Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 787939, 7 pages
doi:10.1155/2011/787939

Research Article

Fractional Quantum Integral Inequalities

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Received 10 November 2010; Revised 19 January 2011; Accepted 16 February 2011

Academic Editor: J. Szabados

The aim of the present paper is to establish some fractional q-integral inequalities on the specific time scale, $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$, and $0 < q < 1$.

1. Introduction

The study of fractional q-calculus in [1] serves as a bridge between the fractional q-calculus in the literature and the fractional q-calculus on a time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}$, where $t_0 \in \mathbb{R}$, and $0 < q < 1$.

Belarbi and Dahmani [2] gave the following integral inequality, using the Riemann-Liouville fractional integral: if f and g are two synchronous functions on $[0, \infty)$, then

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha} f(t) J^{\alpha} g(t), \tag{1.1}$$

for all $t > 0$, $\alpha > 0$.

Moreover, the authors [2] proved a generalized form of (1.1), namely that if f and g are two synchronous functions on $[0, \infty)$, then

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(fg)(t) \geq J^{\alpha} f(t) J^{\beta} g(t) + J^{\beta} f(t) J^{\alpha} g(t), \tag{1.2}$$

for all $t > 0$, $\alpha > 0$, and $\beta > 0$.

Furthermore, the authors [2] pointed out that if $(f_i)_{i=1,2,\dots,n}$ are n positive increasing functions on $[0,\infty)$, then

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(J^{\alpha} f(1)\right)^{1-n} \prod_{i=1}^{n} J^{\alpha} f_i(t), \tag{1.3}$$

for any $t > 0$, $\alpha > 0$.

In this paper, we have obtained fractional q-integral inequalities, which are quantum versions of inequalities (1.1), (1.2), and (1.3), on the specific time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n,$ n a nonnegative integer$\} \cup \{0\}$, where $t_0 \in \mathbb{R}$, and $0 < q < 1$. In general, a time scale is an arbitrary nonempty closed subset of the real numbers [3].

Many authors have studied the fractional integral inequalities and applications. For example, we refer the reader to [4–6].

To the best of our knowledge, this paper is the first one that focuses on fractional q-integral inequalities.

2. Description of Fractional q-Calculus

Let $t_0 \in \mathbb{R}$ and define

$$\mathbb{T}_{t_0} = \left\{t : t = t_0 q^n,\ n \text{ a nonnegative integer}\right\} \cup \{0\}, \quad 0 < q < 1. \tag{2.1}$$

If there is no confusion concerning t_0, we will denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, the nabla q-derivative of f is

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.2}$$

for all $t \in \mathbb{T} \setminus \{0\}$. The q-integral of f is

$$\int_0^t f(s)\nabla s = (1-q)t\sum_{i=0}^{\infty} q^i f\left(tq^i\right). \tag{2.3}$$

The fundamental theorem of calculus applies to the q-derivative and q-integral; in particular,

$$\nabla_q \int_0^t f(s)\nabla s = f(t), \tag{2.4}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla s = f(t) - f(0). \tag{2.5}$$

Let $\mathbb{T}_{t_1}$, $\mathbb{T}_{t_2}$ denote two time scales. Let $f : \mathbb{T}_{t_1} \to \mathbb{R}$ be continuous let $g : \mathbb{T}_{t_1} \to \mathbb{T}_{t_2}$ be q-differentiable, strictly increasing, and $g(0) = 0$. Then for $b \in \mathbb{T}_{t_1}$,

$$\int_0^b f(t)\nabla_q g(t)\nabla t = \int_0^{g(b)} \left(f \circ g^{-1}\right)(s)\nabla s. \tag{2.6}$$

The q-factorial function is defined in the following way: if n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)\left(t-q^2 s\right)\cdots\left(t-q^{n-1}s\right). \tag{2.7}$$

