Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 434175, 7 pages
doi:10.1155/2011/434175

Research Article

Strong Converse Inequality for a Spherical Operator

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Received 2 July 2010; Accepted 8 February 2011

Academic Editor: S. S. Dragomir

In the paper titled as "Jackson-type inequality on the sphere" (2004), Ditzian introduced a spherical nonconvolution operator $O_{t,r}$, which played an important role in the proof of the well-known Jackson inequality for spherical harmonics. In this paper, we give the lower bound of approximation by this operator. Namely, we prove that there are constants C_1 and C_2 such that $C_1\omega_{2r}(f,t)_p \leq \|O_{t,r}f - f\|_p \leq C_2\omega_{2r}(f,t)_p$ for any pth Lebesgue integrable or continuous function f defined on the sphere, where $\omega_{2r}(f,t)_p$ is the $2r$th modulus of smoothness of f.

1. Introduction

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ $(d \geq 3)$ be the unit sphere of $\mathbb{R}^d$ endowed with the usual rotation invariant measure $d\omega(x)$. We denote by $\mathcal{H}_k^d$ the space of all spherical harmonics of degree k on $\mathbb{S}^{d-1}$ and Π_n^d the space of all spherical harmonics of degree at most n. The spaces $\mathcal{H}_k^d (k = 0, 1, \ldots)$ are mutually orthogonal with respect to the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^{d-1}} f(x)g(x)d\omega(x), \tag{1.1}$$

so there holds

$$\Pi_n^d = \mathcal{H}_0^d \oplus \mathcal{H}_1^d \oplus \cdots \oplus \mathcal{H}_n^d. \tag{1.2}$$

By $C(\mathbb{S}^{d-1})$ and $L^p(\mathbb{S}^{d-1})$, $1 \le p < +\infty$, we denote the space of continuous, real-value functions and the space of (the equivalence classes of) p-integrable functions defined on $\mathbb{S}^{d-1}$ endowed with the respective norms

$$\|f\|_{C(\mathbb{S}^{d-1})} := \max_{\mu\in\mathbb{S}^{d-1}} |f(\mu)|, \qquad \|f\|_p := \left(\int_{\mathbb{S}^{d-1}} |f(\mu)|^p d\omega(\mu)\right)^{1/p}, \quad 1 \le p < \infty. \tag{1.3}$$

In the following, $L^p(\mathbb{S}^{d-1})$ will always be one of the spaces $L^p(\mathbb{S}^{d-1})$ for $1 \le p < \infty$, or $C(\mathbb{S}^{d-1})$ for $p = \infty$.

For an arbitrary number θ, $0 < \theta < \pi$, we define the spherical translation operator with step θ as (see [1, 2])

$$S_\theta(f) := S_\theta(f;\mu) = \frac{1}{|\mathbb{S}^{d-2}|\sin^{d-2}\theta} \int_{\mu\cdot\nu=\cos\theta} f(\nu)d\omega_{d-2}(\nu), \tag{1.4}$$

where ω_{d-2} means the $(d-2)$-dimensional surface area of sphere embedded into $\mathbb{R}^{d-1}$. Here we integrate over the family of points $\nu \in \mathbb{S}^{d-2}$ whose spherical distance from the given point $\mu \in \mathbb{S}^{d-1}$ (i.e., the length of minor arc between μ and ν on the great circle passing through them) is equal to θ. Thus $S_\theta(f;\mu)$ can be interpreted as the mean value of the function f on the surface of a $(d-2)$-dimensional sphere with radius $\sin\theta$.

By the help of translation operator, we can define the modulus of smoothness of $f \in L_p(\mathbb{S}^{d-1})$ as (see [3, Chapter 10] or [4])

$$\omega_r(f,t)_p := \sup_{0<\theta_i\le t} \left\|(S_{\theta_1} - I)(S_{\theta_2} - I)\cdots(S_{\theta_r} - I)f\right\|_p. \tag{1.5}$$

Clearly, the modulus is meaningful to describe the approximation degree and the smoothness of f, which has been widely used in the study of approximation on sphere.

