A Study on the $p$-Adic $q$-Integral Representation on $\mathbb{Z}_p$ Associated with the Weighted $q$-Bernstein and $q$-Bernoulli Polynomials

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1 Introduction and Preliminaries

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. Let $q$ be regarded as either a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$. In this paper, we define the $q$-number as $[x]_q = (1 - q^x) / (1 - q)$ (see [1–13]).

Let $C[0,1]$ be the set of continuous functions on $[0,1]$. For $\alpha \in \mathbb{N}$ and $n,k \in \mathbb{Z}_+$, the weighted $q$-Bernstein operator of order $n$ for $f \in C[0,1]$ is defined by

$$B^{(a)}_{n \alpha} (f \mid x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} [x]_q^k (1-x)_q^{n-k} = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B^{(a)}_{k,n} (x,q).$$

Here $B^{(a)}_{k,n} (x,q)$ is called the weighted $q$-Bernstein polynomials of degree $n$ (see [2, 5, 6]).
Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$, which is called the bosonic $q$-integral on $\mathbb{Z}_p$, is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x,$$

(1.2)

(see [10]).

The Carlitz’s $q$-Bernoulli numbers are defined by

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

(1.3)

with the usual convention about replacing $\beta^k$ by $\beta_{k,q}$ (see [3, 9, 10]). In [3], Carlitz also defined the expansion of Carlitz’s $q$-Bernoulli numbers as follows:

$$\tilde{\beta}_{0,q} = \frac{h}{[h]_q}, \quad q^h(q\tilde{\beta} + 1)^n - \tilde{\beta}_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

(1.4)

with the usual convention about replacing $(\tilde{\beta}^h)^n$ by $\tilde{\beta}_{n,q}^h$.

The weighted $q$-Bernoulli numbers are constructed in previous paper [6] as follows: for $a \in \mathbb{N}$,

$$\tilde{\beta}_{a,q} = 1, \quad q^a(q\tilde{\beta}^a + 1)^n - \tilde{\beta}_{a,n,q} = \begin{cases} \frac{a}{[a]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

(1.5)

with the usual convention about replacing $(\tilde{\beta}^a)^n$ by $\tilde{\beta}_{a,n,q}^a$. Let $f_n(x) = f(x + n)$. By the definition (1.2) of $p$-adic $q$-integral on $\mathbb{Z}_p$, we easily get

$$qI_q(f_1) = q \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x + 1)q^x,$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x + \lim_{N \to \infty} \frac{f(p^N)q^{p^N} - f(0)}{[p^N]_q}$$

$$= \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0),$$

(1.6)

Continuing this process, we obtain easily the relation

$$q^n \int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) - \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = (q - 1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} q^l f'(l),$$

(1.7)

where $n \in \mathbb{N}$ and $f'(l) = df(l)/dx$ (see [6]).
Then by (1.2), applying to the function $x \rightarrow [x]_q^n$, we can see that

$$
\tilde{p}_{n,q}^{(a)} = \int_{\mathbb{Z}_p} [x]^n_q \, d\mu_q(x) = -\frac{na}{[a]} \sum_{m=0}^{\infty} q^{ma} [m]_q^{n-1} + (1 - q) \sum_{m=0}^{\infty} q^m [m]_q^n. \quad (1.8)
$$

The weighted $q$-Bernoulli polynomials are also defined by the generating function as follows:

$$
F_q^{(a)}(t, x) = -t \frac{\alpha}{[\alpha]} \sum_{m=0}^{\infty} q^{ma} e^{[m+x]_q t} + (1 - q) \sum_{m=0}^{\infty} q^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} \tilde{p}_{n,q}^{(a)}(x) \frac{t^n}{n!}, \quad (1.9)
$$

(see[6]). Thus, we note that

$$
\tilde{p}_{n,q}^{(a)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{alx} \tilde{p}_{l,q}^{(a)}
$$

$$
= -\frac{na}{[a]} \sum_{m=0}^{\infty} q^{ma} [m + x]_q^{n-1} + (1 - q) \sum_{m=0}^{\infty} q^m [m + x]_q^n. \quad (1.10)
$$

