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Research Article

Some Properties of Orthogonal Polynomials for Laguerre-Type Weights

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Let $\mathbb{R}^+ = [0, \infty)$, let $R : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, nonnegative, and increasing function, and let $p_{n,\rho}(x)$ be the orthonormal polynomials with the weight $w_\rho(x) = x^\rho e^{-R(x)}, \rho > -1/2$. For the zeros $\{x_{k,n,\rho}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$, we estimate $p_{n,\rho}^{(j)}(x_{k,n,\rho})$, where j is a positive integer. Moreover, we investigate the various weighted L_p-norms $(0 < p \leqslant \infty)$ of $p_{n,\rho}(x)$.

1. Introduction and Main Results

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $R : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, nonnegative, and increasing function. Consider the exponential weights $w_\rho(x) = x^\rho \exp(-R(x)), \rho > -1/2$, and then we construct the orthonormal polynomials $\{p_{n,\rho}(x)\}_{n=0}^\infty$ with the weight $w_\rho(x)$. In this paper, for the zeros $\{x_{kn}\}_{k=1}^n$ of $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ we estimate $p_{n,\rho}^{(j)}(x_{kn})$, where j is a positive integer. Moreover, we investigate the various weighted L_p-norms $(0 < p \leqslant \infty)$ of $p_{n,\rho}(x)$.

We say that $f : \mathbb{R} \to \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leqslant Cf(y)$ for $0 < x < y$. The notation $f(x) \sim g(x)$ means that there are positive constants C_1, C_2 such that for the relevant range of x, $C_1 \leqslant f(x)/g(x) \leqslant C_2$. The similar notation is used for sequences and sequences of functions.

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree n by $\mathcal{P}_n$.

First, we introduce some classes of weights.

Levin and Lubinsky [1, 2] introduced the class of weights on $\mathbb{R}^+$ as follows. Let $I = [0, d)$, where $0 < d \leqslant \infty$.

Definition 1.1 (see [1, 2]). We assume that $R : I \to [0,\infty)$ has the following properties. Let $Q(t) = R(t^2)$,

(a) $\sqrt{x}R(x)$ is continuous in I, with limit 0 at 0 and $R(0) = 0$,

(b) $R''(x)$ exists in $(0,d)$, while $Q''(t)$ is positive in $(0,\sqrt{d})$,

(c)

$$\lim_{x\to d-} R(x) = \infty; \tag{1.1}$$

(d) the function

$$T(x) := \frac{xR'(x)}{R(x)} \tag{1.2}$$

is quasi-increasing in $(0,d)$, with

$$T(x) \geqslant \Lambda > \frac{1}{2}, \quad x \in (0,d); \tag{1.3}$$

(e) there exists $C_1 > 0$ such that

$$\frac{|R''(x)|}{R'(x)} \leqslant C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in (0,d). \tag{1.4}$$

Then, we write $w \in \mathcal{L}(C^2)$. If there also exists a compact subinterval $J^* \ni 0$ of $I^* = (-\sqrt{d},\sqrt{d})$, and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \geqslant C_2 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus J^*, \tag{1.5}$$

then we write $w \in \mathcal{L}(C^2+)$.

We consider the case $d = \infty$, that is, the space $\mathbb{R}^+ = [0,\infty)$, and we strengthen Definition 1.1 slightly.

Definition 1.2. We assume that $R : \mathbb{R}^+ \to \mathbb{R}^+$ has the following properties:

(a) $R(x), R'(x)$ are continuous, positive in $\mathbb{R}^+$, with $R(0) = 0$, $R'(0) = 0$,

(b) $R''(x) > 0$ exists in $\mathbb{R}^+ \setminus \{0\}$,

(c)

$$\lim_{x\to\infty} R(x) = \infty, \tag{1.6}$$

(d) the function

$$T(x) := \frac{xR'(x)}{R(x)} \tag{1.7}$$

is quasi-increasing in $\mathbb{R}^+ \setminus \{0\}$, with

$$T(x) \geqslant \Lambda > \frac{1}{2}, \quad x \in \mathbb{R}^+ \setminus \{0\}, \tag{1.8}$$

(e) there exists $C_1 > 0$ such that

$$\frac{R''(x)}{R'(x)} \leqslant C_1 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}. \tag{1.9}$$

There exists a compact subinterval $J \ni 0$ of $\mathbb{R}^+$, and $C_2 > 0$ such that

$$\frac{R''(x)}{R'(x)} \geqslant C_2 \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus J, \tag{1.10}$$

then we write $w \in \mathcal{L}_2$.

Let us consider the weight $w \in \mathcal{L}(C^2+)$ in Definition 1.2. Levin and Lubinsky [2, Theorem 1.3] have given the following theorem.

Theorem A (see [3, Theorem 1.3]). *Let $w \in \mathcal{L}(C^2+)$ and $\rho > -1/2$. There exists n_0 such that uniformly for $n \geqslant n_0$, $1 \leqslant j \leqslant n$,*

$$\left|p'_{n,\rho} w_\rho\right|(x_{jn}) \sim \varphi_n(x_{jn})^{-1}\left[x_{jn}(a_n - x_{jn})\right]^{-1/4}. \tag{1.11}$$

Now, we will estimate the higher-order derivatives of the orthonormal polynomials $p_n(w_\rho^2; x)$. However, we need to focus on a smaller class of weights.

Definition 1.3. Let $w = \exp(-R) \in \mathcal{L}_2$ and $\nu \geqslant 2$ be an integer. For the exponent R, we assume the following:

(a) $R^{(j)}(x) > 0$, for $0 \leqslant j \leqslant \nu$ and $x > 0$, and $R^{(j)}(0) = 0$, $0 \leqslant j \leqslant \nu - 1$.

(b) there exist positive constants $C_i > 0$ such that for $i = 1, 2, \ldots, \nu - 1$

$$R^{(i+1)}(x) \leqslant C_i R^{(i)}(x) \frac{R'(x)}{R(x)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}, \tag{1.12}$$

(c) there exist positive constants C, $0 < c_1 \leqslant 1$ and $0 \leqslant \delta < 1$ such that for $x \in (0, c_1)$

$$R^{(\nu)}(x) \leqslant C\left(\frac{1}{x}\right)^{\delta}, \tag{1.13}$$

(d) there exists $c_2 > 0$ such that we have one among the following:

(d1) $T(x)/\sqrt{x}$ is quasi-increasing on (c_2, ∞),

(d2) $R^{(\nu)}(x)$ is nondecreasing on (c_2, ∞).

Then, we write $w(x) = e^{-R(x)} \in \widetilde{\mathcal{L}}_\nu$.

Example 1.4 (see [1, 4]). Let $\nu \geqslant 2$ be a fixed integer. There are some typical examples satisfying all conditions of Definition 1.3 constructed as follows: let $\alpha > 1$, $l \geqslant 1$, where l is an integer. Then, we define

$$R_{l,\alpha}(x) = \exp_l(x^\alpha) - \exp_l(0), \tag{1.14}$$

where $\exp_l(x) = \exp(\exp(\exp\cdots\exp(x))\cdots)$ is the lth iterated exponential.

(1) When $\alpha > \nu$, we consider $w(x) = e^{-R_{l,\alpha}(x)}$, then $w \in \widetilde{\mathcal{L}}_\nu$.

(2) When $\alpha \leqslant \nu$, α is an integer, we define

$$R_{l,\alpha}(x) = \exp_l(|x|^\alpha) - \exp_l(0) - \sum_{j=1}^{r} \frac{R_{l,\alpha}^{(j)}(0)}{j!} x^j. \tag{1.15}$$

Then, $w(x) = e^{-R_{l,\alpha}(x)} \in \widetilde{\mathcal{L}}_\nu$.

