j and for all ξ in that neighborhood of ξ_0

$$\left|g_j(\xi) - g(\xi)\right| < 1. \tag{3.7}$$

By (3.7), we get

$$K \geq \left|g(\xi_j)\right| \geq \left|g_j(\xi_j)\right| - \left|g_j(\xi_j) - g(\xi_j)\right| \geq \frac{M}{\rho_j^{\alpha} - 1}, \tag{3.8}$$

which is a contradiction since $\rho_j \to 0$ as $j \to \infty$.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D. By Lemma 2.1, for $0 \leq \alpha < k$, there exist $r < 1$, $z_j \in \Delta$, $f_j \in \mathcal{F}$ and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k. By Lemma 2.8, we get

$$g(\xi_0)^n \left(g'(\xi_0)\right)^{n_1} \cdots \left(g^{(k)}(\xi_0)\right)^{n_k} + \frac{a}{g(\xi_0)^m \left(g'(\xi_0)\right)^{m_1} \cdots \left(g^{(k)}(\xi_0)\right)^{m_k}} - b = 0, \tag{4.1}$$

for some $\xi_0 \in \mathbb{C}$.

We can arrive at a contradiction by using the same argument as in the latter part of proof of Theorem 1.1.

This completes the proof of Theorem 1.2.

5. Applications

Proof of Theorem 1.3. We shall divide our argument into two cases.

Case 1 ($a \neq 0$). Let M be a positive constant with $M \leq \sqrt[\gamma_{M_1}]{|a|}$; under the assumptions, we have

$$E_f = \left\{z \in D : M_1\left(f, f', \ldots, f^{(k)}\right) = b\right\} \tag{5.1}$$

and $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$; by Lemmas 2.5 and 2.6, using the similar proof of Theorem 1.1, we obtain the conclusion.

Case 2 ($a = 0$). For $f \in \mathcal{F}$, if $f(z_0) = 0$ for $z_0 \in D$, since $P(f) = 0 \Rightarrow M_1(f, f', \dots, f^{(k)}) = b$, we have $b = 0$, which is a contradiction, hence $f \neq 0$.

If $M_1(f(z_0), f'(z_0), \dots, f^{(k)}(z_0)) = b$ for $z_0 \in D$, since $M_1(f, f', \dots, f^{(k)}) = b \Rightarrow P(f) = 0$, we immediately get $f(z_0) = 0$ and hence $M_1(f, f', \dots, f^{(k)}) = b = 0$, which is still a contradiction, hence $M_1(f, f', \dots, f^{(k)}) \neq b$.

Suppose that $\mathcal{F}$ is not normal in D, by Lemma 2.1, there exist $r < 1$, $z_j \in \Delta$, $f_j \in \mathcal{F}$, and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-\Gamma_{M_1}/\gamma_{M_1}} f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant entire function, all of whose zeros have multiplicity at least k. By Hurwitz's theorem, we have

(i) $g \equiv 0$ or $g \neq 0$, and

(ii) $g^n (g')^{n_1} \cdots (g^{(k)})^{n_k} \equiv b$ or $g^n (g')^{n_1} \cdots (g^{(k)})^{n_k} \neq b$.

Since g is not a constant, we have $g \neq 0$. By Lemma 2.3, g has the order at most 1, so $g(\xi) = e^{c\xi + d}$, where $c(\neq 0)$, d are two constants. Thus

$$g^n(\xi)(g')^{n_1}(\xi) \cdots \left(g^{(k)}\right)^{n_k}(\xi) = c^{\Gamma_{M_1}} e^{\gamma_{M_1}(c\xi + d)}. \tag{5.2}$$

If $g^n (g')^{n_1} \cdots (g^{(k)})^{n_k} \equiv b$, we immediately get a contradiction. Hence

$$g^n (g')^{n_1} \cdots \left(g^{(k)}\right)^{n_k} \neq b, \tag{5.3}$$

but by Lemmas 2.5 and 2.6 we get a contradiction again.

This proves Theorem 5.1. □

Further more, using Theorem 1.3, we obtain a uniqueness theorem related to R. Brück's Conjecture. Firstly, we recall this conjecture.

R. Brück's Conjecture

Let f be a nonconstant entire function such that the hyper-order $\sigma_2(f)$ is not a positive integer and $\sigma_2(f) < \infty$. If f and f' share a finite value a CM, then

$$\frac{f' - a}{f - a} = c, \tag{5.4}$$

where c is a nonzero constant and the hyper-order $\sigma_2(f)$ is defined as follow:

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}. \tag{5.5}$$

Since then, many results related to this conjecture have been obtained. We refer the reader to Brück [23], Gundersen and Yang [24], Yang [25], Chen and Shon [26], Li and Gao [27], and Wang [28].

It's interesting to ask what happens if f is replaced by f^n in Brück's Conjecture. Recently, Yang and Zhang [29] considered this problem and got the following theorem.

Theorem H. *Let f be a nonconstant entire function. $n \geq 7$ be an integer, and let $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{z/n}, \tag{5.6}$$

where c is a nonzero constant.

Lü et al. [30] improves Theorem H and obtained the following theorem.

Theorem I. *Let $Q_1(\not\equiv 0)$ be a polynomial, and let $n \geq 2$ be an intege; let $f(z)$ be a transcendental entire function, and let $F(z) = (f(z))^n$. If $F(z)$ and $F'(z)$ share Q_1 CM, then*

$$f(z) = Ae^{z/n}, \tag{5.7}$$

where A is a nonzero constant.

We obtain a more general result as follows.

