## Research Article

# A Sharp Double Inequality for Sums of Powers 

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It is established that the sequences $n \mapsto S(n):=\sum_{k=1}^{n}(k / n)^{n}$ and $n \mapsto n(e /(e-1)-S(n))$ are strictly increasing and converge to $e /(e-1)$ and $e(e+1) / 2(e-1)^{3}$, respectively. It is shown that there holds the sharp double inequality $(1 /(e-1)) \cdot(1 / n) \leqq e /(e-1)-S(n)<\left(e(e+1) / 2(e-1)^{3}\right) \cdot(1 / n),(n \in \mathbb{N})$.

## 1. Introduction

The proof of the equality

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=\frac{e}{e-1}, \tag{1.1a}
\end{equation*}
$$

published recently in the form [1]

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{n}=\frac{1}{e-1}, \tag{1.1b}
\end{equation*}
$$

was based on the equations $n^{1-k} \cdot n(n-1) \cdots(n-k+2)=(1-1 / n)(1-2 / n) \cdots(1-(k-2) / n)=$ $1+O(1 / n)$ with the false hypothesis that big $O$ is independent of $k$ (see [1, pages 63-64] and [2, pages 54-55]). Deriving (1.1b) the author used the Euler-Maclaurin summation formula and a generating function for the Bernoulli numbers.

Subsequently, Spivey published the correction of his demonstration as the Letter to the Editor [2]. Additionally, Holland [3] published two different derivations of (1.1a) in the same issue as Spivey's correction appeared.

In this note, using only elementary techniques, we demonstrate that the sequence $S(n)$ is strictly increasing and that (1.1a) holds; in addition, we establish a sharp estimate of the rate of convergence.

## 2. Monotone Convergence

The formula (1.1a) is illustrated in Figure 1, where the sequence $n \mapsto S(n):=\sum_{k=1}^{n}(k / n)^{n}$ is depicted. Its monotonicity is seen very clearly.

To prove that the sequence $\left(S_{n}\right)_{n \in \mathbb{N}}$ is strictly increasing, we change the order of summation

$$
\begin{equation*}
S(n) \equiv \sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n} \equiv \sum_{j=0}^{n}\left(\frac{n-j}{n}\right)^{n} \equiv 1+\sum_{j=1}^{n}\left(1+\frac{-j}{n}\right)^{n} . \tag{2.1}
\end{equation*}
$$

Now, consider the function $t \mapsto E(x, t):=(1+x / t)^{t}$ which is, for $x \neq 0$, strictly increasing on the open interval $(-\min \{0, x\}, \infty)$ and $\lim _{t \rightarrow \infty} E(x, t)=\sup _{t>|x|} E(x, t)=e^{x}$, for any $x \in \mathbb{R}$ [4, page 42]. Consequently, the sequence $(S(n))_{n \in \mathbb{N}}$ is strictly increasing. We use Tannery's theorem for series (see [5] or [6, item 49, page 136]) to determine its limit.

Lemma 2.1 (Tannery). Let a double sequence $(j, n) \mapsto z_{j}(n)$ of complex numbers satisfy the following conditions:
(1) The finite limit $z_{\infty}(j):=\lim _{n \rightarrow \infty} z_{n}(j)$ exists for every fixed $j \in \mathbb{N}$.
(2) There exists a sequence of positive constants $M_{1}, M_{2}, M_{3}, \ldots$ such that $\left|z_{n}(j)\right| \leq M_{j}$ for every $(j, n) \in \mathbb{N} \times \mathbb{N}$ satisfying the estimate $j \leq n$, and the series $\sum_{j=1}^{\infty} M_{j}$ converges. (In [6, item 49, page 136], we have the stronger supposition that $\left|z_{n}(j)\right| \leq M_{j}$ for all $(j, n) \in \mathbb{N} \times \mathbb{N})$.

