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Research Article

Bessel and Grüss Type Inequalities in Inner Product Modules over Banach $*$-Algebras

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We give an analogue of the Bessel inequality and we state a simple formulation of the Grüss type inequality in inner product C^*-modules, which is a refinement of it. We obtain some further generalization of the Grüss type inequalities in inner product modules over proper H^*-algebras and unital Banach $*$-algebras for C^*-seminorms and positive linear functionals.

1. Introduction

A proper H^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|\cdot\|)$ where the underlying Banach space is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle$ satisfying the properties $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ba, c\rangle = \langle b, ca^*\rangle$ for all $a, b, c \in \mathcal{A}$. A C^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$. If $\mathcal{A}$ is a proper H^*-algebra or a C^*-algebra and $a \in \mathcal{A}$ is such that $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$, then $a = 0$.

For a proper H^*-algebra $\mathcal{A}$, the trace class associated with $\mathcal{A}$ is $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$. For every positive $a \in \tau(\mathcal{A})$ there exists the square root of a, that is, a unique positive $a^{1/2} \in \mathcal{A}$ such that $(a^{1/2})^2 = a$, the square root of a^*a is denoted by $|a|$. There are a positive linear functional tr on $\tau(\mathcal{A})$ and a norm τ on $\tau(\mathcal{A})$, related to the norm of A by the equality $\mathrm{tr}(a^*a) = \tau(a^*a) = \|a\|^2$ for every $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a proper H^*-algebra or a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is with a mapping $\langle\cdot,\cdot\rangle$ on $X \times X$, which is $\tau(\mathcal{A})$-valued if $\mathcal{A}$ is a proper H^*-algebra, or $\mathcal{A}$-valued if $\mathcal{A}$ is a C^*-algebra, having the following properties:

(i) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in X$,

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in X$, $a \in \mathcal{A}$,

(iii) $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in X$,

(iv) $\langle x, x\rangle \geq 0$ for $x \in X$.

We will say that X is a semi-inner product H^*-module if $\mathcal{A}$ is a proper H^*-algebra and that X is a semi-inner product C^*-module if $\mathcal{A}$ is a C^*-algebra.

If, in addition,

(v) $\langle x, x\rangle = 0$ implies $x = 0$,

then X is called an inner product module over $\mathcal{A}$. The absolute value of $x \in X$ is defined as the square root of $\langle x, x\rangle$ and it is denoted by $|x|$.

Let $\mathcal{A}$ be a $*$-algebra. A seminorm γ on $\mathcal{A}$ is a real-valued function on $\mathcal{A}$ such that for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $\gamma(a) \geq 0$, $\gamma(\lambda a) = |\lambda|\gamma(a)$, $\gamma(a+b) \leq \gamma(a) + \gamma(b)$. A seminorm γ on $\mathcal{A}$ is called a C^*-seminorm if it satisfies the C^*-condition: $\gamma(a^*a) = (\gamma(a))^2$ $(a \in \mathcal{A})$. By Sebestyen's theorem [1, Theorem 38.1] every C^*-seminorm γ on a $*$-algebra $\mathcal{A}$ is submultiplicative, that is, $\gamma(ab) \leq \gamma(a)\gamma(b)$ $(a, b \in \mathcal{A})$, and by [2, Section 39, Lemma 2(i)] $\gamma(a) = \gamma(a^*)$. For every $a \in \mathcal{A}$, the spectral radius of a is defined to be $r(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

The Pták function ρ on $*$-algebra $\mathcal{A}$ is defined to be $\rho : \mathcal{A} \to [0, \infty)$, where $\rho(a) = (r(a^*a))^{1/2}$. This function has important roles in Banach $*$-algebras, for example, on C^*-algebras, ρ is equal to the norm and on Hermitian Banach $*$-algebras ρ is the greatest C^*-seminorm. By utilizing properties of the spectral radius and the Pták function, Pták [3] showed in 1970 that an elegant theory for Banach $*$-algebras arises from the inequality $r(a) \leq \rho(a)$.

This inequality characterizes Hermitian (and symmetric) Banach $*$-algebras, and further characterizations of C^*-algebras follow as a result of Pták theory.

Let $\mathcal{A}$ be a $*$-algebra. We define $\mathcal{A}^+$ by

$$\mathcal{A}^+ = \left\{\sum_{k=1}^{n} a_k^* a_k : n \in \mathbb{N},\ a_k \in \mathcal{A} \text{ for } k = 1, 2, \ldots, n\right\}, \tag{1.1}$$

and call the elements of $\mathcal{A}^+$ positive.

The set $\mathcal{A}^+$ of positive elements is obviously a convex cone (i.e., it is closed under convex combinations and multiplication by positive constants). Hence we call $\mathcal{A}^+$ the positive cone. By definition, zero belongs to $\mathcal{A}^+$. It is also clear that each positive element is Hermitian.

We recall that a Banach $*$-algebra $(\mathcal{A}, \|\cdot\|)$ is said to be an A^*-algebra provided there exists on $\mathcal{A}$ a second norm $|\cdot|$, not necessarily complete, which is a C^*-norm. The second norm will be called an auxiliary norm.