Theorem 5.1. *Let $n, k, n_1, \ldots, n_k$ be nonnegative integers with $n \geq 1$, $k \geq 1$, $n_k \geq 1$, and a, b be two finite nonzero values. Let f be a nonconstant entire function whose zeros have multiplicity at least k. If $f^{n+n_1+\cdots+n_k} = a \rightleftharpoons f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} = b$, then*

$$\frac{f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k} - b}{f^{n+n_1+\cdots+n_k} - a} = c, \tag{5.8}$$

where c is a nonzero constant. Specially, if $a = b$, then $f = c_1 e^{\omega z}$, where c_1 is a nonzero constant, ω is the root of $t^{\Gamma_{M_1}} = 1$.

Proof of Theorem 5.1. First we assert that $\sigma(f) \leq 1$. Let

$$\mathcal{F} = \{g_\omega(z) = f(z+\omega), \omega \in \mathbb{C}\}, \quad z \in D = \Delta. \tag{5.9}$$

Under the assumptions of Theorem 1.3, we get that $\mathcal{F}$ is a normal family of holomorphic functions in D. By Lemma 2.2, there exists a constant M such that

$$f^{\#}(\omega) = \frac{|f'(\omega)|}{1+|f(\omega)|^2} = \frac{|g'_\omega(0)|}{1+|g_\omega(0)|^2} = g^{\#}_\omega(0) \leq M, \tag{5.10}$$

for all $\omega \in \mathbb{C}$. Hence by Lemma 2.3, f has the order at most 1.

Since $f^{n+n_1+\cdots+n_k} = a \rightleftharpoons f^n(f')^{n_1}\cdots(f^{(k)})^{n_k} = b$, f must be a transcendental entire function and

$$\frac{f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}-b}{f^{n+n_1+\cdots+n_k}-a}=e^{\alpha(z)}. \tag{5.11}$$

From (5.11), we have $T(r,e^{\alpha(z)}) = O(T(r,f))$, hence $\sigma(e^{\alpha}) \le \sigma(f) \le 1$ and $\alpha(z)$ is a polynomial with $\deg(\alpha) \le 1$. Note that f is transcendental, we have $M(r,f) \to \infty$, as $r \to \infty$. Let $M(r_n,f) = f(z_n)$, where $z_n = r_n e^{i\theta_n}$, we deduce

$$\lim_{r_n\to\infty}\frac{1}{f(z_n)}=\lim_{r_n\to\infty}\frac{1}{M(r_n,f)}=0. \tag{5.12}$$

By Lemma 2.10, there exists a subset $F_1 \subset (1,\infty)$ of finite logarithmic measure, namely $\int_{F_1} dt/t < +\infty$ such that for some point $z_n = r_n e^{i\theta_n}$ $(\theta_n \in (0,2\pi))$ satisfying $|z_n| = r_n \notin F_1$ and $M(r_n,f) = |f(z_n)|$, we obtain

$$\frac{f^{(k)}(z_n)}{f(z_n)}=\left(\frac{\nu(r_n,f)}{z_n}\right)^k(1+o(1)), \tag{5.13}$$

as $r \to \infty$.

Rewrite (5.11) as

$$\frac{(f'/f)^{n_1}\cdots\left(f^{(k)}/f\right)^{n_k}-b/f^{n+n_1+\cdots+n_k}}{1-a/f^{n+n_1+\cdots+n_k}}=e^{\alpha(z)}, \tag{5.14}$$

it follows from (5.12)–(5.14) and Lemma 2.8 that

$$\begin{aligned}|\alpha(z_n)| &= \left|\log e^{\alpha(z_n)}\right| = \left|\Gamma_{M_1}\left(\log\nu(r_n,f)-\log\left(r_ne^{i\theta_n}\right)\right)+o(1)\right|\\ &= \left|\Gamma_{M_1}(\log\nu(r_n,f)-\log r_n - i\theta(r_n))+o(1)\right|\\ &\le O(\log r_n),\end{aligned} \tag{5.15}$$

as $r_n \to \infty$. Since $\alpha(z)$ is a polynomial, from (5.15), we deduce that $\alpha(z)$ is a constant. Let $e^{\alpha} = c$, then c is a nonzero constant. Thus

$$\frac{f^n(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}-b}{f^{n+n_1+\cdots+n_k}-a}=c. \tag{5.16}$$

Specially, if $a = b$, suppose that f has a zero z_0, by letting $z = z_0$ in (5.16), we get $c = 1$; hence

$$f^{n_1+\cdots+n_k}=(f')^{n_1}\cdots\left(f^{(k)}\right)^{n_k}. \tag{5.17}$$

Suppose that z_0 is a zero of f with multiplicity $p(\geq k)$, then z_0 is a zero of $f^{n_1+\cdots+n_k}$ with multiplicity $(n_1+\cdots+n_k)p$, and a zero of $(f')^{n_1}\cdots(f^{(k)})^{n_k}$ with multiplicity $(n_1+\cdots+n_k)p-\Gamma_{M_1}$, which is a contradiction. So f has no zero, note that f is a transcendental entire function and $\sigma(f)\leq 1$, we have $f=c_1e^{tz}$, where c_1 and t are two finite nonzero values. In view of (5.16) and $a=b$, we deduce that

$$c_1^{\Gamma_{M_1}}\left(t^{\Gamma_{M_1}}-c\right)e^{\gamma_{M_1}tz}=b(1-c); \tag{5.18}$$

hence $c=1$ and $t^{\Gamma_{M_1}}=c=1$. $f=c_1e^{\omega z}$, ω is the root of $t^{\Gamma_{M_1}}=1$.

This completes the proof of Theorem 5.1. □

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