In the rest of this paper, we consider the classes $\mathcal{L}_2$ and $\widetilde{\mathcal{L}}_\nu$; let $w \in \mathcal{L}_2$ or $w \in \widetilde{\mathcal{L}}_\nu$ $(\nu \geqslant 2)$. For $\rho > -1/2$, we set $w_\rho(x) := x^\rho w(x)$. Then, we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_0^\infty p_{n,\rho}(u) p_{m,\rho}(u) w_\rho^2(u) du = \delta_{nm} \quad (\text{Kronecker's delta})\ n, m = 0, 1, 2, \ldots. \tag{1.16}$$

Let us denote the zeros of $p_{n,\rho}(x)$ by

$$0 < x_{n,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \tag{1.17}$$

The Mhaskar-Rahmanov-Saff numbers a_v are defined as follows:

$$v = \frac{1}{\pi} \int_0^1 a_v t R'(a_v t) \{t(1-t)\}^{-1/2} dt, \quad v > 0. \tag{1.18}$$

In this paper, we will consider the orthonormal polynomials $p_{n,\rho}(x)$ with respect to the weight class $\widetilde{\mathcal{L}}_\nu$. Our main themes in this paper are to estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ and to investigate the various weighted L_p-norms $(0 < p \leqslant \infty)$ of $p_{n,\rho}(x)$. More precisely, we will estimate the higher-order derivatives of $p_{n,\rho}(x)$ at all zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. In addition, we will give asymptotic relation of the odd order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ in a certain finite interval. These estimations will play an important role in investigating convergence or divergence of higher-order Hermite-Fejér interpolation polynomials (see [3, 5–17]).

Then, our main purpose is to obtain estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n,\rho})$, $k = 1,2,\ldots,n$, $j = 1,2,\ldots,\nu$ as follows.

Theorem 1.5. *Let $w(x) = \exp(-R(x)) \in \mathcal{L}(C^2+)$ and $\rho > -1/2$. For each $k = 1,2,\ldots,n$ and $j = 0,1,\ldots,\nu-1$ one has*

$$\left|p_{n,\rho}^{(j)}(x_{k,n,\rho})\right| \leqslant C\left(\frac{n}{\sqrt{a_{2n}-x_{k,n,\rho}}}\right)^{j-1} x_{k,n,\rho}^{-(j-1)/2}\left|p_{n,\rho}'(x_{k,n,\rho})\right|. \tag{1.19}$$

Theorem 1.6. *Let $w(x) = \exp(-R(x)) \in \widetilde{\mathcal{L}}_\nu$ and $\rho > -1/2$. Assume that $1 + 2\rho - \delta/2 \geqslant 0$ for $\rho < -1/4$, and if $T(x)$ is bounded, then assume that*

$$a_n \leqslant Cn^{2/(1+\nu-\delta)}, \tag{1.20}$$

where $0 \leqslant \delta < 1$ is defined in (1.13). *For each $k = 1,2,\ldots,n$ and $j = 0,1,\ldots,\nu-1$, one has*

$$\left|p_{n,\rho}^{(j)}(x_{k,n,\rho})\right| \leqslant C\left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n,\rho}}}+\frac{T(a_n)}{\sqrt{a_n}}\right)^{j-1} x_{k,n,\rho}^{-(j-1)/2}\left|p_{n,\rho}'(x_{k,n,\rho})\right|, \tag{1.21}$$

and in particular if j is even, then

$$\begin{aligned}\left|p_{n,\rho}^{(j)}(x_{k,n,\rho})\right| \leqslant{}& C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n,\rho}}}+R'(x_{k,n,\rho})+\frac{1}{x_{k,n,\rho}}\right)\\ &\times\left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n,\rho}}}+\frac{T(a_n)}{\sqrt{a_n}}\right)^{j-2} x_{k,n,\rho}^{-(j-2)/2}\left|p_{n,\rho}'(x_{k,n,\rho})\right|.\end{aligned} \tag{1.22}$$

Theorem 1.7. *Let $w(x) = \exp(-R(x)) \in \widetilde{\mathcal{L}}_\nu$ and $\rho > -1/2$. Let $(1/\varepsilon)(a_n/n^2) \leqslant x_{k,n,\rho} \leqslant \varepsilon a_n$, $0 < \varepsilon < 1/4$, and $\nu = 2,3,\ldots$, $s = 0,1,\ldots,(\nu-1)/2$. Then, under the same conditions as the assumptions of Theorem 1.6, there exist $\beta(n,k)$, $0 < D_1 \leqslant \beta(n,k) \leqslant D_2$ for absolute constants D_1, D_2 such that the following equality holds:*

$$p_{n,\rho}^{(2s+1)}(x_{k,n,\rho}) = (-1)^s\beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}(1+\rho_s(\varepsilon,x_{k,n,\rho},n))p_n'(x_{k,n,\rho})x_{k,n,\rho}^{-s}, \tag{1.23}$$

and $|\rho_s(\varepsilon,x_{k,n,\rho},n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

Define

$$\Phi_n(x) = \max\left\{\eta_n, 1 - \left(\frac{x}{a_n}\right)^{1/2}\right\}, \quad \eta_n = \{nT(a_n)\}^{-2/3},$$
$$z^+ = \begin{cases} z, & z > 0, \\ 0, & z \leqslant 0. \end{cases} \tag{1.24}$$

Let us define

$$\Theta_n(x) = \frac{x/a_n}{1 + x/a_n}. \tag{1.25}$$

We consider the class of weights, $\mathcal{F}(C^2)$, which is defined in Definition 2.1 below. Levin and Lubinsky have obtained the following theorem.

Theorem B (see [18, Theorem 13.6]). *Assume that $W \in \mathcal{F}(C^2)$. Let $0 < p < \infty$. Then uniformly for $n \geqslant 1$,*

$$\|P_n W\|_{L_p(I)} \sim \begin{cases} a_n^{1/p-1/2}, & p < 4, \\ a_n^{-1/4}(\log(n+1))^{1/4}, & p = 4, \\ a_n^{1/p-1/2}(nT(a_n))^{(2/3)(1/4-1/p)}, & p > 4. \end{cases} \tag{1.26}$$

We remark that Levin and Lubinsky have shown Theorem B for more wider class $\mathcal{F}(\mathrm{lip}(1/2)) \supseteq \mathcal{F}(C^2)$. In the following, we investigate the various weighted L_p-norms ($0 < p \leqslant \infty$) of $p_{n,\rho}(x)$.

Theorem 1.8. *Let $w \in \mathcal{L}_2$. Let $0 < p < \infty$ and $\rho > -1/2$. Then one has for $n \geqslant 1$,*

$$\left\|\Theta_n^{1/4}(x) p_{n,\rho}(x) w(x)\left(x + \frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log(1 + nT(a_n)))^{1/4}, & p = 4, \\ (nT(a_n))^{(2/3)(1/4-1/p)}, & p > 4. \end{cases} \tag{1.27}$$

Theorem 1.9. *Let $w \in \mathcal{L}_2$. Let $0 < p < \infty$ and $\rho > -1/2$. Then one has for $n \geqslant 1$,*

$$\left\|p_{n,\rho}(x) w(x)\left(x + \frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \sim a_n^{1/p-1/2} \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ n^{2(1/4-1/p)}, & p > 4. \end{cases} \tag{1.28}$$

Theorem 1.10. *Let $w \in \mathcal{L}_2$, $0 \leqslant s \leqslant r$, and $n \geqslant 1$. Suppose that $\rho > -1/2$. For $0 \leqslant s \leqslant r$ and $n \geqslant 1$, one has*

$$\left\| \Theta_n^{r/4}(x)\Phi_n^{(r/4-1/p)^+}(x) \left| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n}\right)^{\rho}\right|^{s} \right\|_{L_p(\mathbb{R}^+)} \sim \begin{cases} a_n^{1/p-r/2}\log n, & \text{if } s=r,\ pr \geqslant 4, \\ a_n^{1/p-s/2}, & \text{otherwise}, \end{cases} \tag{1.29}$$

and for $p=\infty$

$$\left\| \Theta_n^{r/4}(x)\Phi_n^{(r/4-1/p)^+}(x) \left| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n}\right)^{\rho}\right|^{s} \right\|_{L_p(\mathbb{R}^+)} \sim a_n^{-s/4}. \tag{1.30}$$

This paper is organized as follows. In Section 2, we will introduce the weight class $\widetilde{\mathcal{F}}_\nu$ as an analogy of the class $\widetilde{\mathcal{L}}_\nu$ and the known results of orthonormal polynomials with respect to $\widetilde{\mathcal{F}}_\nu$ in order to prove the main results. In Section 3, we will prove Theorems 1.5, 1.6, and 1.7. Finally, we will prove the results for the various weighted L_p-norms ($0 < p \leqslant \infty$) of $p_{n,\rho}(x)$, that is, Theorems 1.8, 1.9, and 1.10, in Section 4.

2. Preliminaries

Levin and Lubinsky introduced the classes $\mathcal{L}(C^2)$ and $\mathcal{L}(C^2+)$ as an analogy of the classes $\mathcal{F}(C^2)$ and $\mathcal{F}(C^2+)$ which they already defined on $I^* = (-\sqrt{d}, \sqrt{d})$. They defined the following.