Definition 1.1. Let $\mathcal{A}$ be a $*$-algebra. A semi-inner product $\mathcal{A}$-module (or semi-inner product $*$-module) is a complex vector space which is also a right $\mathcal{A}$-module X with a sesquilinear semi-inner product $\langle\cdot,\cdot\rangle : X \times X \to \mathcal{A}$, fulfilling

$$\begin{aligned} \langle x, ya\rangle &= \langle x, y\rangle a \quad \text{(right linearity)} \\ \langle x, x\rangle &\in \mathcal{A}^+ \quad \text{(positivity)} \end{aligned} \tag{1.2}$$

for $x, y \in X$, $a \in \mathcal{A}$. Furthermore, if X satisfies the strict positivity condition

$$x = 0 \quad \text{if } \langle x, x\rangle = 0, \ (\text{strict positivity}) \tag{1.3}$$

then X is called an inner product $\mathcal{A}$-module (or inner product $*$-module).

Let γ be a seminorm or a positive linear functional on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x, x\rangle))^{1/2}$ $(x \in X)$. If Γ is a seminorm on a semi-inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a semi-Hilbert $\mathcal{A}$-module.

If Γ is a norm on an inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a pre-Hilbert $\mathcal{A}$-module.

A pre-Hilbert $\mathcal{A}$-module which is complete with respect to its norm is called a Hilbert $\mathcal{A}$-module.

Since $\langle x + y, x + y\rangle$ and $\langle x + iy, x + iy\rangle$ are self adjoint, therefore we get the following Corollary.

Corollary 1.2. *If X is a semi-inner product $*$-module, then the following symmetry condition holds:*

$$\langle x, y\rangle^* = \langle y, x\rangle \quad \textit{for } x, y \in X \ (\textit{symmetry}). \tag{1.4}$$

Example 1.3. (a) Let $\mathcal{A}$ be a $*$-algebra and γ a positive linear functional or a C^*-seminorm on $\mathcal{A}$. It is known that $(\mathcal{A}, \gamma)$ is a semi-Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a, b\rangle := a^*b$, in this case $\Gamma = \gamma$.

(b) Let $\mathcal{A}$ be a Hermitian Banach $*$-algebra and ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x) = (\rho(\langle x, x\rangle))^{1/2}$ $(x \in X)$, then (X, P) is a semi-Hilbert $\mathcal{A}$-module.

(c) Let $\mathcal{A}$ be a A^*-algebra and $|\cdot|$ be the auxiliary norm on $\mathcal{A}$. If X is an inner product $\mathcal{A}$-module and $|x| = |\langle x, x\rangle|^{1/2}$ $(x \in X)$, then $(X, |\cdot|)$ is a pre-Hilbert $\mathcal{A}$-module.

(d) Let $\mathcal{A}$ be a H^*-algebra and X (a semi-inner product) an inner product $\mathcal{A}$-module. Since tr is a positive linear functional on $\tau(\mathcal{A})$ and for every $x \in X$ we have $\operatorname{tr}(\langle x, x\rangle) = |||x|||^2$; therefore $(X, |||\cdot|||)$ is a (semi-Hilbert) pre-Hilbert $\mathcal{A}$-module.

In the present paper, we give an analogue of the Bessel inequality (2.7) and we obtain some further generalization and a simple form for the Grüss type inequalities in inner product modules over C^*-algebras, proper H^*-algebras, and unital Banach $*$-algebras.

2. Schwarz and Bessel Inequality

If X is a semi-inner product C^*-module, then the following Schwarz inequality holds:

$$\|\langle x, y\rangle\|^2 \le \|\langle x, x\rangle\|\|\langle y, y\rangle\| \quad (x, y \in X). \tag{2.1}$$

(e.g. [4, Lemma 15.1.3]).

If X is a semi-inner product H^*-module, then there are two forms of the Schwarz inequality: for every $x, y \in X$

$$\operatorname{tr}\left(\langle x, y\rangle\right)^2 \leq \operatorname{tr}(\langle x, x\rangle) \operatorname{tr}(\langle y, y\rangle) \quad \text{(the weak Schwarz inequality)}, \tag{2.2}$$

$$\tau(\langle x, y\rangle)^2 \leq \operatorname{tr}(\langle x, x\rangle) \operatorname{tr}(\langle y, y\rangle) \quad \text{(the strong Schwarz inequality)}. \tag{2.3}$$

First Saworotnow in [5] proved the strong Schwarz inequality, but the direct proof of that for a semi-inner product H^*-module can be found in [6].

Now let $\mathcal{A}$ be a $*$-algebra, φ a positive linear functional on $\mathcal{A}$ and let X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X \times X$ by $\sigma(x, y) = \varphi(\langle x, y\rangle)$; the Schwarz inequality for σ implies that

$$\left|\varphi(\langle x, y\rangle)\right|^2 \leq \varphi(\langle x, x\rangle)\varphi(\langle y, y\rangle). \tag{2.4}$$

In [7, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner $\mathcal{A}$-module X, one for positive linear functional φ on $\mathcal{A}$:

$$\varphi(\langle x, y\rangle\langle x, y\rangle) \leq \varphi(\langle x, x\rangle) r(\langle y, y\rangle), \tag{2.5}$$

and another one for C^*-seminorm γ on $\mathcal{A}$:

$$\gamma(\langle x, y\rangle)^2 \leq \gamma(\langle x, x\rangle)\gamma(\langle y, y\rangle). \tag{2.6}$$

The classical Bessel inequality states that if $\{e_i\}_{i \in I}$ is a family of orthonormal vectors in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then

$$\sum_{i \in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \quad (x \in H). \tag{2.7}$$

Furthermore, some results concerning upper bounds for the expression

$$\|x\|^2 - \sum_{i \in I} |\langle x, e_i\rangle|^2 \quad (x \in H) \tag{2.8}$$

and for the expression related to the Grüss-type inequality

$$\left|\langle x, y\rangle - \sum_{i \in I} \langle x, e_i\rangle\langle e_i, y\rangle\right| \quad (x, y \in H) \tag{2.9}$$

have been proved in [8]. A version of the Bessel inequality for inner product H^*-modules and inner product C^*-modules can be found in [9], also there is a version of it for Hilbert C^*-modules in [10, Theorem 3.1]. We provide here an analogue of the Bessel inequality for inner product $*$-modules.