If n is not a positive integer, then

$$(t-s)^{\underline{(n)}} = t^n \prod_{k=0}^{\infty} \frac{1-(s/t)q^k}{1-(s/t)q^{n+k}}. \tag{2.8}$$

The q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(n)}} = \frac{1-q^n}{1-q}(t-s)^{\underline{(n-1)}}, \tag{2.9}$$

and the q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(n)}} = -\frac{1-q^n}{1-q}(t-qs)^{\underline{(n-1)}}. \tag{2.10}$$

The q-exponential function is defined as

$$e_q(t) = \prod_{k=0}^{\infty}\left(1-q^k t\right), \quad e_q(0) = 1. \tag{2.11}$$

Define the q-Gamma function by

$$\Gamma_q(\nu) = \frac{1}{1-q}\int_0^1 \left(\frac{t}{1-q}\right)^{\nu-1} e_q(qt)\nabla t, \quad \nu \in \mathbb{R}^+. \tag{2.12}$$

Note that

$$\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu), \quad \nu \in \mathbb{R}^+, \text{ where } [\nu]_q := \frac{1-q^\nu}{1-q}. \tag{2.13}$$

The fractional q-integral is defined as

$$\nabla_q^{-\nu} f(t) = \frac{1}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)\nabla s. \tag{2.14}$$

Note that

$$\nabla_q^{-\nu}(1) = \frac{1}{\Gamma_q(\nu)} \frac{q-1}{q^{\nu}-1} t^{\underline{(\nu)}} = \frac{1}{\Gamma_q(\nu+1)} t^{\underline{(\nu)}}. \tag{2.15}$$

More results concerning fractional q-calculus can be found in [1, 7–9].

3. Main Results

In this section, we will state our main results and give their proofs.

Theorem 3.1. *Let f and g be two synchronous functions on $\mathbb{T}_{t_0}$. Then for all $t > 0$, $\nu > 0$, we have*

$$\nabla_q^{-\nu}(fg)(t) \geq \frac{\Gamma_q(\nu+1)}{t^{\underline{(\nu)}}} \nabla_q^{-\nu} f(t) \nabla_q^{-\nu} g(t). \tag{3.1}$$

Proof. Since f and g are synchronous functions on $\mathbb{T}_{t_0}$, we get

$$(f(s) - f(\rho))(g(s) - g(\rho)) \geq 0 \tag{3.2}$$

for all $s > 0$, $\rho > 0$. By (3.2), we write

$$f(s)g(s) + f(\rho)g(\rho) \geq f(s)g(\rho) + f(\rho)g(s). \tag{3.3}$$

Multiplying both side of (3.3) by $(t-qs)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$, we have

$$\begin{aligned} &\frac{(t-qs)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)} f(s)g(s) + \frac{(t-qs)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)} f(\rho)g(\rho) \\ &\quad \geq \frac{(t-qs)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)} f(s)g(\rho) + \frac{(t-qs)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)} f(\rho)g(s). \end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) with respect to s on $(0, t)$, we obtain

$$\begin{aligned} &\frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)g(s) \nabla s + \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(\rho)g(\rho) \nabla s \\ &\quad \geq \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)g(\rho) \nabla s + \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(\rho)g(s) \nabla s. \end{aligned} \tag{3.5}$$

So,

$$\begin{aligned}&\nabla_q^{-\nu}(fg)(t)+f(\rho)g(\rho)\frac{1}{\Gamma_q(\nu)}\int_0^t(t-qs)^{\underline{(\nu-1)}}\nabla s\\&\quad\geq\frac{g(\rho)}{\Gamma_q(\nu)}\int_0^t(t-qs)^{\underline{(\nu-1)}}f(s)\nabla s+\frac{f(\rho)}{\Gamma_q(\nu)}\int_0^t(t-qs)^{\underline{(\nu-1)}}g(s)\nabla s.\end{aligned}\tag{3.6}$$