Then we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sum_{j=1}^{n} z_{n}(j)=\sum_{j=1}^{\infty} z_{\infty}(j) \tag{2.2}
\end{equation*}
$$

Proof. Let all the conditions of the Lemma be satisfied and $\varepsilon \in \mathbb{R}^{+}$be given. Then we estimate $\left|z_{\infty}(j)\right| \leq M_{j}$ for $j \in \mathbb{N}$ and $\sum_{j=m_{\varepsilon}+1}^{\infty} M_{j}<\varepsilon / 3$ for some $m_{\varepsilon} \in \mathbb{N}$. Moreover, for any $j \in\left\{1, \ldots, m_{\varepsilon}\right\}$, also $\left|z_{\infty}(j)-z_{n}(j)\right|<\varepsilon /\left(3 m_{\varepsilon}\right)$ for $n \geq n_{\varepsilon}(j)$ at some $n_{\varepsilon}(j) \in \mathbb{N}$. Thus, for $n \geq n_{\varepsilon}:=\max _{1 \leq j \leq m_{\varepsilon}} n_{\varepsilon}(j)$, we estimate

$$
\begin{align*}
\left|\sum_{j=1}^{\infty} z_{\infty}(j)-\sum_{j=1}^{n} z_{n}(j)\right| \leq & \leq \sum_{j=1}^{m_{\varepsilon}}\left|z_{\infty}(j)-z_{n}(j)\right|+\sum_{j=m_{\varepsilon}+1}^{\infty}\left|z_{\infty}(j)\right|+\sum_{j=m_{\varepsilon}+1}^{n}\left|z_{n}(j)\right|  \tag{2.3}\\
& <m_{\varepsilon} \cdot \frac{\varepsilon}{3 m_{\varepsilon}}+\sum_{j=m_{\varepsilon}+1}^{\infty} M_{j}+\sum_{j=m_{\varepsilon}+1}^{n} M_{j}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon .
\end{align*}
$$



Figure 1: The graph of the sequence $n \mapsto S(n) \equiv \sum_{k=1}^{n}(k / n)^{n}$.


Figure 2: The graph of the sequence $n \mapsto n \Delta(n)$.

Now, using (2.1) and putting $z_{n}(j)=(1+-j / n)^{n}$ and $z_{\infty}(j)=e^{-j}$ into Tannery's Lemma, we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} S(n)=1+\sum_{j=1}^{\infty} e^{-j}=\frac{e}{e-1} . \tag{2.4}
\end{equation*}
$$

## 3. The Rate of Convergence

Referring to Figure 1, the convergence of the sequence $(S(n))_{n \in \mathbb{N}}$ appears to be rather slow. The difference

$$
\begin{equation*}
\Delta(n):=\frac{e}{e-1}-S(n) \tag{3.1}
\end{equation*}
$$

determines the sequence $n \mapsto n \Delta(n)$. Its graph, shown in Figure 2, suggests it is monotonic increasing, which we will prove first.

Indeed, according to (3.1) and (2.1), we have

$$
\begin{align*}
\Delta(n) & =\sum_{j=0}^{\infty} e^{-j}-\sum_{j=0}^{n}\left(1-\frac{j}{n}\right)^{n} \\
& =\sum_{j=1}^{n} f_{n}(j)+\sum_{j=n+1}^{\infty} e^{-j}  \tag{3.2}\\
& =\sum_{j=1}^{n} f_{n}(j)+\frac{e^{-n}}{e-1},
\end{align*}
$$

where

$$
\begin{equation*}
f_{n}(x):=e^{-x}-\left(1-\frac{x}{n}\right)^{n} \quad(x \in \mathbb{R}) \tag{3.3}
\end{equation*}
$$

and, for $x \neq 0$, the sequence $n \mapsto f_{n}(x)$ is strictly decreasing and converges to zero [4, (4)]. Thus, we have

$$
\begin{equation*}
n \Delta(n)=\sum_{j=1}^{n} g_{n}(j)+n \frac{e^{-n}}{e-1}=\sum_{j=1}^{n-1} g_{n}(j)+C n e^{-n} \tag{3.4}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{n}(x):=n f_{n}(x), \quad C=\frac{e}{e-1} . \tag{3.5}
\end{equation*}
$$

To examine the monotonicity of the sequence $n \mapsto n \Delta(n)$, we study, using (3.3), (3.4) and (3.5), the difference $(n+1) \Delta(n+1)-n \Delta(n)$, which is equal to

$$
\begin{align*}
& \left(\sum_{j=1}^{n-1} g_{n+1}(j)+g_{n+1}(n)\right)+C \cdot(n+1) e^{-n-1}-\sum_{j=1}^{n-1} g_{n}(j)-C n e^{-n} \\
& \quad=\sum_{j=1}^{n-1}\left(g_{n+1}(j)-g_{n}(j)\right)+(n+1) e^{-n}-\frac{1}{(n+1)^{n}}+\frac{n+1}{e-1} e^{-n}-\frac{e n}{e-1} e^{-n}  \tag{3.6}\\
& \quad=\sum_{j=1}^{n-1}\left((n+1) f_{n+1}(j)-n f_{n}(j)\right)+\left(\frac{e}{e-1} e^{-n}-(n+1)^{-n}\right) \\
& \quad>\sum_{j=1}^{n-1}\left(n f_{n+1}(j)-n f_{n}(j)\right)+0>0 .
\end{align*}
$$