Lemma 2.1. *Let $\mathscr{A}$ be a $*$-algebra, let X be an inner product $\mathscr{A}$-module, and let $\{e_1,\ldots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\ldots,n)$ are idempotent. Then*

$$\langle x,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle \geq 0. \tag{2.10}$$

Proof. By [11, Lemma 1] or a straightforward calculation shows that

$$\begin{aligned}
0 &\leq \left\langle x-\sum_{i=1}^{n}e_i\langle e_i,x\rangle,\ x-\sum_{i=1}^{n}e_i\langle e_i,x\rangle\right\rangle \\
&= \langle x,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle + \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle \\
&= \langle x,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle + \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle \\
&= \langle x,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle.
\end{aligned} \tag{2.11}$$

□

3. Grüss Type Inequalities

Before stating the main results, let us fix the rest of our notation. We assume, unless stated otherwise, throughout this section that $\mathscr{A}$ is a unital Banach $*$-algebra. Also if X is a semi-inner product $\mathscr{A}$-module and γ is a C^*-seminorm on $\mathscr{A}$, we put $\Gamma(x) = (\gamma(\langle x,x\rangle))^{1/2}$ $(x \in X)$, and if φ is a positive linear functional on $\mathscr{A}$, we put $\Phi(x) = (\varphi(\langle x,x\rangle))^{1/2}$ $(x \in X)$. Let $\{e_1,\ldots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\ldots,n)$ be idempotent, we set $G_{x,y} := \langle x,y\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,y\rangle$ and $G_x := \langle x,x\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle$.

Dragomir in [8, Lemma 4] shows that in a Hilbert space H, the condition

$$\operatorname{Re}\left\langle \sum_{i=1}^{n}\alpha_i e_i - x, x-\sum_{i=1}^{n}\beta_i e_i\right\rangle \geq 0, \tag{3.1}$$

is equivalent to the condition

$$\left\| x-\sum_{i=1}^{n}\left(\frac{\alpha_i+\beta_i}{2}\right)e_i\right\| \leq \frac{1}{2}\left(\sum_{i=1}^{n}\|\alpha_i-\beta_i\|^2\right)^{1/2}, \tag{3.2}$$

where $x,e_1,\ldots,e_n \in H$ and $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n \in \mathbb{C}$. But for semi-inner product $\mathscr{A}$-modules we have the following lemma, which is a generalization of [7, Lemma 1].

Lemma 3.1. *Let X be a semi-inner product $\mathcal{A}$-module and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ $x, y_1, \ldots, y_n \in X$. Then*

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle \geq 0 \tag{3.3}$$

if and only if

$$\left\langle x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2}\right), x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2}\right) \right\rangle \leq \frac{1}{4} \sum_{i=1}^{n} (a_i - b_i)^* \langle y_i, y_i \rangle (a_i - b_i). \tag{3.4}$$

Proof. Follows from the equalities:

$$\begin{aligned}
&\operatorname{Re}\left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle \\
&\quad = \frac{1}{2}\left(\left\langle \sum_{i=1}^{n} y_i a_i - x, x - \sum_{i=1}^{n} y_i b_i \right\rangle + \left\langle x - \sum_{i=1}^{n} y_i b_i, \sum_{i=1}^{n} y_i a_i - x \right\rangle\right) \\
&\quad = \sum_{i=1}^{n} \frac{a_i^* + b_i^*}{2} \langle y_i, x \rangle - \frac{1}{2} \sum_{i=1}^{n} \left(a_i^* \langle y_i, y_i \rangle b_i + b_i^* \langle y_i, y_i \rangle a_i\right) \\
&\qquad - \langle x, x \rangle + \sum_{i=1}^{n} \langle x, y_i \rangle \frac{a_i + b_i}{2} \\
&\quad = \frac{1}{4} \sum_{i=1}^{n} (a_i - b_i)^* \langle y_i, y_i \rangle (a_i - b_i) \\
&\qquad - \left\langle x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2}\right),\ x - \sum_{i=1}^{n} y_i \left(\frac{a_i + b_i}{2}\right) \right\rangle.
\end{aligned} \tag{3.5}$$

□

Remark 3.2. By making use of the previous Lemma 3.1, we may conclude the following statements.

