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Research Article

Some New Double Sequence Spaces Defined by Orlicz Function in n-Normed Space

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The aim of this paper is to introduce and study some new double sequence spaces with respect to an Orlicz function, and also some properties of the resulting sequence spaces were examined.

1. Introduction

We recall that the concept of a 2-normed space was first given in the works of Gähler ([1, 2]) as an interesting nonlinear generalization of a normed linear space which was subsequently studied by many authors (see, [3, 4]). Recently, a lot of activities have started to study summability, sequence spaces, and related topics in these nonlinear spaces (see, e.g., [5–9]). In particular, Savaş [10] combined Orlicz function and ideal convergence to define some sequence spaces using 2-norm.

In this paper, we introduce and study some new double-sequence spaces, whose elements are form n-normed spaces, using an Orlicz function, which may be considered as an extension of various sequence spaces to n-normed spaces. We begin with recalling some notations and backgrounds.

Recall in [11] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is continuous, convex, and nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently, Orlicz function was used to define sequence spaces by Parashar and Choudhary [12] and others. An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x p(t)dt$, where p is the known kernel of M, right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is nondecreasing, and $p(t) \to \infty$ as $t \to \infty$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called Modulus function, which was presented and discussed by Ruckle [13] and Maddox [14].

Remark 1.1. If M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let $n \in \mathbb{N}$ and X be real vector space of dimension d, where $n \leq d$. An n-norm on X is a function $\|\cdot,\ldots,\cdot\| : X \times X \times \cdots \times X \to \mathbb{R}$ which satisfies the following four conditions:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \ldots, x_n\|$ are invariant under permutation,

(iii) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$, $\alpha \in \mathbb{R}$,

(iv) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$.

The pair $(X, \|\cdot,\ldots,\cdot\|)$ is then called an n-normed space [3].

Let $X = \mathbb{R}^d$ $(d \leq n)$ be equipped with the n-norm, then $\|x_1, x_2, \ldots, x_{n-1}, x_n\|_S :=$ the volume of the n-dimensional parallelepiped spanned by the vectors, $x_1, x_2, \ldots, x_{n-1}, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_{n-1}, x_n\|_S = \begin{vmatrix} \langle x_1, x_2 \rangle \cdots & \langle x_1, x_n \rangle \\ . & \\ . & \cdots \\ . & \\ \langle x_n, x_1 \rangle \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Let $(X, \|\cdot,\ldots,\cdot\|)$ be an n-normed space of dimension $d \geq n$ and $\{a_1, a_2, \ldots, a_n\}$ a linearly independent set in X. Then, the function $\|\cdot,\cdot\|_\infty$ on X^{n-1} is defined by

$$\|x_1, x_2, \ldots, x_{n-1}, x_n\|_\infty := \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : \ i = 1, 2, \ldots, n\}, \tag{1.2}$$

is defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$ (see, [15]).

Definition 1.2 (see [7]). A sequence (x_k) in n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is aid to be convergent to an x in X (in the n-norm) if

$$\lim_{k\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - x\| = 0, \tag{1.3}$$

for every $x_1, x_2, \ldots, x_{n-1} \in X$.

Definition 1.3 (see [16]). Let X be a linear space. Then, a map $g : X \to \mathbb{R}$ is called a paranorm (on X) if it is satisfies the following conditions for all $x, y \in X$ and λ scalar:

(i) $g(\theta) = 0$ $(\theta = (0, 0, \ldots, 0 \ldots)$ is zero of the space),

(ii) $g(x) = g(-x)$,

(iii) $g(x+y) \le g(x) + g(y)$,

(iv) $|\lambda^n - \lambda| \to 0$ $(n \to \infty)$ and $g(x^n - x) \to 0$ $(n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0$ $(n \to \infty)$.

2. Main Results

Let $(X, \|\cdot, \ldots, \cdot\|)$ be any n-normed space, and let $S''(n - X)$ denote X-valued sequence spaces. Clearly $S''(n - X)$ is a linear space under addition and scalar multiplication.