Hence, we have

$$\nabla_q^{-\nu}(fg)(t)+f(\rho)g(\rho)\nabla_q^{-\nu}(1)\geq g(\rho)\nabla_q^{-\nu}(f)(t)+f(\rho)\nabla_q^{-\nu}(g)(t).\tag{3.7}$$

Multiplying both side of (3.7) by $(t-q\rho)^{\underline{(\nu-1)}}/\Gamma_q(\nu)$, we obtain

$$\begin{aligned}&\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\nabla_q^{-\nu}(fg)(t)+\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}f(\rho)g(\rho)\nabla_q^{-\nu}(1)\\&\quad\geq\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}g(\rho)\nabla_q^{-\nu}f(t)+\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}f(\rho)\nabla_q^{-\nu}g(t).\end{aligned}\tag{3.8}$$

Integrating both side of (3.8) with respect to ρ on $(0,t)$, we get

$$\begin{aligned}&\nabla_q^{-\nu}(fg)(t)\int_0^t\frac{(t-q\rho)^{\underline{(\nu-1)}}}{\Gamma_q(\nu)}\nabla\rho+\frac{\nabla_q^{-\nu}(1)}{\Gamma_q(\nu)}\int_0^t f(\rho)g(\rho)(t-q\rho)^{\underline{(\nu-1)}}\nabla\rho\\&\quad\geq\frac{\nabla_q^{-\nu}f(t)}{\Gamma_q(\nu)}\int_0^t(t-q\rho)^{\underline{(\nu-1)}}g(\rho)\nabla\rho+\frac{\nabla_q^{-\nu}g(t)}{\Gamma_q(\nu)}\int_0^t(t-q\rho)^{\underline{(\nu-1)}}f(\rho)\nabla\rho.\end{aligned}\tag{3.9}$$

Obviously,

$$\nabla_q^{-\nu}(fg)(t)\geq\frac{1}{\nabla_q^{-\nu}(1)}\nabla_q^{-\nu}f(t)\nabla_q^{-\nu}g(t)=\frac{\Gamma_q(\nu+1)}{t^{\underline{(\nu)}}}\nabla_q^{-\nu}f(t)\nabla_q^{-\nu}g(t)\tag{3.10}$$

and the proof is complete. □

The following result may be seen as a generalization of Theorem 3.1.

Theorem 3.2. *Let f and g be as in Theorem 3.1. Then for all $t>0$, $\nu>0$, $\mu>0$ we have*

$$\frac{t^{\underline{(\nu)}}}{\Gamma_q(\nu+1)}\nabla_q^{-\mu}(fg)(t)+\frac{t^{\underline{(\mu)}}}{\Gamma_q(\mu+1)}\nabla_q^{-\nu}(fg)(t)\geq\nabla_q^{-\nu}f(t)\nabla_q^{-\mu}g(t)+\nabla_q^{-\mu}f(t)\nabla_q^{-\nu}g(t).\tag{3.11}$$

Proof. By making similar calculations as in Theorem 3.1 we have

$$\begin{aligned} &\frac{(t-q\rho)^{\underline{(\mu-1)}}}{\Gamma_q(\mu)}\nabla_q^{-\nu}(fg)(t)+\nabla_q^{-\nu}(1)\frac{(t-q\rho)^{\underline{(\mu-1)}}}{\Gamma_q(\mu)}f(\rho)g(\rho)\\ &\quad\geq\frac{(t-q\rho)^{\underline{(\mu-1)}}}{\Gamma_q(\mu)}g(\rho)\nabla_q^{-\nu}f(t)+\frac{(t-q\rho)^{\underline{(\mu-1)}}}{\Gamma_q(\mu)}f(\rho)\nabla_q^{-\nu}g(t). \end{aligned} \tag{3.12}$$