Definition 2.1 (see [18]). We assume that $Q : I^* \to [0,\infty)$ has the following properties:

(a) $Q(t)$ is continuous in I^*, with $Q(0)=0$,

(b) $Q''(t)$ exists and is positive in $I^* \setminus \{0\}$,

(c)

$$\lim_{t \to \sqrt{d}-} Q(t) = \infty, \tag{2.1}$$

(d) the function

$$T^*(t) := \frac{tQ'(t)}{Q(t)} \tag{2.2}$$

is quasi-increasing in $(0, \sqrt{d})$, with

$$T^*(t) \geqslant \Lambda^* > 1, \quad t \in I^* \setminus \{0\}, \tag{2.3}$$

(e) there exists $C_1 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \leqslant C_1 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus \{0\}. \tag{2.4}$$

Then, we write $W \in \mathcal{F}(C^2)$. If there also exists a compact subinterval $J^* \ni 0$ of I^*, and $C_2 > 0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \geqslant C_2 \frac{|Q'(t)|}{Q(t)}, \quad \text{a.e. } t \in I^* \setminus J^*, \tag{2.5}$$

then, we write $W \in \mathcal{F}(C^2+)$.

Then we see that $w \in \mathcal{L}(C^2) \Leftrightarrow \mathcal{F}(C^2)$ and $w \in \mathcal{L}(C^2+) \Leftrightarrow \mathcal{F}(C^2+)$ from [1, Lemma 2.2]. In addition, we easily have the following.

Lemma 2.2. *Let $Q(t) = R(t^2), x = t^2$. Then one has*

$$w \in \mathcal{L}_2 \Longrightarrow W \in \mathcal{F}\left(C^2+\right), \tag{2.6}$$

where $W(t) = w(x), x = t^2$.

On $\mathbb{R}$, we can consider the corresponding class to $\widetilde{\mathcal{L}}_\nu$ as follows.

Definition 2.3 (cf. [19]). Let $W = \exp(-Q) \in \mathcal{F}(C^2+)$ and $\nu \geqslant 2$ be an integer. Let Q be a continuous and even function on $\mathbb{R}$. For the exponent Q, we assume the following:

(a) $Q^{(j)}(t) > 0$, for $0 \leqslant j \leqslant \nu$ and $t \in \mathbb{R}^+ \setminus \{0\}$,

(b) there exist positive constants $C_i > 0$ such that for $i = 1, 2, \ldots, \nu - 1$

$$Q^{(i+1)}(t) \leqslant C_i Q^{(i)}(t) \frac{Q'(t)}{Q(t)}, \quad \text{a.e. } x \in \mathbb{R}^+ \setminus \{0\}, \tag{2.7}$$

(c) there exist positive constants $C, c_1 > 0$ and $0 \leqslant \delta^* < 1$ such that for $t \in (0, c_1)$

$$Q^{(\nu)}(t) \leqslant C\left(\frac{1}{t}\right)^{\delta^*}, \tag{2.8}$$

(d) there exists $c_2 > 0$ such that one has one among the following:

(d1) $T^*(t)/t$ is quasi-increasing on (c_2, ∞),

(d2) $Q^{(\nu)}(t)$ is nondecreasing on (c_2, ∞).

Then, we write $W(t) = e^{-Q(t)} \in \widetilde{\mathcal{F}}_\nu$.

Let $W \in \widetilde{\mathcal{F}}_\nu$ and $\nu \geqslant 2$. For $\rho^* > -1/2$, we set

$$W_{\rho^*}(t) := |t|^{\rho^*} W(t). \tag{2.9}$$

Then, we can construct the orthonormal polynomials $P_{n,\rho^*}(t) = P_n(W_{\rho^*}^2; t)$ of degree n with respect to $W_{\rho^*}(t)$. That is,

$$\int_{-\infty}^{\infty} P_{n,\rho^*}(v) P_{m,\rho^*}(v) W_{\rho^*}^2(v) dt = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots. \tag{2.10}$$

Let us denote the zeros of $P_{n,\rho^*}(t)$ by

$$-\infty < t_{nn} < \cdots < t_{2n} < t_{1n} < \infty. \tag{2.11}$$

Jung and Sakai [5, Theorems 3.3 and 3.6] estimate $P_{n,\rho^*}^{(j)}(t_{k,n})$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \nu$, and we will obtain analogous estimations with respect to $p_{n,\rho}^{(j)}(x_{k,n})$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \nu$ in Theorems 1.6 and 1.7.

There are many properties of $P_{n,\rho^*}(t) = P_n(W_{\rho^*}; t)$ with respect to $W_{\rho^*}(t)$, $W \in \widetilde{\mathcal{F}}_\nu$, $\nu = 2, 3, \ldots$ of Definition 2.3 in [4–6, 19–21]. They were obtained by transformations from the results in [1, 2]. In this paper, we consider $w = \exp(-R) \in \widetilde{\mathcal{L}}_\nu$ and $p_{n,\rho}(x) = p_n(w_\rho; x)$. In [5] we got the estimations of $P_{n,\rho^*}^{(j)}(t_{kn})$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \nu - 1$ with the weight $W_{\rho^*}(t) \in \widetilde{\mathcal{F}}_\nu$. By a transformation of the results with respect to $P_{n,\rho^*}(t)$, we estimate $p_{n,\rho}^{(j)}(x_{kn})$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \nu - 1$. In order to it we will give the transformation theorems in this section. In the following, we will give some applications of them.

Theorem 2.4 (see [21, Theorem 2.1]). *Let $W(t) = w(x)$ with $x = t^2$. Then, the orthonormal polynomials $P_{n,\rho^*}(t)$ on $\mathbb{R}$ can be entirely reduced to the orthonormal polynomials $p_{n,\rho}(x)$ in $\mathbb{R}^+$ as follows: for $n = 0, 1, 2, \ldots$,*

$$P_{2n,2\rho+(1/2)}(t) = p_{n,\rho}(x), \qquad P_{2n+1,2\rho-(1/2)}(t) = t p_{n,\rho}(x). \tag{2.12}$$

In this paper, we will use the fact that $w_\rho(x) = x^\rho \exp(-R(x))$ is transformed into $W_{2\rho+1/2}(t) = |t|^{2\rho+1/2} \exp(-Q(t))$ as meaning that

$$\begin{aligned}\int_0^\infty p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx &= 2\int_0^\infty p_{n,\rho}\left(t^2\right) p_{m,\rho}\left(t^2\right) t^{4\rho+1} W^2(t) dt \\ &= \int_{-\infty}^{\infty} P_{2n,2\rho+1/2}(t) P_{2m,2\rho+1/2}(t)\ W_{2\rho+1/2}^2(t) dt.\end{aligned} \tag{2.13}$$

Theorem 2.5. *Let $Q(t) = R(x)$, $x = t^2$. Then one has*

$$w(x) = \exp(-R(x)) \in \widetilde{\mathcal{L}}_\nu \Longrightarrow W(t) = \exp(-Q(t)) \in \widetilde{\mathcal{F}}_\nu. \tag{2.14}$$

In particular, one has

$$Q^{(\nu)}(t) \leqslant C\left(\frac{1}{t}\right)^{\delta}, \tag{2.15}$$

where $0 \leqslant \delta < 1$ *is defined in* (1.13).