Hence:
The sequence $n \longmapsto n \Delta(n)$ is strictly increasing.
Next, we examine also the question of convergence of the above sequence. First, referring to (3.3), (3.5), and [4, page 29, equation (16)], there exists the limit

$$
\begin{equation*}
g_{\infty}(j):=\lim _{n \rightarrow \infty} g_{n}(j)=\frac{e^{-j} j^{2}}{2} \quad(j \in \mathbb{N}) . \tag{3.8}
\end{equation*}
$$

Moreover, according to (3.3), (3.5), and [4, (15)], the estimates

$$
\begin{equation*}
g_{n}(j)<\frac{e^{-j} j^{2}}{2} \cdot \frac{n}{n-j} \leq \frac{e^{-j} j^{2}}{2} \cdot(1+j) \tag{3.9}
\end{equation*}
$$

hold true for $j \leq n-1$. Additionally, $g_{n}(n)=n e^{-n}$, due to (3.3) and (3.5). Thus, the estimate

$$
\begin{equation*}
g_{n}(j) \leq M_{j}:=\frac{(j+1) j^{2}}{2} \cdot e^{-j} \tag{3.10}
\end{equation*}
$$

is being valid for $n \in \mathbb{N}$ and $j \leq n$ with

$$
\begin{equation*}
\sum_{j=1}^{\infty} M_{j}=\sum_{j=1}^{\infty} \frac{(j+1) j^{2}}{2} \cdot e^{-j}<\infty \tag{3.11}
\end{equation*}
$$

According to (3.8) and differentiating the appropriate power series resulting from the geometric series, we obtain

$$
\begin{equation*}
\sum_{j=1}^{\infty} g_{\infty}(j)=\sum_{j=1}^{\infty} \frac{e^{-j} j^{2}}{2}=\frac{e^{2}+e}{2(e-1)^{3}} \tag{3.12}
\end{equation*}
$$

Now, referring to (3.4) and (3.8)-(3.12), and applying Tannery's Lemma-equation (2.2), with $z_{n}(j) \equiv g_{n}(j)$, we obtain the result

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n \Delta(n)=\sum_{j=1}^{\infty} g_{\infty}(j)+0=\frac{e(e+1)}{2(e-1)^{3}} \tag{3.13}
\end{equation*}
$$

Therefore, using (3.1) and (3.7), we find the following sharp inequality

$$
\begin{equation*}
\frac{e}{e-1}-S(n)<\frac{e(e+1)}{2(e-1)^{3}} \cdot \frac{1}{n} \tag{3.14}
\end{equation*}
$$

true for every $n \in \mathbb{N}$. In addition, we have also the estimate

$$
\begin{equation*}
\frac{e}{e-1}-S(n) \geq m\left(\frac{e}{e-1}-S(m)\right) \cdot \frac{1}{n} \tag{3.15}
\end{equation*}
$$

valid for every $m, n \in \mathbb{N}$ such that $n \geq m$.
We have $e(e+1) / 2(e-1)^{3}=0.996147 \ldots$, and for the function $P(m):=m \Delta(m)=$ $m(e /(e-1)-S(m))$ we calculate $P(1)=0.581976 \ldots$, and $P(999)=0.995149 \ldots$. This way we obtain simple and rather accurate estimates

$$
\begin{gather*}
0.581 \cdot \frac{1}{n}<\frac{e}{e-1}-S(n)<0.996 \cdot \frac{1}{n}, \quad \text { for } n \geq 1 \\
0.995 \cdot \frac{1}{n}<\frac{e}{e-1}-S(n)<0.997 \cdot \frac{1}{n}, \quad \text { for } n \geq 1000 \tag{3.16}
\end{gather*}
$$

Consequently, we get, for example, a simple double inequality

$$
\begin{equation*}
\frac{e}{e-1}-\frac{1}{n}<S(n)<\frac{e}{e-1}-\frac{1}{2 n}, \quad \text { for } n \geq 1 \tag{3.17}
\end{equation*}
$$

Open Question. Are the sequences $n \mapsto S(n)$ and $n \mapsto n \Delta(n)$ strictly concave?

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