From (1.2) and the previous equalities, we obtain the Witt’s formula for the weighted $q$-Bernoulli polynomials as follows:

$$
\tilde{p}_{n,q}^{(a)}(x) = \int_{\mathbb{Z}_p} [x + y]^n_q \, d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} q^{alx} [x]_q^{n-l} \int_{\mathbb{Z}_p} [y]^l_q \, d\mu_q(y). \quad (1.11)
$$

By using (1.2) and the weighted $q$-Bernoulli polynomials, we easily get

$$
q^n \tilde{p}_{m,q}^{(a)}(n) - \tilde{p}_{m,q}^{(a)} = (q - 1) \sum_{i=0}^{n-1} q^i [i]_q^m + \frac{ma}{[a]} \sum_{i=0}^{n-1} q^{al+i} [i]_q^{m-1}, \quad (1.12)
$$

where $n, \alpha \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, (see [6]).

In this paper, we consider the weighted $q$-Bernstein polynomials to express the bosonic $q$-integral on $\mathbb{Z}_p$ and investigate some properties of the weighted $q$-Bernstein polynomials associated with the weighted $q$-Bernoulli polynomials by using the expression of $p$-adic $q$-integral on $\mathbb{Z}_p$ of those polynomials.

### 2. Weighted $q$-Bernstein Polynomials and $q$-Bernoulli Polynomials

In this section, we assume that $\alpha \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

Now we consider the $p$-adic weighted $q$-Bernstein operator as follows:

$$
\mathbb{B}_{n,q}^{(a)}(f \mid x)(fx) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]^k_q [1 - x]_{q^k}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{(a)}(x, q). \quad (2.1)
$$
The $p$-adic $q$-Bernstein polynomials with weight $\alpha$ of degree $n$ are given by

$$B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k} [x]^k_q [1 - x]^{n-k}_q, \quad (2.2)$$

where $x \in \mathbb{Z}_p$, $\alpha \in \mathbb{N}_+$, and $n, k \in \mathbb{Z}_+$ (see [6, 7]). Note that $B_{k,n}^{(\alpha)}(x, q) = B_{n-k,n}^{(\alpha)}(1 - x, 1/q)$. That is, the weighted $q$-Bernstein polynomials are symmetric.

From the definition of the weighted $q$-Bernoulli polynomials, we have

$$\tilde{p}_{n,q}^{(\alpha)}(1 - x) = (-1)^n q^{\alpha n} \tilde{p}_{n,q}^{(\alpha)}(x). \quad (2.3)$$

By the definition of $p$-adic $q$-integral on $\mathbb{Z}_p$, we get

$$\int_{\mathbb{Z}_p} [1 - x]^n_q d\mu_q(x) = q^{\alpha n} (-1)^n \int_{\mathbb{Z}_p} [1 + x]^n_q d\mu_q(x)$$

$$\quad = \int_{\mathbb{Z}_p} \left(1 - [x]_q^n\right) d\mu_q(x). \quad (2.4)$$

From (2.3) and (2.4), we have

$$\int_{\mathbb{Z}_p} [1 - x]^n_q d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{p}_{l,q}^{(\alpha)} = q^{\alpha n} (-1)^n \tilde{p}_{n,q}^{(\alpha)}(-1) = \tilde{p}_{n,q}^{(\alpha)}(2). \quad (2.5)$$

Therefore, we obtain the following lemma.