Proof. Let $w \in \widetilde{\mathcal{L}}_2$. Then, from Lemma 2.2, one has $W \in \mathcal{F}(C^2+)$. Let $[x]$ denote the maximum integer as $[x] \leqslant x$ (Gaussian symbol). For $1 \leqslant j \leqslant \nu$, one has

$$Q^{(j)}(t) = \sum_{i=0}^{[j/2]} c_{ij} R^{(j-i)}(x) t^{j-2i}, \quad c_{ij} > 0 \ \left(i = 0, 1, \ldots, [j/2]\right), \ x = t^2. \tag{2.16}$$

Therefore, we easily see that (a) of Definition 2.3 holds. Let $x = t^2$. Since $R^{(\ell)}(x)$ is increasing for $x > 0$ and $\ell = 0, 1, \ldots, \nu - 1$, there exists ξ with $0 < \xi < x$ such that for $k = 0, 1, \ldots, \nu - 2$,

$$\frac{R^{(k)}(x)}{x} = R^{(k+1)}(\xi) \leqslant C R^{(k+1)}(x). \tag{2.17}$$

Then, since for $0 \leqslant k < j \leqslant \nu - 1$,

$$R^{(k)}(x) \leqslant C x^{j-k} R^{(j)}(x), \tag{2.18}$$

one has by (b) of Definition 1.3 that

$$\begin{aligned} Q^{(j)}(t) &= \sum_{i=0}^{[j/2]} c_{ij} R^{(j-i)}(x) t^{j-2i} \leqslant C R^{(j)}(x) t^{j} \leqslant C R^{(j-1)}(x) t^{j-1} \left(\frac{t R'(x)}{R(x)}\right) \\ &\leqslant C Q^{(j-1)}(t) \frac{Q'(t)}{Q(t)}, \qquad 1 \leqslant j \leqslant \nu - 1. \end{aligned} \tag{2.19}$$

Similarly, one has by (2.16), (d) of Definition 1.2, and (b) of Definition 1.3 that

$$\begin{aligned} Q^{(\nu)}(t) &= c_{0,\nu} R^{(\nu)}(x) t^{\nu} + \sum_{i=1}^{[\nu/2]} c_{i,\nu} R^{(\nu-i)}(x) t^{\nu-2i} \\ &\leqslant c_{0,\nu} R^{(\nu)}(x) t^{\nu} + C R^{(\nu-1)}(x) t^{\nu-2} \\ &\leqslant C R^{(\nu-1)}(x) t^{\nu-1} \left(\frac{t R'(x)}{R(x)}\right) \\ &\leqslant C Q^{(\nu-1)}(t) \frac{Q'(t)}{Q(t)}. \end{aligned} \tag{2.20}$$

Consequently, one has (b) in Definition 2.3. We know that

$$\sum_{i=1}^{[\nu/2]} c_{i,\nu} R^{(\nu-i)}(x) t^{\nu-2i} \leqslant C, \quad t \in (0, c_1), \tag{2.21}$$

and since $t^{\nu-\delta} \leqslant C$ on $t \in (0, c_1)$, one has from (1.13) that

$$R^{(\nu)}(x) t^{\nu} \leqslant C\left(\frac{1}{t^2}\right)^{\delta} t^{\nu} \leqslant C\left(\frac{1}{t^2}\right)^{\delta} t^{\delta} \leqslant C\left(\frac{1}{t}\right)^{\delta}. \tag{2.22}$$

Therefore, one has by (2.16)

$$Q^{(\nu)}(t) \leqslant C\left(\frac{1}{t}\right)^{\delta}, \tag{2.23}$$

where $0 \leqslant \delta < 1$ is defined in (1.13). The inequalities (d1) and (d2) of Definition 2.3 follow easily from (d1) and (d2) of Definition 1.3. Therefore, one has (2.14). □

3. Proofs of Theorems 1.5, 1.6, and 1.7

For convenience, in the rest of this paper, we put as follows:

$$\rho > -\frac{1}{2}, \qquad \rho^* := 2\rho + \frac{1}{2}, \qquad p_n(x) := p_{n,\rho}(x), \qquad P_n(t) := P_{n,\rho^*}(t), \tag{3.1}$$

and $x_{kn} = x_{k,n,\rho}, t_{kn} = t_{k,n,\rho^*}$. Then, we know that $\rho^* > -1/2$ and

$$p_n(x) = P_{2n}(t), \qquad x = t^2, \qquad x_{kn} = t_{k,2n}^2, \quad t_{k,2n} > 0, \;\; k = 1, 2, \ldots, n. \tag{3.2}$$

In the following, we introduce some useful notations.

(a) The Mhaskar-Rahmanov-Saff numbers a_v and a_u^* are defined as the positive roots of the following equations:

$$\begin{aligned} v &= \frac{1}{\pi} \int_0^1 a_v t R'(a_v t) \{t(1-t)\}^{-1/2} dt, \quad v > 0, \\ u &= \frac{2}{\pi} \int_0^1 a_u^* t Q'(a_u^* t) \left(1 - t^2\right)^{-1/2} dt, \quad u > 0. \end{aligned} \tag{3.3}$$

(b) Let

$$\eta_n = \{n T(a_n)\}^{-2/3}, \qquad \eta_n^* = \{n T^*(a_n^*)\}^{-2/3}. \tag{3.4}$$

Then, one has the following.

Lemma 3.1 (see [1, (2.5), (2.7), (2.9)]).

$$a_n = {a_{2n}^*}^2, \qquad \eta_n = 4^{2/3}\eta_{2n}^*, \qquad T(a_n) = \frac{1}{2}T^*(a_{2n}^*). \tag{3.5}$$

To prove Theorem 1.6, we need some lemmas as follows.

Lemma 3.2 (see [21, Theorem 2.2, Lemma 3.7]). *For the minimum positive zero $t_{[n/2],n}$ ($[n/2]$ is the largest integer $n/2$), one has*

$$t_{[n/2],n} \sim a_n^* n^{-1}, \tag{3.6}$$

and for the maximum zero x_{1n}, one has for large enough n,

$$1 - \frac{t_{1n}}{a_n^*} \sim \eta_n^*, \qquad \eta_n^* = (nT^*(a_n^*))^{-2/3}. \tag{3.7}$$

Moreover, for some constant $0 < \varepsilon \leqslant 2$, one has

$$T^*(a_n^*) \leqslant Cn^{2-\varepsilon}. \tag{3.8}$$

Lemma 3.3 (see [6, Theorem 2.5]). *Let $W \in \mathcal{F}(C^2+)$ and $r = 1, 2, \ldots$. Then, uniformly for $1 \leqslant k \leqslant n$,*

$$\left|\frac{P_{n,\rho}^{(r)}(t_{k,\rho,n})}{P_{n,\rho}'(t_{k,\rho,n})}\right| \leqslant C\left(\frac{n}{\sqrt{{a_{2n}^*}^2 - t_{k,\rho,n}^2}}\right)^{r-1}. \tag{3.9}$$

Lemma 3.4 (see [5, Theorem 3.6 and Lemma 3.7 (3.20)]). *Let $\rho^* > -1/2$ and $W(x) = \exp(-Q(x)) \in \tilde{\mathcal{F}}_\nu$, $\nu \geqslant 2$. Assume that $1 + 2\rho^* - \delta^* \geqslant 0$ for $\rho^* < 0$ and if $T^*(t)$ is bounded, then assume*

$$a_n^* \leqslant Cn^{1/(1+\nu-\delta^*)}, \tag{3.10}$$

where $0 \leqslant \delta^* < 1$ is defined in (2.8). *If $t_{kn} \neq 0$, then one has for $j = 1, 2, \ldots, \nu$*

$$\left|P_n^{(j)}(t_{kn})\right| \leqslant C\left(\frac{n}{a_{2n}^* - |t_{kn}|} + \frac{T^*(a_n^*)}{a_n^*}\right)^{j-1}\left|P_n'(t_{kn})\right|, \tag{3.11}$$

and in particular, if j is even, then

$$\left|P_n^{(j)}(t_{kn})\right| \leqslant C\left(\frac{T^*(a_n^*)}{a_n^*} + \left|Q'(t_{kn})\right| + \frac{1}{|t_{kn}|}\right)\left(\frac{n}{a_{2n}^* - |t_{kn}|} + \frac{T^*(a_n^*)}{a_n^*}\right)^{j-2}\left|P_n'(t_{kn})\right|. \tag{3.12}$$

Remark 3.5. Let $W(t) \in \mathcal{F}(C^2+)$. Then, from [19, Theorem 1.6] we know that when $T^*(t)$ is unbounded, for any $\eta > 0$, there exists $C(\eta) > 0$ such that for $t \geq 1$,

$$a_t^* \leqslant C(\eta) t^{\eta}. \tag{3.13}$$

In addition, since $T(x) = T^*(t)/2$, we know that

(i) $T(x)$ is bounded $\Leftrightarrow T^*(t)$ is bounded,

(ii) $T(x)$ is unbounded $\Rightarrow a_n \leqslant Cn^{\eta}$ for any $\eta > 0$,

(iii) $T(a_n) \leqslant Cn^{2-\varepsilon}$ for some constant $0 < \varepsilon \leqslant 2$.