(i) Let X be an inner product C^*-module and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i \rangle$ $(i = 1, \ldots, n)$ are idempotent, then inequality (3.3) implies that

$$\left\| x - \sum_{i=1}^{n} e_i \left(\frac{a_i + b_i}{2}\right) \right\| \leq \frac{1}{2} \left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \right)^{1/2} \|\langle e_i, e_i \rangle\| = \frac{1}{2} \left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \right)^{1/2}. \tag{3.6}$$

(ii) Let X be an inner product $\mathcal{A}$-module and $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. If γ is a C^*-seminorm on $\mathcal{A}$ then inequality (3.3) implies that

$$\Gamma\left(x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right)\leq\frac{1}{2}\left(\sum_{i=1}^{n}\gamma(a_i-b_i)^2\right)^{1/2}\Gamma(e_i)\leq\frac{1}{2}\left(\sum_{i=1}^{n}\gamma(a_i-b_i)^2\right)^{1/2}, \tag{3.7}$$

and if φ is a positive linear functional on $\mathcal{A}$ from inequality (3.3) and [2, Section 37 Lemma 6(iii)], we get

$$\begin{aligned}\Phi\left(x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right)^2 &\leq \frac{1}{4}\sum_{i=1}^{n}\varphi\left((a_i-b_i)^*\langle e_i,e_i\rangle(a_i-b_i)\right)\\ &\leq \frac{1}{4}\sum_{i=1}^{n}\varphi\left((a_i-b_i)^*(a_i-b_i)\right)r(\langle e_i,e_i\rangle).\end{aligned} \tag{3.8}$$

(iii) Let $\mathcal{A}$ be a proper H^*-algebra, let X be an inner product $\mathcal{A}$-module, and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. Since for every $a \in H$, $\operatorname{tr}(a^*a) = \|a\|^2$ inequality (3.3) is valid only if

$$\left|\left|\left|x-\sum_{i=1}^{n}e_i\left(\frac{a_i+b_i}{2}\right)\right|\right|\right|\leq\frac{1}{2}\left(\sum_{i=1}^{n}\|a_i-b_i\|^2\right)^{1/2}. \tag{3.9}$$

We are able now to state our first main result.

Theorem 3.3. *Let X be an inner product C^*-module and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$, r, s are real numbers and $x, y \in X$ such that*

$$\left\|x-\sum_{i=1}^{n}e_ia_i\right\|\leq r,\qquad \left\|y-\sum_{i=1}^{n}e_ib_i\right\|\leq s \tag{3.10}$$

hold, then one has the inequality

$$\|G_{x,y}\|\leq rs-\sqrt{r^2-\|G_x\|}\sqrt{s^2-\|G_y\|}. \tag{3.11}$$

Proof. By [11, Lemma 2] or, a straightforward calculation shows that for every $a_1, \ldots, a_n \in \mathcal{A}$

$$\begin{aligned}G_x = \langle x,x\rangle-\sum_{i=1}^{n}\langle x,e_i\rangle\langle e_i,x\rangle &= \left\langle x-\sum_{i=1}^{n}e_ia_i, x-\sum_{i=1}^{n}e_ia_i\right\rangle\\ &-\left\langle \sum_{i=1}^{n}e_i(a_i-\langle e_i,x\rangle), \sum_{i=1}^{n}e_i(a_i-\langle e_i,x\rangle)\right\rangle.\end{aligned} \tag{3.12}$$

Therefore

$$G_x \leq \left\langle x - \sum_{i=1}^{n} e_i a_i, x - \sum_{i=1}^{n} e_i a_i \right\rangle. \tag{3.13}$$

Analogously, for every $b_1, \ldots, b_n \in \mathcal{A}$, we have

$$G_y \leq \left\langle y - \sum_{i=1}^{n} e_i b_i, y - \sum_{i=1}^{n} e_i b_i \right\rangle. \tag{3.14}$$

The equalities (3.10), (3.13), and (3.14) imply that

$$\|G_x\| \leq \left\| x - \sum_{i=1}^{n} e_i a_i \right\|^2 \leq r^2, \tag{3.15}$$

$$\|G_y\| \leq \left\| y - \sum_{i=1}^{n} e_i b_i \right\|^2 \leq s^2. \tag{3.16}$$

Since

$$G_{x,y} = \left\langle x - \sum_{i=1}^{n} e_i \langle e_i, x \rangle, y - \sum_{i=1}^{n} e_i \langle e_i, y \rangle \right\rangle, \tag{3.17}$$

therefore the Schwarz's inequality (2.1) holds, that is,

$$\left\|G_{x,y}\right\|^2 \leq \|G_x\| \left\|G_y\right\|. \tag{3.18}$$

Finally, using the elementary inequality for real numbers

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2 \tag{3.19}$$

on

$$m = r, \qquad n = \sqrt{r^2 - \|G_x\|}, \qquad p = s, \qquad q = \sqrt{s^2 - \|G_y\|}, \tag{3.20}$$

we get

$$\left\|G_{x,y}\right\|^2 \leq \|G_x\| \left\|G_y\right\| \leq \left(rs - \sqrt{r^2 - \|G_x\|}\sqrt{s^2 - \left\|G_y\right\|} \right)^2. \tag{3.21}$$

$\square$

Remark 3.4. (i) Let X be an inner product C^*-module and let $\{e_1,\dots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\dots,n)$ are idempotent. If $a_i,b_i,c_i,d_i\in\mathscr{A}$ $(i=1,\dots,n)$ and $x,y\in X$ are such that

$$\begin{gathered}\left\|x-\sum_{i=1}^n e_i\left(\frac{a_i+b_i}{2}\right)\right\|\le\frac{1}{2}\left(\sum_{i=1}^n\|a_i-b_i\|^2\right)^{1/2},\\ \left\|y-\sum_{i=1}^n e_i\left(\frac{c_i+d_i}{2}\right)\right\|\le\frac{1}{2}\left(\sum_{i=1}^n\|c_i-d_i\|^2\right)^{1/2}\end{gathered}\tag{3.22}$$

and if we put $r=(1/2)(\sum_{i=1}^n\|a_i-b_i\|^2)^{1/2}$, and $s=(1/2)(\sum_{i=1}^n\|c_i-d_i\|^2)^{1/2}$, then, by (3.15) and (3.16), we have

$$\|G_x\|\le\left\|x-\sum_{i=1}^n e_i\left(\frac{a_i+b_i}{2}\right)\right\|^2\le r^2,\qquad \|G_y\|\le\left\|y-\sum_{i=1}^n e_i\left(\frac{c_i+d_i}{2}\right)\right\|^2\le s^2.\tag{3.23}$$