Definition 2.1. Let M be an Orlicz function and $(X, \|\cdot, \ldots, \cdot\|)$ any n-normed space. Further, let $p = (p_{k,l})$ be a bounded sequence of positive real numbers. Now, we define the following new double sequence space as follows:

$$l''(M, p, \|\cdot, \ldots, \cdot\|) := \left\{ x \in S''(n - X) : \sum_{k,l=1}^{\infty,\infty} \left[M\left(\left\| \frac{x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \infty,\ \rho > 0 \right\}, \tag{2.1}$$

for each $z_1, z_2, \ldots, z_{n-1} \in X$.

The following inequalities will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \le \sup_{k,l} p_{k,l} = H$, and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have as in Maddox [16]

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le D\left(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}} \right). \tag{2.2}$$

Theorem 2.2. *$l''(M, p, \|\cdot, \ldots, \cdot\|)$ sequences space is a linear space.*

Proof. Now, assume that $x, y \in l''$ $(M, p, \|\cdot, \ldots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then,

$$\begin{gathered}
\sum_{k,l=1}^{\infty,\infty} \left[M\left(\left\| \frac{x_{k,l}}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \infty \quad \text{for some } \rho_1 > 0, \\
\sum_{k,l=1,1}^{\infty,\infty} \left[M\left(\left\| \frac{x_{k,l}}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \infty \quad \text{for some } \rho_2 > 0.
\end{gathered} \tag{2.3}$$

Since $\|\cdot,\ldots,\cdot\|$ is a n-norm on X, and M is an Orlicz function, we get

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{\alpha x_{k,l}+\beta y_{k,l}}{\max(|\alpha|\rho_1,|\beta|\rho_2)},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\quad\le D\sum_{k,l=1,1}^{\infty,\infty}\left[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\qquad+D\sum_{k,l=1,1}^{\infty}\left[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\quad\le DF\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\qquad+DF\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}},
\end{aligned}\tag{2.4}$$

where

$$F=\max\left[1,\left(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H,\left(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\right)^H\right],\tag{2.5}$$

and this completes the proof. □

Theorem 2.3. *$l''(M,p,\|\cdot,\ldots,\cdot\|)$ space is a paranormed space with the paranorm defined by g : $l''(M,p,\|\cdot,\ldots,\cdot\|)\to\mathbb{R}$*

$$g(x)=\inf\left\{\rho^{p_{k,l}/H}:\left(\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{1/M^*}<\infty\right\},\tag{2.6}$$

where $0<p_{k,l}\le\sup p_{k,l}=H$, $M^*=\max(1,H)$.

Proof. (i) Clearly, $g(\theta)=0$ and (ii) $g(-x)=g(x)$. (iii) Let $x_{k,l},y_{k,l}\in l''(M,p,\|\cdot,\ldots,\cdot\|)$, then there exists $\rho_1,\rho_2>0$ such that

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}<\infty,\\
&\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}<\infty.
\end{aligned}\tag{2.7}$$

So, we have

$$\begin{aligned}
M&\left(\left\|\frac{x_{k,l}+y_{k,l}}{\rho_1+\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\\
&\le M\left(\left\|\frac{x_{k,l}}{\rho_1+\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|+\left\|\frac{y_{k,l}}{\rho_1+\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\\
&\le \frac{\rho_1}{\rho_1+\rho_2}M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\\
&\quad+\frac{\rho_1}{\rho_1+\rho_2}M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right),
\end{aligned}\tag{2.8}$$

and thus

$$\begin{aligned}
g(x+y)&=\inf\left\{(\rho_1+\rho_2)^{p_{k,l}/H}:\left(\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{x_{k,l}+y_{k,l}}{\rho_1+\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{1/M^*}\right\}\\
&\le\inf\left\{(\rho_1)^{p_{k,l}/H}:\left(\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{1/M^*}\right\}\\
&\quad+\inf\left\{(\rho_2)^{p_{k,l}/H}:\left(\sum_{k=1}^{\infty}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{1/M^*}\right\}.
\end{aligned}\tag{2.9}$$

(iv) Now, let $\lambda\to 0$ and $g(x^n-x)\to 0$ $(n\to\infty)$. Since

$$g(\lambda x)=\inf\left\{\left(\frac{\rho}{|\lambda|}\right)^{p_{k,l}/H}:\left(\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{\lambda x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{1/M^*}<\infty\right\}.\tag{2.10}$$