The Laplace-Beltrami operator Δ is defined by (see [5, 6])

$$\Delta f := \sum_{i=1}^{d} \frac{\partial^2 g(x)}{\partial x_i^2}\bigg|_{|x|=1}, \quad g(x) = f\left(\frac{x}{|x|}\right), \tag{1.6}$$

where $|x| = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$, $x = (x_1, x_2, \ldots, x_d)$. We also need a K-functional on sphere $\mathbb{S}^{d-1}$ defined by (see [3])

$$K_{2r}(f;t)_p := \inf_{g\in C^{2r}(\mathbb{S}^{d-1})} \left\{\|f-g\|_p + t^{2r}\|\Delta^r g\|_p\right\}, \tag{1.7}$$

where $\Delta^k := \Delta^{k-1}\Delta$. For the modulus of smoothness and K-functional, the following equivalent relationship has been proved (see [3, Section 10.6])

$$\omega_{2r}(f,t)_p \sim K_{2r}(f,t)_p. \tag{1.8}$$

Throughout this paper, we denote by C_i $(i = 1,2,\ldots)$ the positive constants independent of f and n and by $C(a)$ the positive constants depending only on a. Their value will be different at different occurrences, even within the same formula. By $A \sim B$ we denote that there are positive constants C_1 and C_2 such that $C_1 B \le A \le C_2 B$.

In [3], Ditzian introduced a spherical operator $O_{t,r}$ and used it to prove the well-known Jackson type inequality for spherical harmonics. Before giving the definition of $O_{t,r}$, we need to introduce some preliminaries. Denote

$$T(\rho)f(x) := f(\rho x) \quad \text{for } \rho \in \mathrm{SO}(d),\ x \in \mathbb{S}^{d-1}, \tag{1.9}$$

where $\mathrm{SO}(d)$ denotes the group of orthogonal matrices on $\mathbb{R}^d$ with determinants 1. We denote further

$$\overline{\Delta}_{\rho}^{2r} f(x) := \left(T(\rho) - 2I + T\left(\rho^{-1}\right)\right)^{r} f(x). \tag{1.10}$$

For an orthogonal matrix Q with determinant 1, we define

$$M(t,Q) := Q^{-1}\begin{pmatrix} \cos t & \sin t & 0 & 0 & \cdots & 0 \\ -\sin t & \cos t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{1.11}$$

Now we are in the position to define the operator $O_{t,r}$. At first we define the averaging operator $A_{t,r}f$ by (see [3])

$$f - A_{t,r}f := \frac{1}{\binom{2r}{r}} \int_{Q\in \mathrm{SO}(d)} \overline{\Delta}_{M(t,Q)}^{2r} f\, dQ, \tag{1.12}$$

where dQ represents the Haar measure on $\mathrm{SO}(d)$ normalized so that

$$\int_{Q\in \mathrm{SO}(d)} dQ = 1, \tag{1.13}$$

where the definition of the Haar measure can be found in [7]. Furthermore, for a measure $\mu_t(u)$ supported in $[0,t]$ (t being fixed and u is the variable) such that $\int d\mu_t(u) = 1 (d\mu_t(u) = 0$ for $u > |t|)$, the operator $O_{t,r}$ is defined by

$$O_{t,r}f := \int A_{u,r}f(x)\, d\mu_t(u). \tag{1.14}$$

In [3], Ditzian gave a converse inequality for $O_{t,r}$ as follows.

Theorem A. *For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, and some fixed $\eta > 1$, there holds*

$$C^{-1}\|f - O_{t,r}f\|_p \le K_{2r}(f;t)_p \le C\Big(\|f - O_{t,rf}\|_p + \|f - O_{\eta t,r}f\|_p\Big). \tag{1.15}$$

In this paper, we improve this result. Motivated by [8, 9], we obtain the following Theorem 1.1.