Lemma 3.6. *For $j = 1, 2, 3, \ldots$, one has*

$$p_n^{(j)}(x) = \sum_{i=1}^{j} (-1)^{j-i} c_{j,i} P_{2n}^{(i)}(t) t^{-2j+i}, \tag{3.14}$$

where $c_{j,i} > 0$ satisfy that for $k = 1, 2, \ldots$,

$$c_{k+1,1} = \frac{2k-1}{2} c_{k,1}, \qquad c_{k+1,k+1} = \frac{1}{2^{k+1}}, \qquad c_{1,1} = \frac{1}{2}, \tag{3.15}$$

and for $2 \leqslant i \leqslant k$

$$c_{k+1,i} = \frac{c_{k,i-1} + (2k-i) c_{k,i}}{2}. \tag{3.16}$$

Proof. It is easily proved, using the mathematical induction on j. □

Proof of Theorem 1.5. By Lemmas 3.3, 3.6 and (3.2), one has

$$\begin{aligned} \left| p_n^{(j)}(x_{kn}) \right| &\leqslant C \sum_{i=1}^{j} \left| P_{2n}^{(i)}(t_{k,2n}) \right| \left| t_{k,2n}^{-2j+i} \right| \\ &\leqslant C \sum_{i=1}^{j} \left(\frac{2n}{\sqrt{{a_{4n}^*}^2 - t_{k,2n}^2}} \right)^{i-1} \left| P_{2n}'(t_{k,2n}) \right| \left| t_{k,2n}^{-2j+i} \right|. \end{aligned} \tag{3.17}$$

Since by Lemma 3.2

$$\begin{aligned}
\sum_{i=1}^{j}\left(\frac{2n}{\sqrt{{a_{4n}^{*}}^{2}-t_{k,2n}^{2}}}\right)^{i-1}\left|t_{k,2n}^{i-1}\right| &\leqslant C\left\{1+\left(\frac{2n}{\sqrt{{a_{4n}^{*}}^{2}-t_{k,2n}^{2}}}\right)^{j-1}\left|t_{k,2n}^{j-1}\right|\right\} \\
&\leqslant C\left\{\left|t_{k,2n}^{-j+1}\right|+\left(\frac{2n}{\sqrt{{a_{4n}^{*}}^{2}-t_{k,2n}^{2}}}\right)^{j-1}\right\}\left|t_{k,2n}^{j-1}\right| \qquad (3.18)\\
&\leqslant C\left(\frac{n}{\sqrt{{a_{4n}^{*}}^{2}-t_{k,2n}^{2}}}\right)^{j-1}\left|t_{k,2n}^{j-1}\right|,
\end{aligned}$$

one has from Lemma 3.1 that

$$\begin{aligned}
\left|p_{n}^{(j)}(x_{kn})\right| &\leqslant C\left(\frac{n}{\sqrt{{a_{4n}^{*}}^{2}-t_{k,2n}^{2}}}\right)^{j-1}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j}\right| \\
&\leqslant C\left(\frac{n}{\sqrt{a_{2n}-x_{k,n}}}\right)^{j-1}x_{k,n}^{-(j-1)/2}\left|p_{n}'(x_{k,n})\right|.
\end{aligned} \qquad (3.19)$$

□

Proof of Theorem 1.6. Since $w(x)\in\widetilde{\mathcal{L}}_{\nu}$, we know that $W(t)\in\widetilde{\mathcal{F}}_{\nu}$ and we know that $\delta^{*}=\delta$ by Theorem 2.5 and from (3.1), (3.2), and Lemma 3.1 that

(i) $\rho>-1/2\Rightarrow\rho^{*}>-1/2$,

(ii) $1+2\rho-\delta/2\geqslant 0$ for $\rho<-1/4\Rightarrow 1+2\rho^{*}-\delta^{*}\geqslant 0$ for $\rho^{*}<0$,

(iii) $a_{n}\leqslant Cn^{2/(1+\nu-\delta)}\Rightarrow a_{n}^{*}\leqslant Cn^{1/(1+\nu-\delta^{*})}$.

Then, using Remark 3.5, we can apply Lemma 3.4 to $p_{n}(x)=P_{2n,\rho^{*}}(t)$, $x=t^{2}$. In a similar way to the proof of Theorem 1.5, one has from Lemma 3.4 and Lemma 3.1

$$\begin{aligned}
\left|p_{n}^{(j)}(x_{kn})\right| &\leqslant C\sum_{i=1}^{j}\left|P_{2n}^{(i)}(t_{k,2n})\right|\left|t_{k,2n}^{-2j+i}\right| \\
&\leqslant C\sum_{i=1}^{j}\left(\frac{n}{a_{4n}^{*}-|t_{k,2n}|}+\frac{T^{*}(a_{2n}^{*})}{a_{2n}^{*}}\right)^{i-1}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-2j+i}\right| \\
&\leqslant C\left(\frac{n}{a_{4n}^{*}-|t_{k,2n}|}+\frac{T^{*}(a_{2n}^{*})}{a_{2n}^{*}}\right)^{j-1}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j}\right| \\
&\leqslant C\left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}}+\frac{T(a_{n})}{\sqrt{a_{n}}}\right)^{j-1}x_{k,n}^{-(j-1)/2}\left|p_{n}'(x_{k,n})\right|.
\end{aligned} \qquad (3.20)$$

Let j be even. Then, one has from Lemma 3.4 that

$$\begin{aligned}\sum_{i:\text{even}}\left|P_{2n}^{(i)}(t_{k,2n})\right|\left|t_{k,2n}^{-2j+i}\right| &\leqslant C\left(\frac{T^*(a_{2n}^*)}{a_{2n}^*}+\left|Q'(t_{k,2n})\right|+\frac{1}{|t_{k,2n}|}\right)\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-2j+2}\right| \\ &\quad\times\sum_{i:\text{even}}\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{i-2}\left|t_{k,2n}^{i-2}\right|.\end{aligned}\tag{3.21}$$

Since by Lemma 3.2 and

$$\sum_{i:\text{even}}\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{i-2}\left|t_{k,2n}^{i-2}\right| \leqslant C\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{j-2}\left|t_{k,2n}^{j-2}\right|,\tag{3.22}$$

one has

$$\begin{aligned}\sum_{i:\text{even}}\left|P_{2n}^{(i)}(t_{k,2n})\right|\left|t_{k,2n}^{-2j+i}\right| &\leqslant C\left(\frac{T^*(a_{2n}^*)}{a_{2n}^*}+\left|Q'(t_{k,2n})\right|+\frac{1}{|t_{k,2n}|}\right)\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j}\right| \\ &\quad\times\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{j-2}, \\ \sum_{i:\text{odd}}^{j-1}\left|P_{2n}^{(i)}(t_{k,2n})\right|\left|t_{k,2n}^{-2j+i}\right| &\leqslant C\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{j-2}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j-1}\right| \\ &\leqslant C\left(\frac{T^*(a_{2n}^*)}{a_{2n}^*}+\left|Q'(t_{k,2n})\right|+\frac{1}{|t_{k,2n}|}\right) \\ &\quad\times\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{j-2}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j}\right|.\end{aligned}\tag{3.23}$$

Therefore, when j is even, one has by Lemma 3.1 that

$$\begin{aligned}\left|p_n^{(j)}(x_{kn})\right| &\leqslant C\left(\frac{T^*(a_{2n}^*)}{a_{2n}^*|t_{k,2n}|}+\frac{|Q'(t_{k,2n})|}{|t_{k,2n}|}+\frac{1}{t_{k,2n}^2}\right) \\ &\quad\times\left(\frac{n}{a_{4n}^*-|t_{k,2n}|}+\frac{T^*(a_{2n}^*)}{a_{2n}^*}\right)^{j-2}\left|P_{2n}'(t_{k,2n})\right|\left|t_{k,2n}^{-j+1}\right| \\ &\leqslant C\left(\frac{T(a_n)}{\sqrt{a_n x_{k,n}}}+R'(x_{k,n})+\frac{1}{x_{k,n}}\right) \\ &\quad\times\left(\frac{n}{\sqrt{a_{2n}}-\sqrt{x_{k,n}}}+\frac{T(a_n)}{\sqrt{a_n}}\right)^{j-2}x_{k,n}^{-(j-2)/2}\left|p_n'(x_{k,n})\right|.\end{aligned}\tag{3.24}$$

□

Next, we will prove Theorem 1.7. To prove it, we need two lemmas as follows.

Lemma 3.7 ([5, Theorem 3.3]). *Let $W(x) = \exp(-Q(x)) \in \widetilde{\mathcal{F}}_\nu$, $\nu \geqslant 2$. Let $(1/\varepsilon)(a_n^*/n) \leqslant |t_{kn}| \leqslant \varepsilon a_n^*$, $0 < \varepsilon < 1/2$, and $s = 1, 2, \ldots, (\nu - 1)/2$. Then, under the same conditions as the assumptions of Lemma 3.4, there exist $\beta(n,k)$, $0 < D_1 \leqslant \beta(n,k) \leqslant D_2$ for absolute constants D_1, D_2 such that the following equality holds:*

$$P_n^{(2s+1)}(t_{kn}) = (-1)^s \beta_n^s(n,k) \left(\frac{n}{a_n^*}\right)^{2s} \left(1 + \widetilde{\rho}_{2s+1}(\varepsilon, t_{kn}, n)\right) P_n'(t_{kn}), \tag{3.25}$$

and $|\widetilde{\rho}_{2s+1}(\varepsilon, t_{kn}, n)| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

From Lemma 3.3, we easily have the following.