This gives us $g(\lambda x^n)\to 0$ $(n\to\infty)$. $\square$

Theorem 2.4. *If $0<p_{k,l}<q_{k,l}<\infty$ for each k and l, then $l''(M,p,\|\cdot,\ldots,\cdot\|)\subseteq l''(M,q,\|\cdot,\ldots,\cdot\|)$.*

Proof. If $x\in l''(M,p,\|\cdot,\ldots,\cdot\|)$, then there exists some $\rho>0$ such that

$$\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}<\infty.\tag{2.11}$$

This implies that

$$M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)<1,\tag{2.12}$$

for sufficiently large values of k and l. Since M is nondecreasing, we are granted

$$\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{q_{k,l}} \leq \sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty. \tag{2.13}$$

Thus, $x \in l''(M,q,\|\cdot,\ldots,\cdot\|)$. This completes the proof. □

The following result is a consequence of the above theorem.

Corollary 2.5. (i) *If* $0 < p_{k,l} < 1$ *for each* k *and* l, *then*

$$l''(M,p,\|\cdot,\ldots,\cdot\|) \subseteq l''(M,\|\cdot,\ldots,\cdot\|), \tag{2.14}$$

(ii) *If* $p_{k,l} \geq 1$ *for each* k *and* l, *then*

$$l''(M,\|\cdot,\ldots,\cdot\|) \subseteq l''(M,p,\|\cdot,\ldots,\cdot\|). \tag{2.15}$$

Theorem 2.6. $u = (u_{k,l}) \in l''_\infty \Rightarrow ux \in l''(M,p,\|\cdot,\ldots,\cdot\|)$, *where* l''_∞ *is the double space of bounded sequences and* $ux = (u_{k,l}x_{k,l})$.

Proof. $u = (u_{k,l}) \in l''_\infty$. Then, there exists an $A > 1$ such that $|u_{k,l}| \leq A$ for each k, l. We want to show $(u_{k,l}x_{k,l}) \in l''(M,p,\|\cdot,\ldots,\cdot\|)$. But

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{u_{k,l}x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\quad=\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(|u_{k,l}|\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_{k,l}}\\
&\quad\leq (KA)^H\sum_{k,l=1,1}^{\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-2},z_{n-1}\right\|\right)\right]^{p_{k,l}},
\end{aligned} \tag{2.16}$$

and this completes the proof. □

Theorem 2.7. *Let* M_1 *and* M_2 *be Orlicz function. Then, we have*

$$l''(M_1,p,\|\cdot,\ldots,\cdot\|) \bigcap l''(M_2,p,\|\cdot,\ldots,\cdot\|) \subseteq l''(M_1+M_2,p,\|\cdot,\ldots,\cdot\|). \tag{2.17}$$

Proof. We have

$$\begin{aligned}
&\left[(M_1+M_2)\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\quad=\left[M_1\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)+M_2\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\quad\leq D\left[M_1\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}+D\left[M_2\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}.
\end{aligned} \tag{2.18}$$

Let $x\in l''(M_1,p,\|\cdot,\ldots,\cdot\|)\bigcap l''(M_2,p,\|\cdot,\ldots,\cdot\|)$; when adding the above inequality from $k,l=0,0$ to ∞,∞ we get $x\in l''(M_1+M_2,p,\|\cdot,\ldots,\cdot\|)$ and this completes the proof. □

Definition 2.8 (see [10]). Let X be a sequence space. Then, X is called solid if $(\alpha_k x_k)\in X$ whenever $(x_k)\in X$ for all sequences (α_k) of scalars with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$.

Definition 2.9. Let X be a sequence space. Then, X is called monotone if it contains the canonical preimages of all its step spaces (see, [17]).

Theorem 2.10. *The sequence space $l''(M,p,\|\cdot,\ldots,\cdot\|)$ is solid.*

Proof. Let $(x_{k,l})\in l''(M,p,\|\cdot,\ldots,\cdot\|)$; that is,

$$\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}<\infty. \tag{2.19}$$

Let $(\alpha_{k,l})$ be double sequence of scalars such that $|\alpha_{k,l}|\leq 1$ for all $k,l\in\mathbb{N}\times\mathbb{N}$. Then, the result follows from the following inequality:

$$\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{\alpha_{k,l}x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq\sum_{k,l=1,1}^{\infty,\infty}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}, \tag{2.20}$$

and this completes the proof. □

We have the following result in view of Remark 1.1 and Theorem 2.10.