Theorem 1.1. *For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, there holds*

$$\|O_{t,r}f - f\|_p \sim \omega_{2r}(f,t)_p \sim K_{2r}(f,t)_p. \tag{1.16}$$

2. The Proof of Main Result

Before proceeding the proof, we state some useful lemmas at first. The first one can be find in [3, page 6].

Lemma 2.1. *For any $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, there exists a constant $C(r)$ depending only on r such that*

$$\|O_{t,r}(f)\|_p \le C(r)\|f\|_p. \tag{2.1}$$

The following three lemmas reveal some important properties of $O_{t,r}(f)$. Their proofs can be found in [3, Theorem 6.1], [3, Theorem 6.2], and [3, equation (4.8)], respectively.

Lemma 2.2. *For $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, and $m \ge 2k$, one has*

$$\left\|\Delta^k O_{t,r}^m f\right\|_p \le \frac{C(k)}{t^{2k}}\left\|O_{t,r}^{m-2k} f\right\|_p, \tag{2.2}$$

where $O_{t,r}^m f := O_{t,r}^{m-1} O_{t,r} f$.

Lemma 2.3. *For $g \in C^{2r+2}(\mathbb{S}^{d-1})$, and $1 \le p \le \infty$, there holds*

$$\left\|O_{t,r}g - g - t^{2r}P_r(\Delta)g\right\|_p \le Ct^{2r+2}\left\|\Delta^{r+1}g\right\|_p, \tag{2.3}$$

where $P_r(\Delta) := \sum_{i=1}^r a_i\Delta^i g$ is a polynomial of degree r in Δ. Moreover, $P_r(\Delta)g = 0$ only for $g =$ const.

Lemma 2.4. *For any $g \in C^{2r+2}(\mathbb{S}^{d-1})$, any $k \le r$, and $m \in \mathbb{Z}$, there holds*

$$O_{t,r}^m \Delta^k f = \Delta^k O_{t,r}^m f. \tag{2.4}$$

From (1.8) and [10, Theorem 3.2] (see also [3, page 16]) we deduce the following Lemma 2.5 easily.

Lemma 2.5. *Let $P_r(\Delta)$ be defined in* (2.3) *and* $1 \le p \le \infty$, *then one has*

$$K_{2r}(f,t)_p \sim \omega_{2r}(f,t)_p \sim \inf_{g\in C^{2r}} \left(\|f-g\|_p + t^{2r}\|P_r(\Delta)g\|_p\right). \tag{2.5}$$

Now, we give the last lemma, which can easily be deduced from [10, Theorem 3.1].

Lemma 2.6. *Let $P_r(\Delta)$ be defined in* (2.3) *and* $1 \le p \le \infty$, *then one has*

$$t^{2r+2}\left\|\Delta^{r+1}O_{t,r}^{m} f\right\|_p \le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m-2} f\right\|_p. \tag{2.6}$$

We now give the proof of Theorem 1.1. It has been shown in (1.15) and (1.8) that there exists a constant C_1 such that

$$\|f-O_{t,r}(f)\|_p \le C_1\omega_{2r}(f,t)_p, \tag{2.7}$$

hence we only need to prove that there exists a constant C_2 such that

$$\omega_{2r}(f,t)_p \le C_2\|f-O_{t,r}(f)\|_p. \tag{2.8}$$

From (2.5) it is sufficient to prove that, for $m \ge 2r+1$, there holds

$$\left\|f-O_{t,r}^{m}(f)\right\|_p + t^{2r}\left\|P_r(\Delta)O_{t,r}^{m}(f)\right\|_p \le C_3\|f-O_{t,r}(f)\|_p. \tag{2.9}$$