These and (3.11) imply that

$$\begin{aligned}\|G_{x,y}\|&\le rs-\sqrt{r^2-\|G_x\|}\sqrt{s^2-\|G_y\|}\\ &\le\frac{1}{4}\left(\sum_{i=1}^n\|a_i-b_i\|^2\sum_{i=1}^n\|c_i-d_i\|^2\right)^{1/2}\\ &\quad-\left(\frac{1}{4}\sum_{i=1}^n\|a_i-b_i\|^2-\left\|x-\sum_{i=1}^n e_i\left(\frac{a_i+b_i}{2}\right)\right\|^2\right)^{1/2}\\ &\quad\times\left(\frac{1}{4}\sum_{i=1}^n\|c_i-d_i\|^2-\left\|y-\sum_{i=1}^n e_i\left(\frac{c_i+d_i}{2}\right)\right\|^2\right)^{1/2}\\ &\le\frac{1}{4}\left(\sum_{i=1}^n\|a_i-b_i\|^2\sum_{i=1}^n\|c_i-d_i\|^2\right)^{1/2}=rs.\end{aligned}\tag{3.24}$$

Therefore, (3.11) is a refinement and a simple formulation of [9, Theorem 4.1.]

(ii) If for $i=1,\dots,n$, we set

$$\begin{aligned}a_i&=\alpha_i\langle e_i,e_i\rangle, & b_i&=\beta_i\langle e_i,e_i\rangle,\\ c_i&=\lambda_i\langle e_i,e_i\rangle, & d_i&=\mu_i\langle e_i,e_i\rangle,\end{aligned}\tag{3.25}$$

then similarly (3.11) is a refinement and a simple form of [9, Corollary 4.3].

Corollary 3.5. *Let $\mathcal{A}$ be a Banach $*$-algebra, let X be an inner product $\mathcal{A}$-module, and let $\{e_1,\dots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\dots,n)$ are idempotent. If $a_1,\dots a_n$, $b_1,\dots,b_n\in\mathcal{A}$, r,s are real numbers and $x,y\in X$ such that*

$$\Gamma\left(x-\sum_{i=1}^{n}e_ia_i\right)\le r,\qquad \Gamma\left(y-\sum_{i=1}^{n}e_ib_i\right)\le s \tag{3.26}$$

hold, then one has the inequality

$$\gamma(G_{x,y})\le rs-\sqrt{r^2-\gamma(G_x)}\sqrt{s^2-\gamma(G_y)}. \tag{3.27}$$

Proof. Using the schwarz's inequality (2.6), we have

$$\gamma(G_{x,y})^2\le\gamma(G_x)\gamma(G_y). \tag{3.28}$$

The assumptions (3.26) and the elementary inequality for real numbers (3.19) will provide the desired result (3.27). □

Example 3.6. Let $\mathcal{A}$ be a Hermitian Banach $*$-algebra and let ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x)=(\rho(\langle x,x\rangle))^{1/2}$ $(x\in X)$ with the properties that

$$P\left(x-\sum_{i=1}^{n}e_ia_i\right)\le r,\qquad P\left(y-\sum_{i=1}^{n}e_ib_i\right)\le s, \tag{3.29}$$

then we have

$$\rho(G_{x,y})\le rs-\sqrt{r^2-\rho(G_x)}\sqrt{s^2-\rho(G_y)}. \tag{3.30}$$

That is interesting in its own right.

Corollary 3.7. *Let $\mathcal{A}$ be a proper H^*-algebra, let X be an inner product $\mathcal{A}$-module, and let $\{e_1,\dots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\dots,n)$ are idempotent. If $a_1,a_n,b_1,\dots,b_n\in\mathcal{A}$, r,s are real numbers and $x,y\in X$ such that*

$$\left|\left|\left|x-\sum_{i=1}^{n}e_ia_i\right|\right|\right|\le r,\qquad \left|\left|\left|y-\sum_{i=1}^{n}e_ib_i\right|\right|\right|\le s \tag{3.31}$$

hold, then one has the inequality

$$\tau(G_{x,y})\le rs-\sqrt{r^2-\operatorname{tr}(G_x)}\sqrt{s^2-\operatorname{tr}(G_y)}. \tag{3.32}$$

Proof. Using the strong Schwarz's inequality (2.3), we have

$$\tau(G_{x,y})^2 \leq \operatorname{tr}(G_x)\operatorname{tr}(G_y). \tag{3.33}$$

The assumptions (3.31) and the elementary inequality for real numbers (3.19) will provide (3.32). □

The following companion of the Grüss inequality for positive linear functionals holds.