Corollary 2.11. *The sequence space $l''(M,p,\|\cdot,\ldots,\cdot\|)$ is monotone.*

Definition 2.12 (see [18]). Let $A=(a_{m,n,k,l})$ denote a four-dimensional summability method that maps the complex double sequences x into the double-sequence Ax, where the mnth term to Ax is as follows:

$$(Ax)_{m,n}=\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}x_{k,l}. \tag{2.21}$$

Such transformation is said to be nonnegative if $a_{m,n,k,l}$ is nonnegative for all m,n,k, and l.

Definition 2.13. Let $A = (a_{m,n,k,l})$ be a nonnegative matrix. Let M be an Orlicz function and $p_{k,l}$ a factorable double sequence of strictly positive real numbers. Then, we define the following sequence spaces:

$$\begin{aligned}&\omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|)\\&\quad = \left\{ x \in S''(n-1) : \lim_{m,n\to\infty,\infty} \sum_{k,l=1,1}^{\infty,\infty} \left[M\left(\left\| \frac{a_{m,n,k,l}x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right]^{p_{k,l}} = 0 \right\}.\end{aligned} \tag{2.22}$$

for each $z_1, z_2, \ldots, z_{n-1} \in X$. If $x - le \in \omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|)$, then we say x is $\omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|)$ summable to l, where $e = (1, 1, \ldots)$.

If we take $M(x) = x$ and $p_{k,l} = 1$ for all (k, l), then we have

$$\omega_0''(A, p, \|\cdot,\ldots,\cdot\|) = \left\{ x \in S''(n-1) : \lim_{m,n\to\infty} \sum_{k,l=1,1}^{\infty,\infty} \|a_{m,n,k,l}x_{k,l}, z_1, z_2, \ldots, z_{n-2}, z_{n-1}\| = 0 \right\}. \tag{2.23}$$

Theorem 2.14. $\omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|)$ *is linear spaces.*

Proof. This can be proved by using the techniques similar to those used in Theorem 2.2. □

Theorem 2.15. (1) *If* $0 < \inf p_{k,l} \le p_{k,l} < 1$*, then*

$$\omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|) \subset \omega_0''(M, A, \|\cdot,\ldots,\cdot\|). \tag{2.24}$$

(2) *If* $1 \le p_{k,l} \le \sup p_{k,l} < \infty$*, then*

$$\omega_0''(M, A, \|\cdot,\ldots,\cdot\|) \subset \omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|). \tag{2.25}$$

Proof. (1) Let $x \in \omega_0''(M, A, p, \|\cdot,\ldots,\cdot\|)$; since $0 < \inf p_{k,l} \le 1$, we have

$$\begin{aligned}&\sum_{k,l=1,1}^{\infty,\infty} \left[M\left(\left\| \frac{a_{m,n,k,l}x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right]\\&\quad \le \sum_{k,l=1}^{\infty,\infty} \left[M\left(\left\| \frac{a_{m,n,k,l}x_{k}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right]^{p_{k,l}},\end{aligned} \tag{2.26}$$

and hence $x \in \omega_0''(M, A, \|\cdot,\ldots,\cdot\|)$.

(2) Let $p_{k,l} \ge 1$ for each (k, l) and $\sup_{k,l} p_{k,l} < \infty$. Let $x \in \omega_0''(M, A, \|\cdot,\ldots,\cdot\|)$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sum_{k,l=1,1}^{\infty,\infty} \left[M\left(\left\| \frac{a_{m,n,k,l}x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right] \le \epsilon < 1, \tag{2.27}$$

for all $m, n \geq \mathbb{N}$. This implies that

$$\sum_{k,l=1,1}^{\infty,\infty} \left[M\left(\left\| \frac{a_{m,n,k,l} x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq \sum_{k,l=1}^{\infty} \left[M\left(\left\| \frac{a_{m,n,k,l} x_{k,l}}{\rho}, z_1, z_2, \ldots, z_{n-2}, z_{n-1} \right\| \right) \right]. \tag{2.28}$$

Thus, $x \in \omega_0''(M, A, p, \|\cdot, \ldots, \cdot\|)$, and this completes the proof. □

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