**Lemma 2.1.** For $n \in \mathbb{Z}_+$, one has

$$\int_{\mathbb{Z}_p} [1 - x]^n_q d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{p}_{l,q}^{(\alpha)} = q^{\alpha n} (-1)^n \tilde{p}_{n,q}^{(\alpha)}(-1) = \tilde{p}_{n,q}^{(\alpha)}(2), \quad (2.6)$$

$$\tilde{p}_{n,q}^{(\alpha)}(1 - x) = (-1)^n q^{-\alpha n} \tilde{p}_{n,q}^{(\alpha)}(x).$$

By (2.2), (2.3), and (2.4), we get

$$q^2 \tilde{p}_{n,q}^{(\alpha)}(2) = n \frac{\alpha}{[\alpha]_q} q^{1+\alpha} + q^2 - q + \tilde{p}_{n,q}^{(\alpha)}, \quad \text{if } n > 1. \quad (2.7)$$

Thus, we have

$$\tilde{p}_{n,q}^{(\alpha)}(2) = \frac{1}{q^2} \tilde{p}_{n,q}^{(\alpha)} + \frac{n\alpha}{[\alpha]_q} q^{\alpha-1} + 1 - \frac{1}{q}, \quad \text{if } n > 1. \quad (2.8)$$

Therefore, by (2.8), we obtain the following proposition.
Proposition 2.2. For $n \in \mathbb{N}$ with $n > 1$, one has

$$f_{n,q}^{(a)}(2) = \frac{1}{q} f_{n,q}^{(a)} + \frac{n\alpha}{[\alpha]_q} q^{a-1} + 1 - \frac{1}{q}.$$  \hfill (2.9)

By using Proposition 2.2 and Lemma 2.1, we obtain the following corollary.

Corollary 2.3. For $n \in \mathbb{N}$ with $n > 1$, one has

$$\int_{\mathbb{Z}_p} [1 - x]^n_{q^a} d\mu_q(x) = q^2 f_{n,q}^{(a)} + \frac{n\alpha}{[\alpha]_q} + 1 - q.$$ \hfill (2.10)

$$\int_{\mathbb{Z}_p} [1 - x]^n_{q^a} d\mu_q(x) = \frac{n\alpha}{[\alpha]_q} + 1 + q^2 \sum_{l=0}^{n-k} [x]_{q^a}^l [1 - x]_{q^a}^{n-k} d\mu_q(x).$$ \hfill (2.11)

Taking the bosonic $q$-integral on $\mathbb{Z}_p$ for one weighted $q$-Bernstein polynomials in (2.1), we have

$$\int_{\mathbb{Z}_p} B_{k,n}^{(a)}(x,q) d\mu_q(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]_{q^a}^k [1 - x]_{q^a}^{n-k} d\mu_q(x)$$

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^a}^l d\mu_q(x)$$ \hfill (2.12)

$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l f_{k+l,q}^{(a)}.$$

By the symmetry of $q$-Bernstein polynomials, we get

$$\int_{\mathbb{Z}_p} B_{k,n}^{(a)}(x,q) d\mu_q(x) = \int_{\mathbb{Z}_p} B_{n-k,n}^{(a)} \left(1 - x, \frac{1}{q} \right) d\mu_q(x)$$

$$= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1 - x]_{q^a}^{n-l} d\mu_q(x).$$ \hfill (2.13)
For \( n > k + 1 \), by (2.11) and (2.13), we have

\[
\int_{\mathbb{Z}_p} B_{k,n}^{(a)}(x, q) d\mu_q(x) = \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( \frac{na}{[\alpha]_q} + 1 - q + q^2 \int_{\mathbb{Z}_p} [x]^n \mu_{q^{-1}}(x) \right)
\]

\[
= \left\{ \begin{array}{ll}
\frac{na}{[\alpha]_q} + 1 - q + q^2 \beta_{n,q^{-1}}, & \text{if } k = 0, \\
\left( \frac{n}{k} \right) q^2 \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \beta_{n,q^{-1}}, & \text{if } k > 0.
\end{array} \right.
\]  

By comparing the coefficients on the both sides of (2.12) and (2.14), we obtain the following theorem.