Theorem 3.8. *Let X be an inner product $\mathcal{A}$-module, let φ be a positive linear functional on $\mathcal{A}$, and let $\{e_1,\ldots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\ldots,n)$ are idempotent. If $a_1,\ldots a_n$, $b_1,\ldots,b_n \in \mathcal{A}$, r, s are real numbers and $x,y\in X$ such that*

$$\Phi\left(x-\sum_{i=1}^{n}e_ia_i\right)\leq r,\qquad \Phi\left(y-\sum_{i=1}^{n}e_ib_i\right)\leq s \tag{3.34}$$

hold, then one has the inequality

$$\left|\varphi(G_{x,y})\right|\leq rs-\sum_{i=1}^{n}\Phi(e_ia_i-e_i\langle e_i,x\rangle)\Phi(e_ib_i-e_i\langle e_i,y\rangle). \tag{3.35}$$

Proof. By taking φ on both sides of (3.12), we have

$$\begin{aligned}\varphi(G_x)&=\Phi\left(x-\sum_{i=1}^{n}e_ia_i\right)^2-\sum_{i=1}^{n}\Phi(e_ia_i-e_i\langle e_i,x\rangle)^2\\&\leq r^2-\sum_{i=1}^{n}\Phi(e_ia_i-e_i\langle e_i,x\rangle)^2.\end{aligned} \tag{3.36}$$

Analogously

$$\begin{aligned}\varphi(G_y)&=\Phi\left(y-\sum_{i=1}^{n}e_ib_i\right)^2-\sum_{i=1}^{n}\Phi(e_ib_i-e_i\langle e_i,y\rangle)^2\\&\leq s^2-\sum_{i=1}^{n}\Phi(e_ib_i-e_i\langle e_i,y\rangle)^2.\end{aligned} \tag{3.37}$$

Now, using Aczél's inequality for real numbers, that is, we recall that

$$\left(a^2-\sum_{i=1}^{n}a_i^2\right)\left(b^2-\sum_{i=1}^{n}b_i^2\right)\leq\left(ab-\sum_{i=1}^{n}a_ib_i\right)^2, \tag{3.38}$$

and the Schwarz's inequality for positive linear functionals, that is,

$$\varphi(G_{x,y})^2 \leq \varphi(G_x)\varphi(G_y), \tag{3.39}$$

we deduce (3.35). □

4. Some Related Results

Theorem 4.1. *Let X be an inner product C^*-module and let $\{e_1, \dots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i \rangle$ $(i = 1, \dots, n)$ are idempotent. Let $x, y \in X$ and if we define*

$$\begin{aligned} r_0 &= \inf\left\{ \left\| x - \sum_{i=1}^{n} e_i a_i \right\| : (a_1, \dots, a_n) \in \mathcal{A}^n \right\}, \\ s_0 &= \inf\left\{ \left\| y - \sum_{i=1}^{n} e_i a_i \right\| : (a_1, \dots, a_n) \in \mathcal{A}^n \right\}, \end{aligned} \tag{4.1}$$

then we have

$$\|G_{x,y}\| \leq r_0 s_0 - \sqrt{r_0^2 - \|G_x\|}\sqrt{s_0^2 - \|G_y\|}. \tag{4.2}$$

Proof. For every $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{A}$, by (3.13) and (3.14), we have

$$\|G_x\| \leq \left\| x - \sum_{i=1}^{n} e_i a_i \right\|^2, \qquad \|G_y\| \leq \left\| y - \sum_{i=1}^{n} e_i b_i \right\|^2. \tag{4.3}$$

Therefore

$$\|G_x\| \leq r_0^2, \qquad \|G_y\| \leq s_0^2. \tag{4.4}$$

Now, using the elementary inequality for real numbers

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2 \tag{4.5}$$

on

$$m = r_0, \qquad n = \sqrt{r_0^2 - \|G_x\|}, \qquad p = s_0, \qquad q = \sqrt{s_0^2 - \|G_y\|}, \tag{4.6}$$

we get

$$\|G_{x,y}\|^2 \leq \|G_x\|\|G_y\| \leq \left(r_0 s_0 - \sqrt{r_0^2 - \|G_x\|}\sqrt{s_0^2 - \|G_y\|}\right)^2. \tag{4.7}$$

□

Corollary 4.2. *Let $\mathcal{A}$ be a Banach $*$-algebra, let X be an inner product $\mathcal{A}$-module, and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. Let $x, y \in X$ and put*

$$\begin{aligned} r_0 &= \inf\left\{\Gamma\left(x - \sum_{i=1}^n e_i a_i\right) : (a_1, \ldots, a_n) \in \mathcal{A}^n\right\}, \\ s_0 &= \inf\left\{\Gamma\left(y - \sum_{i=1}^n e_i a_i\right) : (a_1, \ldots, a_n) \in \mathcal{A}^n\right\}, \end{aligned} \tag{4.8}$$

then

$$\gamma(G_{x,y}) \leq r_0 s_0 - \sqrt{r_0^2 - \gamma(G_x)}\sqrt{s_0^2 - \gamma(G_y)}. \tag{4.9}$$

Corollary 4.3. *Let $\mathcal{A}$ be a proper H^*-algebra, let X be an inner product $\mathcal{A}$-module, and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. Let $x, y \in X$ and if we consider*

$$\begin{aligned} r_0 &= \inf\left\{\left|\left|\left|\left(x - \sum_{i=1}^n e_i a_i\right)\right|\right|\right| : (a_1, \ldots, a_n) \in \mathcal{A}^n\right\}, \\ s_0 &= \inf\left\{\left|\left|\left|\left(y - \sum_{i=1}^n e_i a_i\right)\right|\right|\right| : (a_1, \ldots, a_n) \in \mathcal{A}^n\right\}, \end{aligned} \tag{4.10}$$

then

$$\tau(G_{x,y}) \leq r_0 s_0 - \sqrt{r_0^2 - \operatorname{tr}(G_x)}\sqrt{s_0^2 - \operatorname{tr}(G_y)}. \tag{4.11}$$

From a different perspective, we can state the following result as well.