Lemma 3.8. *Let $W \in \mathcal{F}(C^2+)$ and $j = 1, 2, \ldots$. Then, uniformly for $|t_{kn}| \leqslant a_n^*/2$,*

$$\left|P_n^{(j)}(t_{kn})\right| \leqslant C \left(\frac{n}{a_n^*}\right)^{j-1} \left|P_n'(t_{kn})\right|. \tag{3.26}$$

Proof of Theorem 1.7. By Lemmas 3.4, 3.6 and Theorem 2.4, one has

$$\begin{aligned} p_n^{(2s+1)}(x_{kn}) &= \sum_{i=1}^{2s+1} (-1)^{2s+1-i} c_{2s+1,i} P_{2n}^{(i)}(t_{k,2n}) t_{k,2n}^{-2(2s+1)+i} \\ &= \sum_{p=0}^{s} c_{2s+1,2p+1} P_{2n}^{(2p+1)}(t_{k,2n}) t_{k,2n}^{-4s+2p-1} - \sum_{p=1}^{s} c_{2s+1,2p} P_{2n}^{(2p)}(t_{k,2n}) t_{k,2n}^{-4s+2p-2} \\ &=: \sum_{\text{odd}} - \sum_{\text{even}}. \end{aligned} \tag{3.27}$$

Since we know that

$$\frac{1}{\varepsilon}\frac{a_n}{n^2} \leqslant x_{k,n} \leqslant \varepsilon a_n \Longrightarrow \frac{2}{\sqrt{\varepsilon}}\frac{a_{2n}^*}{2n} \leqslant |t_{k,2n}| \leqslant \sqrt{\varepsilon} a_{2n}^*, \quad 0 < \sqrt{\varepsilon} < \frac{1}{2}, \tag{3.28}$$

by the same reason as the proof of Theorem 1.6, we can apply Lemma 3.7 to $P_{2n}^{(2p+1)}(t_{k,2n})$.

Then, using Lemmas 3.7 and 3.6, one has

$$
\begin{aligned}
\sum_{\text{odd}} &= \sum_{p=0}^{s} c_{2s+1,2p+1}(-1)^p \beta_{2n}^p(2n,k)\left(\frac{2n}{a_{2n}^*}\right)^{2p}\left(1+\tilde{\rho}_{2p+1}\right)P_{2n}'(t_{k,2n})t_{k,2n}^{-4s+2p-1} \\
&= (-1)^s \beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}\left(\frac{1}{2}+\frac{\tilde{\rho}_{2s+1}}{2}\right)P_{2n}'(t_{k,2n})t_{k,2n}^{-2s-1} \\
&\quad + \sum_{p=0}^{s-1} c_{2s+1,2p+1}(-1)^p \beta_{2n}^p(2n,k)\left(\frac{2n}{a_{2n}^*}\right)^{2p}\left(1+\tilde{\rho}_{2p+1}\right)P_{2n}'(t_{k,2n})t_{k,2n}^{-4s+2p-1} \\
&= (-1)^s \beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}\left(\frac{1}{2}+\frac{\tilde{\rho}_{2s+1}}{2}\right)P_{2n}'(t_{k,2n})t_{k,2n}^{-2s-1} \\
&\quad + (-1)^s \beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}P_{2n}'(t_{k,2n})t_{k,2n}^{-2s-1} \\
&\quad \times \sum_{p=0}^{s-1} c_{2s+1,2p+1}(-1)^{p-s}2^{2s}\beta_{2n}^{p-s}(2n,k)\left(\frac{2n}{a_{2n}^*}\right)^{2(p-s)}\left(1+\tilde{\rho}_{2p+1}\right)t_{k,2n}^{2(p-s)}.
\end{aligned}
\tag{3.29}
$$

Here, $\tilde{\rho}_{2p+1} := \tilde{\rho}_{2p+1}(\varepsilon, t_{k,2n}, 2n)$, $p = 0, 1, \ldots, s$. Since from (3.28) we see that for $0 \leqslant p \leqslant s-1$,

$$
\left(\frac{a_{2n}^*}{2n}\frac{1}{t_{k,2n}}\right)^{2(s-p)} \leqslant \left(\frac{\sqrt{\varepsilon}}{2}\right)^{2(s-p)} \leqslant \left(\frac{1}{4}\right)^{s-p}\varepsilon,
\tag{3.30}
$$

one has that

$$
\left|\sum_{p=0}^{s-1} c_{2s+1,2p+1}(-1)^{p-s}2^{2s}\beta_{2n}^{p-s}(2n,k)\left(\frac{a_{2n}^*}{2n}\frac{1}{t_{k,2n}}\right)^{2(s-p)}\left(1+\tilde{\rho}_{2p+1}\right)\right| \longrightarrow 0,
\tag{3.31}
$$

as $n \to \infty$ and $\varepsilon \to 0$. If we let

$$
\xi_{n,1}'(s; x_{kn}) := \tilde{\rho}_{2s+1} + \sum_{p=0}^{s-1} c_{2s+1,2p+1}(-1)^{p-s}2^{2s+1}\beta_{2n}^{p-s}(2n,k)\left(\frac{2n}{a_{2n}^*}\right)^{2(p-s)}\left(1+\tilde{\rho}_{2p+1}\right)t_{k,2n}^{2(p-s)},
\tag{3.32}
$$

then one has

$$
\begin{aligned}
\sum_{\text{odd}} &= (-1)^s \beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}\left(\frac{1}{2}+\frac{\xi_{n,1}'(s; x_{k,n})}{2}\right)P_{2n}'(t_{k,2n})t_{k,2n}^{-2s-1} \\
&= (-1)^s \beta_{2n}^s(2n,k)\left(\frac{n}{\sqrt{a_n}}\right)^{2s}\left(1+\xi_{n,1}'(s; x_{k,n})\right)p_n'(x_{k,n})x_{k,n}^{-s},
\end{aligned}
\tag{3.33}
$$

and $|\xi'_{n,1}(s;x_{kn})| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$. On the other hand, we obtain

$$\begin{aligned}
\sum_{\text{even}} &= \sum_{p=1}^{s} c_{2s+1,2p} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} P'_{2n}(t_{k,2n}) t_{k,2n}^{-4s+2p-2} \\
&= \sum_{p=1}^{s} 2c_{2s+1,2p} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} p'_n(x_{k,n}) t_{k,2n}^{-4s+2p-1} \\
&= (-1)^s \beta_{2n}^{s}(2n,k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} p'_n(x_{k,n}) x_{k,n}^{-s} \\
&\quad \times \sum_{p=1}^{s} 2c_{2s+1,2p} (-1)^s \beta_{2n}^{-s}(2n,k) \left(\frac{n}{a_{2n}^*} \right)^{-2s} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} t_{k,2n}^{-2s+2p-1} \\
&:= \xi'_{n,2}(s;x_{k,n}) (-1)^s \beta_{2n}^{s}(2n,k) \left(\frac{n}{\sqrt{a_n}} \right)^{2s} p'_n(x_{k,n}) x_{k,n}^{-s}.
\end{aligned} \tag{3.34}$$

Here, one has from Lemma 3.8 and (3.28) that

$$\begin{aligned}
\left|\xi'_{n,2}(s;x_{k,n})\right| &= \left| \sum_{p=1}^{s} 2c_{2s+1,2p} (-1)^s \beta_{2n}^{-s}(2n,k) \left(\frac{n}{a_{2n}^*} \right)^{-2s} \frac{P_{2n}^{(2p)}(t_{k,2n})}{P'_{2n}(t_{k,2n})} t_{k,2n}^{-2s+2p-1} \right| \\
&\leqslant C \sum_{p=1}^{s} \left(\frac{a_{2n}^*}{n} \frac{1}{t_{k,2n}} \right)^{2s-2p+1} \leqslant C\sqrt{\varepsilon}.
\end{aligned} \tag{3.35}$$