Integrating both side of (3.12) with respect to ρ on $(0,t)$, we obtain

$$\begin{aligned} &\nabla_q^{-\nu}(fg)(t)\int_0^t\frac{(t-q\rho)^{\underline{(\mu-1)}}}{\Gamma_q(\mu)}\nabla\rho+\frac{\nabla_q^{-\nu}(1)}{\Gamma_q(\mu)}\int_0^t f(\rho)g(\rho)(t-q\rho)^{\underline{(\mu-1)}}\nabla\rho\\ &\quad\geq\frac{\nabla_q^{-\nu}f(t)}{\Gamma_q(\mu)}\int_0^t(t-q\rho)^{\underline{(\mu-1)}}g(\rho)\nabla\rho+\frac{\nabla_q^{-\nu}g(t)}{\Gamma_q(\mu)}\int_0^t(t-q\rho)^{\underline{(\mu-1)}}f(\rho)\nabla\rho. \end{aligned} \tag{3.13}$$

Thus, (3.11) holds for all $t>0$, $\nu>0$, $\mu>0$, so the proof is complete. □

Remark 3.3. The inequalities (3.1) and (3.11) are reversed if the functions are asynchronous on $\mathbb{T}_{t_0}$ (i.e., $(f(x)-f(y))(g(x)-g(y))\leq 0$, for any $x,y\in\mathbb{T}_{t_0}$).

Theorem 3.4. *Let $(f_i)_{i=1,\dots,n}$ be n positive increasing functions on $\mathbb{T}_{t_0}$. Then for any $t>0$, $\nu>0$ we have*

$$\nabla_q^{-\nu}\left(\prod_{i=1}^n f_i\right)(t)\geq\left(\nabla_q^{-\nu}(1)\right)^{1-n}\prod_{i=1}^n\nabla_q^{-\nu}f_i(t). \tag{3.14}$$

Proof. We prove this theorem by induction.

Clearly, for $n=1$, we have

$$\nabla_q^{-\nu}(f_1)(t)\geq\nabla_q^{-\nu}(f_1)(t), \tag{3.15}$$

for all $t>0$, $\nu>0$.

For $n=2$, applying (3.1), we obtain

$$\nabla_q^{-\nu}(f_1f_2)(t)\geq\left(\nabla_q^{-\nu}(1)\right)^{-1}\nabla_q^{-\nu}(f_1)(t)\nabla_q^{-\nu}(f_2)(t), \tag{3.16}$$

for all $t>0$, $\nu>0$.

Suppose that

$$\nabla_q^{-\nu}\left(\prod_{i=1}^{n-1} f_i\right)(t)\geq\left(\nabla_q^{-\nu}(1)\right)^{2-n}\prod_{i=1}^{n-1}\nabla_q^{-\nu}f_i(t),\quad t>0,\ \nu>0. \tag{3.17}$$

Since $(f_i)_{i=1,\ldots,n}$ are positive increasing functions, then $(\prod_{i=1}^{n-1} f_i)(t)$ is an increasing function. Hence, we can apply Theorem 3.1 to the functions $\prod_{i=1}^{n-1} f_i = g$, $f_n = f$. We obtain

$$\nabla_q^{-\nu}\left(\prod_{i=1}^{n} f_i\right)(t) = \nabla_q^{-\nu}(fg)(t) \geq \left(\nabla_q^{-\nu}(1)\right)^{-1}\nabla_q^{-\nu}\left(\prod_{i=1}^{n-1} f_i\right)(t)\nabla_q^{-\nu}(f_n)(t). \tag{3.18}$$

Taking into account the hypothesis (3.17), we obtain

$$\nabla_q^{-\nu}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(\nabla_q^{-\nu}(1)\right)^{-1}\left(\left(\nabla_q^{-\nu}(1)\right)^{2-n}\left(\prod_{i=1}^{n-1}\nabla_q^{-\nu} f_i\right)(t)\right)\nabla_q^{-\nu}(f_n)(t) \tag{3.19}$$

and this ends the proof. □

Acknowledgment

The authors thank referees for suggestions which have improved the final version of this paper.

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