Theorem 4.4. *Let X be an inner product C^*-module and let $\{e_1, \ldots, e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i, e_i\rangle$ $(i = 1, \ldots, n)$ are idempotent. If $a_1, \ldots a_n \in \mathcal{A}$, $r \in \mathbb{R}$, $\lambda \in (0,1)$ and $x, y \in X$ such that*

$$\left\|\lambda x + (1-\lambda)y - \sum_{i=1}^n e_i a_i\right\| \leq r, \tag{4.12}$$

then we have the inequality

$$\left\|\operatorname{Re}\left(G_{x,y}\right)\right\| \le \frac{1}{4}\cdot\frac{1}{\lambda(1-\lambda)}r^2. \tag{4.13}$$

Proof. We know that for any $a,b\in X$ and $\lambda\in(0,1)$ one has

$$\operatorname{Re}\langle a,b\rangle = \frac{1}{2}(\langle a,b\rangle+\langle b,a\rangle) \le \frac{1}{4\lambda(1-\lambda)}\langle \lambda a+(1-\lambda)b,\lambda a+(1-\lambda)b\rangle. \tag{4.14}$$

Put $a = x-\sum_{i=1}^{n} e_i\langle e_i,x\rangle$, $b = y-\sum_{i=1}^{n} e_i\langle e_i,y\rangle$, and since

$$G_{x,y} = \left\langle x-\sum_{i=1}^{n}e_i\langle e_i,x\rangle, y-\sum_{i=1}^{n}e_i\langle e_i,y\rangle\right\rangle = \langle a,b\rangle \tag{4.15}$$

using (4.14), we have

$$\begin{aligned}\left\|\operatorname{Re}\left(G_{x,y}\right)\right\| &= \|\operatorname{Re}(\langle a,b\rangle)\| \le \frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b\|^2\\ &\le \frac{1}{4\lambda(1-\lambda)}\left\|\lambda x+(1-\lambda)y-\sum_{i=1}^{n}e_i\langle e_i,\lambda x+(1-\lambda)y\rangle\right\|^2\\ &= \frac{1}{4\lambda(1-\lambda)}\left\|G_{\lambda x+(1-\lambda)y}\right\|^2.\end{aligned} \tag{4.16}$$

Now, inequality (4.13) follows from inequalities (3.15) and (4.16). □

The following companion of the Grüss inequality for positive linear functionals holds.

Theorem 4.5. *Let X be an inner product $\mathcal{A}$-module, let φ be a positive linear functional on $\mathcal{A}$, and let $\{e_1,\dots,e_n\}$ be a finite set of orthogonal elements in X such that $\langle e_i,e_i\rangle$ $(i=1,\dots,n)$ are idempotent. If $a_1,\dots a_n\in\mathcal{A}$, $r\in\mathbb{R}$, $\lambda\in(0,1)$ and $x,y\in X$ are such that*

$$\Phi\left(\lambda x+(1-\lambda)y-\sum_{i=1}^{n}e_ia_i\right)\le r, \tag{4.17}$$

then we have the inequality

$$\left|\varphi\left(\operatorname{Re}\left(G_{x,y}\right)\right)\right| \le \frac{1}{4}\cdot\frac{1}{\lambda(1-\lambda)}\left(r^2-\sum_{i=1}^{n}\Phi\left(e_ia_i-e_i\langle e_i,\lambda x+(1-\lambda)y\rangle\right)^2\right). \tag{4.18}$$

Proof. The inequality (4.14) for $a = x - \sum_{i=1}^{n} e_i\langle e_i, x\rangle$, $b = y - \sum_{i=1}^{n} e_i\langle e_i, y\rangle$ implies that

$$\begin{aligned}
\left|\varphi(\mathrm{Re}(G_{x,y}))\right| = \left|\varphi(\mathrm{Re}(\langle a,b\rangle))\right| &\le \frac{1}{4\lambda(1-\lambda)}\Phi(\lambda a + (1-\lambda)b)^2 \\
&\le \frac{1}{4\lambda(1-\lambda)}\Phi\left(\lambda x + (1-\lambda)y - \sum_{i=1}^{n} e_i\langle e_i, \lambda x + (1-\lambda)y\rangle\right)^2 \\
&= \frac{1}{4\lambda(1-\lambda)}\varphi\left(G_{\lambda x+(1-\lambda)y}\right)^2.
\end{aligned} \tag{4.19}$$

By making use of inequality (3.12) for $\lambda x + (1-\lambda)y$ instead of x and taking φ on both sides, we have

$$\begin{aligned}
\varphi\left(G_{\lambda x+(1-\lambda)y}\right) &= \Phi\left(\lambda x + (1-\lambda)y - \sum_{i=1}^{n} e_i a_i\right)^2 - \sum_{i=1}^{n}\Phi\left(e_i a_i - e_i\langle e_i, \lambda x + (1-\lambda)y\rangle\right)^2 \\
&\le r^2 - \sum_{i=1}^{n}\Phi\left(e_i a_i - e_i\langle e_i, \lambda x + (1-\lambda)y\rangle\right)^2.
\end{aligned} \tag{4.20}$$