Finally, if we let $\rho_s(\varepsilon, x_{kn}, n) := \xi'_{n,1}(s;x_{k,n}) - \xi'_{n,2}(s;x_{k,n})$, then the result is proved. □

4. Proofs of Theorems 1.8, 1.9, and 1.10

Lemma 4.1. *Let $W(t) \in \mathcal{F}(C^2)$, and let $0 < p < \infty$ and $\rho^* > -1/2$. Then, one has for $n \geqslant 1$ that*

$$\begin{aligned}
\left\| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right\|_{L_p(\mathbb{R})} &\sim \left\| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right\|_{L_p(a_n^*/2 \leqslant |t| \leqslant 2a_n^*)} \\
&\sim a_n^{*\,1/p-1/2} \begin{cases} 1, & p < 4, \\ \left\{ \log\left(1 + nT^*(a_n^*)\right) \right\}^{1/4}, & p = 4, \\ \left\{ nT^*(a_n^*) \right\}^{2/3(1/4-1/p)}, & p > 4. \end{cases}
\end{aligned} \tag{4.1}$$

Proof. In [21, theorem 2.6] we showed that

$$\left\| P_{n,\rho^*}(t)W(t)\left(|t|+\frac{a_n^*}{n}\right)^{\rho^*}\right\|_{L_p(\mathbb{R})} \sim a_n^{*\,1/p-1/2}\begin{cases} 1, & p<4, \\ \left\{\log\left(1+nT^*(a_n^*)\right)\right\}^{1/4}, & p=4, \\ \left\{nT^*(a_n^*)\right\}^{2/3(1/4-1/p)}, & p>4. \end{cases} \tag{4.2}$$

But, seeing our proof of [21, Theorem 2.6] carefully, we can easily prove the first equivalence. □

Lemma 4.2 (see [21, Theorem 2.4]). *Let $W(t)\in\mathcal{F}(C^2)$, $0<p\leqslant\infty$ and $L\geqslant 0$. Let $\beta\in\mathbb{R}$. Then, given $r>1$, there exists a positive constant C_2 such that one has for any polynomial $P\in\mathcal{P}_n$ that*

$$\left\|(PW_\beta)(t)\right\|_{L_p(a_{rn}^*\leqslant|t|)} \leqslant \exp(-C_2 n^\alpha)\left\|(PW_\beta)(t)\right\|_{L_p(L(a_n^*/n)\leqslant|t|\leqslant a_n^*(1-L\eta_n))}. \tag{4.3}$$

Proof of Theorem 1.8. From Theorem 2.4 and Lemmas 4.2 and 4.1, one has

$$\begin{aligned} &\left\|\Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \\ &\quad\leqslant C\left\|\left(\frac{t^2}{a_{2n}^{*2}}\right)^{1/4}P_{2n,\rho^*}(t)W(t)\left(t^2+\frac{a_{2n}^{*2}}{n^2}\right)^{\rho}|t|^{1/p}\right\|_{L_p(\mathbb{R})} \\ &\quad\leqslant C\left(\frac{1}{a_{2n}^*}\right)^{1/2}\left\|P_{2n,\rho^*}(t)W(t)\left(|t|+\frac{a_{2n}^*}{n}\right)^{\rho^*}|t|^{1/p}\right\|_{L_p(|t|\leqslant 2a_{2n}^*)} \\ &\quad\leqslant C{a_{2n}^*}^{1/p-1/2}\left\|P_{2n,\rho^*}(t)W(t)\left(|t|+\frac{a_{2n}^*}{n}\right)^{\rho^*}\right\|_{L_p(\mathbb{R})} \\ &\quad\sim {a_{2n}^*}^{2/p-1}\begin{cases} 1, & p<4, \\ \left\{\log\left(1+nT^*(a_{2n}^*)\right)\right\}^{1/4}, & p=4, \\ \left\{nT^*(a_{2n}^*)\right\}^{2/3(1/4-1/p)}, & p>4. \end{cases} \end{aligned} \tag{4.4}$$

On the other hand, one has by Theorem 2.4 and Lemma 4.1 that

$$\begin{aligned}
&\left\| \Theta_n^{1/4}(x) p_{n,\rho}(x) w(x) \left(x + \frac{a_n}{n^2} \right)^{\rho} \right\|_{L_p(\mathbb{R}^+)} \\
&\geq \left\| \Theta_n^{1/4}\left(t^2\right) P_{2n,\rho^*}(t) W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^{\rho} |t|^{1/p} \right\|_{L_p(a_{2^*n}/2 \leqslant |t| \leqslant 2a_{2n}^*)} \\
&\sim {a_{2n}^*}^{1/p-1/2} \left\| P_{2n,\rho^*}(t) W(t) \left(|t| + \frac{a_{2n}^*}{n} \right)^{\rho^*} \right\|_{L_p(a_{2^*n}/2 \leqslant |t| \leqslant 2a_{2n}^*)} \\
&\sim {a_{2n}^*}^{2/p-1} \begin{cases} 1, & p < 4, \\ \left\{ \log\left(1 + nT^*\left(a_{2n}^*\right)\right) \right\}^{1/4}, & p = 4, \\ \left\{ nT^*\left(a_{2n}^*\right) \right\}^{2/3(1/4-1/p)}, & p > 4. \end{cases}
\end{aligned} \tag{4.5}$$

Consequently, using Lemma 3.1, one has the result. □

Lemma 4.3. *Let $\rho > -1/2$, and let $w(x) \in \mathcal{L}(C^2+)$. Then, uniformly for $n \geq 1$ and $1 \leqslant j \leqslant n$, one has the following:*

(a)

$$\sup_{x \in I} \left| p_{n,\rho}(x) w(x) \right| \left(x + \frac{a_n}{n^2} \right)^{\rho} \left| \left(x + a_n n^{-2} \right) \left(a_n - x \right) \right|^{1/4} \sim 1, \tag{4.6}$$

(b) *for $j \leqslant n-1$ and $x \in [x_{j+1,n}, x_{jn}]$,*

$$\left| p_{n,\rho}(x) \right| w(x) \left(x + \frac{a_n}{n^2} \right)^{\rho} \sim \min\left\{ \left| x - x_{jn} \right|, \left| x - x_{j+1,n} \right| \right\} \varphi_n\left(x_{jn}\right)^{-1} \left[x_{jn}\left(a_n - x_{jn}\right) \right]^{-1/4}, \tag{4.7}$$

(c) *for $1 \leqslant j \leqslant n-1$,*

$$x_{jn} - x_{j+1,n} \sim \varphi_n\left(x_{jn}\right), \tag{4.8}$$

where

$$\varphi_u(x) = \begin{cases} \dfrac{\sqrt{x + a_u u^{-2}}\left(a_{2u} - x\right)}{u\sqrt{a_u - x + a_u \eta_u}}, & 0 \leqslant x \leqslant a_u, \\ \varphi_u\left(a_u\right), & a_u < x. \end{cases} \tag{4.9}$$

Proof. (a) It is from [1, Theorem 1.2]. (b) It is from [2, Theorem 1.3]. (c) It is from [2, Theorem 1.4]. □

Proof of Theorem 1.9. By Theorem 1.8, one has for $0 < p \leqslant 4$,

$$\begin{aligned}\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} &\geqslant C\left\|\Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \\ &\sim {a_n}^{1/p-1/2}\begin{cases}1, & p<4, \\ \left\{\log\left(nT(a_n)\right)\right\}^{1/4}, & p=4.\end{cases}\end{aligned} \tag{4.10}$$

For $p > 4$, we know by (4.7) and (4.8) that

$$\begin{aligned}&\int_{x_{nn}}^{a_n/3}\left|p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right|^p dx \\ &\quad\geqslant C\sum_{x_{nn}\leqslant x_{jn}\leqslant a_n/3}\int_{x_{j+1,n}}^{x_{jn}}\left|x-x_{j+1,n}\right|^p dx\varphi_n^{-p}(x_{jn})\left[x_{jn}(a_n-x_{jn})\right]^{-p/4} \\ &\quad\sim a_n^{-p/4}\sum_{x_{nn}\leqslant x_{jn}\leqslant a_n/3}\varphi_n(x_{jn})x_{jn}^{-p/4} \\ &\quad\sim a_n^{-p/4}\int_{a_n n^{-2}}^{a_n/3}t^{-p/4}dt \\ &\quad\sim a_n^{-p/2+1}n^{2(p/4-1)}.\end{aligned} \tag{4.11}$$

Then, for $p > 4$

$$\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \geqslant Ca_n^{-1/2+1/p}n^{2(1/4-1/p)}. \tag{4.12}$$