**Theorem 2.4.** For \( n, k \in \mathbb{Z}_+ \) with \( n > k + 1 \), one has

\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{k+l} \beta_{k+l,q} = q^2 \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \beta_{n-l,q^{-1}}, \quad \text{if } k \neq 0.
\]  

In particular, when \( k = 0 \), one has

\[
\frac{na}{[\alpha]_q} + 1 - q + q^2 \beta_{n,q^{-1}} = \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \beta_{l,q}.
\]  

Let \( m, n, k \in \mathbb{Z}_+ \), with \( m + n > 2k + 1 \). Then we see that

\[
\int_{\mathbb{Z}_p} P_{k,n}^{(a)}(x, q) P_{k,m}^{(a)}(x, q) d\mu_q(x)
\]

\[
= \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \int_{\mathbb{Z}_p} [x]^{2k} [1-x]^{n+m-2k} d\mu_q(x)
\]

\[
= \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1-x]^{n+m-l} d\mu_q(x).
\]

Therefore, by (2.17), we obtain the following theorem.
Theorem 2.5. For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, one has

\[
\int_{\mathbb{Z}_p} B_{k,n}^{(a)}(x,q) B_{k,m}(x,q) \, d\mu_q(x) = \begin{cases} 
\frac{n\alpha}{[\alpha]}_q + 1 - q + q^2 \tilde{\beta}_{n+m,q}^{(a)}, & \text{if } k = 0, \\
\binom{n}{k} \binom{m}{k} q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \tilde{\beta}_{n+m-l,q}^{(a)}, & \text{if } k \neq 0. 
\end{cases}
\]

(2.18)

For $m, n, k \in \mathbb{Z}_+$, we have

\[
\int_{\mathbb{Z}_p} B_{k,n}^{(a)}(x,q) B_{k,m}(x,q) \, d\mu_q(x)
= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]^{2k} [1-x]^{n+m-2k} \, d\mu_q(x)
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]^{2k+l} \, d\mu_q(x)
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \tilde{\beta}_{l+2k,q}^{(a)}.
\]

(2.19)

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.6. For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, one has

\[
\frac{n\alpha}{[\alpha]}_q + 1 - q + q^2 \tilde{\beta}_{n+m,q}^{(a)} = \sum_{l=0}^{n+m} \binom{n+m}{l} (-1)^l \tilde{\beta}_{l,q}^{(a)}. 
\]

(2.20)

Furthermore, for $k \neq 0$, one has

\[
\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \tilde{\beta}_{l+2k,q}^{(a)} = q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \tilde{\beta}_{n+m-l,q}^{(a)}. 
\]

(2.21)

By the induction hypothesis, we obtain the following theorem.

Theorem 2.7. For $s \in \mathbb{N}$ and $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$, one has

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x,q) \right) \, d\mu_q(x) = \begin{cases} 
\frac{n\alpha}{[\alpha]}_q + 1 - q + q^2 \tilde{\beta}_{n_1+\cdots+n_s,q}^{(a)}, & \text{if } k = 0, \\
\binom{s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^l \tilde{\beta}_{n_1+\cdots+n_s-l,q}^{(a)}, & \text{if } k \neq 0. 
\end{cases}
\]

(2.22)
For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then we show that

$$\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} \beta_{k,n_i}(x,q) \right) d\mu_q(x) = \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \tilde{\beta}^{(a)}_{l+sk,q}. \tag{2.23}$$

Therefore, by Theorem 2.7 and (2.23), we obtain the following theorem.

**Theorem 2.8.** For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then one sees that

$$\sum_{i=0}^{n_1+\cdots+n_s} \binom{n_1+\cdots+n_s}{l} (-1)^l \tilde{\beta}^{(a)}_{l,q} = \frac{n\alpha}{\alpha} + 1 - q + q^2 \tilde{\beta}^{(a)}_{n_1+\cdots+n_s,q}. \tag{2.24}$$

For $k \neq 0$, one has

$$\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \tilde{\beta}^{(a)}_{n_1+\cdots+n_s-l,q} = \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^l \tilde{\beta}^{(a)}_{l+sk,q}. \tag{2.25}$$

**References**