From (4.19) and (4.20), we easily deduce (4.18). □

Remark 4.6. (i) The constant 1 coefficient of rs in (3.11) is sharp, in the sense that it cannot be replaced by a smaller quantity. If the submodule of H generated by $e_1, \ldots, e_n$ is not equal to X, then there exists $t \in X$ such that $t \neq \sum_{i=1}^{n} e_i\langle e_i, t\rangle$. We put $z = t - \sum_{i=1}^{n} e_i\langle e_i, t\rangle$, then $0 \neq z \in X$ and for any $j \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
\langle z, e_j\rangle &= \langle t, e_j\rangle - \sum_{i=1}^{n}\langle t, e_j\rangle\langle e_i, e_j\rangle \\
&= \langle t, e_j\rangle - \langle t, e_j\rangle\langle e_j, e_j\rangle = 0.
\end{aligned} \tag{4.21}$$

For every $\epsilon > 0$, if we put

$$x_\epsilon = \frac{rz}{\|z\| + \epsilon} + \sum_{i=1}^{n} e_i a_i, \qquad y_\epsilon = \frac{sz}{\|z\| + \epsilon} + \sum_{i=1}^{n} e_i b_i, \tag{4.22}$$

then

$$\begin{aligned}
G_{x_\epsilon, y_\epsilon} &= \langle x_\epsilon, y_\epsilon\rangle - \sum_{j=1}^{n}\langle x_\epsilon, e_j\rangle\langle e_j, y_\epsilon\rangle \\
&= \frac{rs}{(\|z\| + \epsilon)^2}\langle z, z\rangle + \sum_{i=1}^{n} a_i^*\langle e_i, e_i\rangle b_i - \sum_{j=1}^{n} a_i^*\langle e_j, e_j\rangle\langle e_j, e_j\rangle b_i \\
&= \frac{rs}{(\|z\| + \epsilon)^2}\langle z, z\rangle,
\end{aligned} \tag{4.23}$$

therefore

$$\left\| G_{x_\epsilon, y_\epsilon} \right\| = \frac{rs}{(\|z\| + \epsilon)^2} \|z\|^2. \tag{4.24}$$

Now if c is a constant such that $0 < c < 1$, then there is a $\epsilon > 0$ such that $\|z\|^2/(\|z\| + \epsilon)^2 > c$; therefore

$$\|G_{x_\epsilon, y_\epsilon}\| > crs. \tag{4.25}$$

(ii) Similarly, the constant 1 coefficient of rs in (3.32) is best possible, it is sufficient instead of (4.22) to put

$$x_\epsilon = \frac{rz}{\left\| |z|^2 \right\| + \epsilon} + \sum_{i=1}^{n} e_i a_i, \qquad y_\epsilon = \frac{sz}{\left\| |z|^2 \right\| + \epsilon} + \sum_{i=1}^{n} e_i b_i. \tag{4.26}$$

(iii) If there is a nonzero element z in X such that $z \perp \{e_1, \ldots, e_n\}$ and $\Gamma(z) \neq 0$ (resp. $\Phi(z) \neq 0$) then the constant 1 coefficient of rs in (3.27) (resp. (3.35)) is best possible. Also similarly, the inequalities in Theorem 4.1, Corollaries 4.2 and 4.3, and Theorems 4.4 and 4.5 are sharp. However, the details are omitted.

References

[1] R. S. Doran and V. A. Belfi, *Characterizations of C^*-Algebras*, vol. 101 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1986.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, vol. 8 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, New York, NY, USA, 1973.

[3] V. Pták, "On the spectral radius in Banach algebras with involution," *The Bulletin of the London Mathematical Society*, vol. 2, pp. 327–334, 1970.

[4] N. E. Wegge-Olsen, *K-Theory and C^*-Algebras: A Friendly Approach*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, NY, USA, 1993.

[5] P. P. Saworotnow, "A generalized Hilbert space," *Duke Mathematical Journal*, vol. 35, pp. 191–197, 1968.

[6] L. Molnár, "A note on the strong Schwarz inequality in Hilbert A-modules," *Publicationes Mathematicae Debrecen*, vol. 40, no. 3-4, pp. 323–325, 1992.

[7] A. G. Ghazanfari and S. S. Dragomir, "Schwarz and Grüss type inequalities for C^*-seminorms and positive linear functionals on Banach $*$-modules," *Linear Algebra and Its Applications*, vol. 434, no. 4, pp. 944–956, 2011.

[8] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science, Hauppauge, NY, USA, 2005.

[9] S. Banić, D. Ilišević, and S. Varošanec, "Bessel-and Grüss-type inequalities in inner product modules," *Proceedings of the Edinburgh Mathematical Society*, vol. 50, pp. 23–36, 2007.

[10] S. S. Dragomir, M. Khosravi, and M. S. Moslehian, "Bessel-type inequalities in hilbert C^*-modules," *Linear and Multilinear Algebra*, vol. 58, no. 8, pp. 967–975, 2010.

[11] D. Bakić and B. Guljaš, "Hilbert C^*-modules over C^*-algebras of compact operators," *Acta Scientiarum Mathematicarum*, vol. 68, no. 1-2, pp. 249–269, 2002.