In order to prove (2.9), we first prove

$$\left\|f-O_{t,r}^{m} f\right\|_p \le C(m)\|f-O_{t,r}\|_p. \tag{2.10}$$

Indeed, from (2.1), we have

$$\begin{aligned}\left\|f-O_{t,r}^{m}f\right\|_{p} &\le \|f-O_{t,r}f\|_{p}+\sum_{k=1}^{m-1}\left\|O_{t,r}^{k}f-O_{t,r}^{k+1}f\right\|_{p}\\ &\le \|f-O_{t,r}f\|_{p}+\sum_{k=1}^{m-1}\left\|O_{t,r}^{k}(f-O_{t,r}f)\right\|_{p}\\ &\le \|f-O_{t,r}f\|_{p}+C\sum_{k=1}^{m-1}\left\|O_{t,r}^{k-1}(f-O_{t,r}f)\right\|_{p}\\ &\le \cdots \le \|f-O_{t,r}f\|_{p}+C\sum_{k=1}^{m-1}\|O_{t,r}(f-O_{t,r}f)\|_{p}\\ &\le C(m)\|f-O_{t,r}f\|_{p}.\end{aligned} \tag{2.11}$$

Now we turn to prove

$$t^{2r}\left\|P_r(\Delta)O_{t,r}^{m}(f)\right\|_{p} \le C_4\|f-O_{t,r}(f)\|_{p}. \tag{2.12}$$

In fact, from (2.3), we obtain

$$t^{2r}\left\|P_r(\Delta)O_{t,r}^{m}(f)\right\|_{p} \le \left\|O_{t,r}O_{t,r}^{m}(f)-O_{t,r}^{m}(f)\right\|_{p}+C_5t^{2r+2}\left\|\Delta^{r+1}O_{t,r}^{m}(f)\right\|_{p}. \tag{2.13}$$

In order to estimate $t^{2r+2}\|\Delta^{r+1}O_{t,r}^{m}(f)\|_{p}$, we use (2.6) and obtain that

$$\begin{aligned}t^{2r+2}\left\|\Delta^{r+1}O_{t,r}^{m}(f)\right\|_{p} &\le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m-2}f\right\|_{p}\\ &\le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m}f\right\|_{p}+Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m}f-P_r(\Delta)O_{t,r}^{m-2}f\right\|_{p}\\ &\le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m}f\right\|_{p}+Ct^{2r}\left\|a_r\Delta^{r}O_{t,r}^{m-2}\left(f-O_{t,r}^{2}f\right)\right\|_{p}\\ &\quad+Ct^{2r}\left\|a_{r-1}\Delta^{r-1}O_{t,r}^{m-4}\left(O_{t,r}^{2}f-O_{t,r}^{4}f\right)\right\|_{p}\\ &\quad+\cdots+Ct^{2r}\left\|a_1\Delta O_{t,r}^{m-2r}\left(O_{t,r}^{2r-2}f-O_{t,r}^{2r}f\right)\right\|_{p}.\end{aligned} \tag{2.14}$$

Using (2.2) again and (2.10), we have

$$\begin{aligned} t^{2r+2}\left\|\Delta^{r+1}O_{t,r}^{m}(f)\right\|_p &\le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m}f\right\|_p + C_r\left\|O_{t,r}^{m-2-2r}\left(f-O_{t,r}^{2}f\right)\right\|_p \\ &\quad + C_{r-1}t^2\left\|O_{t,r}^{m-2r-2}\left(O_{t,r}^{2}f-O_{t,r}^{4}f\right)\right\|_p \\ &\quad + \cdots + C_1 t^{2r-2}\left\|O_{t,r}^{m-2r-2}\left(O_{t,r}^{2r-2}f-O_{t,r}^{4}f\right)\right\|_p \\ &\le Ct^{2r}\left\|P_r(\Delta)O_{t,r}^{m}f\right\|_p + C'\left\|f-O_{t,r}f\right\|_p. \end{aligned} \tag{2.15}$$

The above inequality together with (2.13) and (2.10) yields (2.12). Then we can deduce (2.9) from (2.12) and (2.10) easily. Therefore (2.8) holds. This completes the proof of Theorem 1.1.

Acknowledgment

The research was supported by the National Natural Science Foundation of China (no. 60873206).

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