Therefore, one has

$$\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \geqslant C{a_n}^{1/p-1/2}\begin{cases}1, & p<4, \\ \left\{\log\left(nT(a_n)\right)\right\}^{1/4}, & p=4, \\ n^{2(1/4-1/p)}, & p>4.\end{cases} \tag{4.13}$$

On the other hand, one has from Theorem 1.8 that

$$\begin{aligned}
\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(x\geq a_n/3)} &\sim \left\|\Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(x\geq a_n/3)} \\
&\leqslant \left\|\Theta_n^{1/4}(x)p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \\
&\sim {a_n}^{1/p-1/2}\begin{cases} 1, & p<4, \\ \{\log(nT(a_n))\}^{1/4}, & p=4, \\ \{nT(a_n)\}^{2/3(1/4-1/p)}, & p>4, \end{cases}
\end{aligned} \tag{4.14}$$

and by (4.6) that

$$\begin{aligned}
\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(x\leqslant a_n/3)} &\leqslant Ca_n^{-1/4}\left(\int_0^{a_n/3}\left(x+\frac{a_n}{n^2}\right)^{-p/4}dx\right)^{1/p} \\
&\sim {a_n}^{1/p-1/2}\begin{cases} 1, & p<4, \\ \{\log a_n\}^{1/4}, & p=4, \\ n^{2(1/4-1/p)}, & p>4. \end{cases}
\end{aligned} \tag{4.15}$$

Therefore, one has

$$\left\| p_{n,\rho}(x)w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right\|_{L_p(\mathbb{R}^+)} \sim {a_n}^{1/p-1/2}\begin{cases} 1, & p<4, \\ \{\log(nT(a_n))\}^{1/4}, & p=4, \\ n^{2(1/4-1/p)}, & p>4. \end{cases} \tag{4.16}$$

From Remark 3.5(iii), we see that $nT(a_n) < Cn^3$. So, consequently, one has the result. □

Let

$$\Phi_n^*(t) = \max\left\{\eta_n^*, 1-\frac{|t|}{a_n^*}\right\}, \qquad \eta_n^* = \{nT^*(a_n^*)\}^{-2/3}. \tag{4.17}$$

Then, we obtain by Lemma 3.1 that

$$\Phi_n(x) \sim \Phi_{2n}^*(t), \quad x = t^2. \tag{4.18}$$

Lemma 4.4 (see [21, Theorem 2.7]). *Let $W_{\rho^*} \in \mathcal{F}(C^2)$ and $\rho^* > -1/2$. For $0 \leqslant s \leqslant r$ and $n \geqslant 1$, one has*

$$\begin{aligned}
&\left\| \Phi_n^*(t)^{(r/4-1/p)^+} \left| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right|^s \right\|_{L_p(\mathbb{R})} \\
&\quad \sim \left\| \Phi_n^*(t)^{(r/4-1/p)^+} \left| P_{n,\rho^*}(t) W(t) \left(|t| + \frac{a_n^*}{n} \right)^{\rho^*} \right|^s \right\|_{L_p(a_n^*/2 \leqslant |t| \leqslant 2a_n^*)} \\
&\quad \sim \begin{cases} a_n^{*1/p-r/2} \log n, & s = r,\ 4 \leqslant pr < \infty, \\ a_n^{*1/p-s/2}, & \text{otherwise}. \end{cases}
\end{aligned} \tag{4.19}$$

Proof of Theorem 1.10. By Theorem 2.4, we can transform $p_{n,\rho}(x)$ on $\mathbb{R}^+$ to $P_{2n,\rho^*}(t)$ on $\mathbb{R}$.

$$\begin{aligned}
&\left\| \Theta_n^{r/4}(x) \Phi_n^{(r/4-1/p)^+}(x) \left| p_n w(x) \left(x + \frac{a_n}{n^2} \right)^{\rho} \right|^s \right\|_{L_p(\mathbb{R}^+)} \\
&\quad = \left\| \Theta_n^{r/4}\left(t^2\right) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^{\rho} \right|^s (2|t|)^{1/p} \right\|_{L_p(\mathbb{R})}.
\end{aligned} \tag{4.20}$$

Using Lemma 4.4 and noting (3.1), one has

$$\begin{aligned}
&\left\| \Theta_n^{r/4}\left(t^2\right) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^{\rho} \right|^s (2|t|)^{1/p} \right\|_{L_p(\mathbb{R})} \\
&\quad \geqslant \left\| \Theta_n^{r/4}\left(t^2\right) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^{\rho} \right|^s (2|t|)^{1/p} \right\|_{L_p(a_{2n}^*/2 \leqslant |t| \leqslant 2a_{2n}^*)} \\
&\quad \geqslant C \left\| \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n^2} \right)^{\rho^*} \right|^s |t|^{1/p-s/2} \right\|_{L_p(a_{2n}^*/2 \leqslant |t| \leqslant 2a_{2n}^*)} \\
&\quad \sim a_{2n}^{*1/p-s/2} \left\| \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n} W(t) \left(|t| + \frac{a_{2n}^*}{n^2} \right)^{\rho^*} \right|^s \right\|_{L_p(a_{2n}^*/2 \leqslant |t| \leqslant 2a_{2n}^*)} \\
&\quad \sim \begin{cases} a_{2n}^{*2/p-r} \log n, & \text{if } s = r,\ 4 \leqslant pr < \infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise}. \end{cases}
\end{aligned} \tag{4.21}$$

On the other hand, by Lemma 4.2, we see

$$\left\| \Theta_n^{r/4}(x) \Phi_{2n}^{*(r/4-1/p)^+}(t) \left| P_{2n}(t) W(t) \left(t^2 + \frac{a_{2n}^{*2}}{n^2} \right)^{\rho} \right|^s (2t)^{1/p} \right\|_{L_p(a_{2n}^* \leqslant |t|)} \leqslant C \eta_{2n}^{(r/4-1/p)^+} e^{-cn}, \tag{4.22}$$

where $c > 0$ is a constant. Therefore, using Lemma 4.4 and noting (3.1) and the definition of Θ_n, one has

$$
\begin{aligned}
&\left\| \Theta_n^{r/4}\left(t^2\right)\Phi_{2n}^{*(r/4-1/p)^+}(t)\left|P_{2n}W(t)\left(t^2+\frac{a_{2n}^{*2}}{n^2}\right)^{\rho}\right|^s(2|t|)^{1/p}\right\|_{L_p(\mathbb{R})} \\
&\leqslant Ca_{2n}^{*-r/2}\left\|\Phi_{2n}^{*(r/4-1/p)^+}(t)\left|P_{2n}W(t)\left(|t|+\frac{a_{2n}^{*}}{n}\right)^{\rho^*}\right|^s|t|^{1/p+r/2-s/2}\right\|_{L_p(|t|\leqslant 2a_{2n}^*)} \\
&\leqslant Ca_{2n}^{*1/p-s/2}\left\|\Phi_{2n}^{*(r/4-1/p)^+}(t)\left|P_{2n}W(t)\left(|t|+\frac{a_{2n}^{*}}{n}\right)^{\rho^*}\right|^s\right\|_{L_p(|t|\leqslant 2a_{2n}^*)} \\
&\leqslant Ca_{2n}^{*1/p-s/2}\left\|\Phi_{2n}^{*(r/4-1/p)^+}(t)\left|P_{2n}W(t)\left(|t|+\frac{a_{2n}^{*}}{n}\right)^{\rho^*}\right|^s\right\|_{L_p(\mathbb{R})} \\
&\sim\begin{cases} a_{2n}^{*2/p-r}\log n, & \text{if } s=r,\ 4\leqslant pr<\infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise}. \end{cases}
\end{aligned}
\tag{4.23}
$$

Therefore, one has

$$
\begin{aligned}
&\left\|\Theta_n^{r/4}(x)\Phi_n^{(r/4-1/p)^+}(x)\left|p_n w(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right|^s\right\|_{L_p(\mathbb{R}^+)} \\
&\sim\begin{cases} a_{2n}^{*2/p-r}\log n, & \text{if } s=r,\ 4\leqslant pr<\infty, \\ a_{2n}^{*2/p-s}, & \text{otherwise}, \end{cases} \\
&\sim\begin{cases} a_{n}^{1/p-r/2}\log n, & \text{if } s=r,\ 4\leqslant pr<\infty, \\ a_{n}^{1/p-s/2}, & \text{otherwise}. \end{cases}
\end{aligned}
\tag{4.24}
$$

□

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