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Research Article

Notes on $|N, p, q|_k$ Summability Factors of Infinite Series

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New result concerning $|N, p, q|_k$ summability of the infinite series $\sum a_n \lambda_n$ is presented.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). Let (T_n) denote the sequence of (N, p, q) means of (s_n). The (N, p, q) transform of (s_n) is defined by

$$T_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v, \tag{1.1}$$

where

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v \neq 0, \quad \text{for any } n \left(p_{-1} = q_{-1} = R_{-1} = 0\right). \tag{1.2}$$

Necessary and sufficient conditions for the (N, p, q) method to be regular are

(i) $\lim_{n \to \infty} p_{n-v} q_n / R_n = 0$ for each v,

(ii) $\sum_{v=0}^{n} |p_{n-v} q_v| < K|R_n|$, where K is a positive constant independent of n.

The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\varphi_n - \varphi_{n-1}\right|^k < \infty, \tag{1.3}$$

where

$$\varphi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \tag{1.4}$$

where $P_n = p_1 + p_2 + \cdots + p_n \to \infty$ as $n \to \infty$.

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty, \tag{1.5}$$

where

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v, \tag{1.6}$$

and it is said to be summable $|N, p, q|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{1.7}$$

where T_n is as defined by (1.1).

For $k = 1$, $|N, p, q|_k$ summability reduces to $|N, p, q|$ summability.

The series $\sum a_n$ is said to be (N, p, q) bounded or $\sum a_n = \mathrm{O}(1)(N, p, q)$ if

$$t_n = \sum_{v=1}^{n} p_{n-v} q_v s_v = \mathrm{O}(R_n) \quad \text{as } n \longrightarrow \infty. \tag{1.8}$$

By M, we denote the set of sequences $p = (p_n)$ satisfying

$$\frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad p_n > 0, \; n = 0, 1, \ldots. \tag{1.9}$$

It is known (Das [1]) that for $p \in M$, (1.5) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n P_n} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right| < \infty. \tag{1.10}$$

For $p \in M$, the series $\sum a_n$ is said to be $|N, p_n|_k$-summable, $k \geq 1$, (Sulaiman [2]), if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty, \tag{1.11}$$

where $P_n = p_1 + p_2 + \cdots + p_n \to \infty$ as $n \to \infty$.

It is quite reasonable to give the following definition.

For $p \in M$, the series $\sum a_n$ is said to be $|N, p, q|_k$-summable, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| \sum_{v=1}^{n} v p_{n-v} q_v a_v \right|^k < \infty, \tag{1.12}$$

where $P_n = p_1 + p_2 + \cdots + p_n \to \infty$ as $n \to \infty$.

We also assume that (p_n), (q_n) are positive sequences of numbers such that

$$\begin{aligned} P_n &= p_0 + p_1 + \cdots + p_n \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ Q_n &= q_0 + q_1 + \cdots + q_n \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.13}$$

A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi-f-power increasing sequence, $f = (f_n)$, if there exists a constant $K = K(\alpha, f)$ such that

$$K f_n \alpha_n \geq f_m \alpha_m, \tag{1.14}$$

holds for $n \geq m \geq 1$ (see [3]).

Das [1], in 1966, proved the following result.

Theorem 1.1. *Let $(p_n) \in M$, $q_n \geq 0$. Then if $\sum a_n$ is $|N, p, q|$-summable, it is $|\overline{N}, q_n|$-summable.*

Recently Singh and Sharma [4] proved the following theorem.

Theorem 1.2. *Let $(p_n) \in M$, $q_n > 0$ and let (q_n) be a monotonic nondecreasing sequence for $n \geq 0$. The necessary and sufficient condition that $\sum a_n \lambda_n$ is $|\overline{N}, q_n|$-summable whenever*

$$\begin{aligned} &\sum a_n = O(1)(N, p, q), \\ &\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\lambda_n| < \infty, \\ &\sum_{n=0}^{\infty} |\Delta \lambda_n| < \infty, \\ &\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} \left| \Delta^2 \lambda_n \right| < \infty, \end{aligned} \tag{1.15}$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n||\lambda_n| < \infty. \tag{1.16}$$

2. Lemmas

Lemma 2.1. *Let (p_n) be nonincreasing, $n = O(P_n)$. Then for $r > 0$, $k \geq 1$,*

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} = O\left(\frac{1}{v^{r+k-1}}\right). \tag{2.1}$$

Proof. Since p_n is nonincreasing, then $np_n = O(P_n)$.

$$\begin{aligned}
\sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} &= \sum_{n=v+1}^{2v} \frac{p_{n-v}^k}{n^r P_n^k} + \sum_{n=2v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k}, \\
\sum_{n=v+1}^{2v} \frac{p_{n-v}^k}{n^r P_n^k} &= O(1)\frac{1}{v^r P_v^k} \sum_{n=v+1}^{2v} p_{n-v}^k = O(1)\frac{1}{v^r P_v^k} \sum_{m=1}^{v} p_m^k \\
&= O(1)\frac{1}{v^r P_v^k} \sum_{m=1}^{v} p_m = O(1)\frac{1}{v^r P_v^{k-1}} = O\left(\frac{1}{v^{r+k-1}}\right), \\
\sum_{n=2v+1}^{\infty} \frac{p_{n-v}^k}{n^r P_n^k} &= O(1) \sum_{m=v+1}^{\infty} \frac{p_m^k}{(m+v)^r P_{m+v}^k} = O(1) \sum_{m=v+1}^{\infty} \frac{p_m^k}{m^r P_m^k} \\
&= O(1) \sum_{m=v+1}^{\infty} \frac{1}{m^{r+k}} = O(1) \int_v^{\infty} x^{-r-k} dx = O\left(\frac{1}{v^{r+k-1}}\right).
\end{aligned} \tag{2.2}$$

Therefore

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v}^k}{n^r \ P_n^k} = O\left(\frac{1}{v^{r+k-1}}\right). \tag{2.3}$$

□

Lemma 2.2. *For $p \in M$,*

$$\sum_{v=0}^{\infty} \left|\Delta_v p_{n-v}\right| < \infty. \tag{2.4}$$

Proof. Since $p \in M$, then (p_n) is nonincreasing and hence

$$\sum_{v=0}^{m} \left|\Delta_v p_{n-v}\right| = \sum_{v=0}^{\infty} (p_{n-v-1} - p_{n-v}) = p_n - p_{m-v-1} = O(1). \tag{2.5}$$

□

Lemma 2.3 (see [3]). *If (X_n) is a quasi-f-increasing sequence, where $f = (f_n) = (n^{\beta}(\log n)^{\gamma})$, $\gamma > 0$, $0 < \beta < 1$, then under the conditions*

$$\begin{gathered} X_m|\lambda_m| = O(1), \quad m \longrightarrow \infty, \\ \sum_{n=1}^{m} nX_n\left|\Delta^2\lambda_n\right| = O(1), \quad m \longrightarrow \infty, \end{gathered} \tag{2.6}$$

one has

$$\begin{gathered} nX_n|\Delta\lambda_n| = O(1), \\ \sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \end{gathered} \tag{2.7}$$

3. Result

Our aim is to present the following new general result.

Theorem 3.1. *Let $p \in M$, and let (X_n) be a quasi-f-increasing sequence, where $f = (f_n) = (n\log^{\gamma} n)$, $\gamma > 0$, $0 < \beta < 1$ and* (2.6)*, and*

$$\begin{gathered} \sum_{v=1}^{n} \frac{q_v|s_v|^k}{vX_v^{k-1}} = O(X_n), \\ \Delta q_v = O\left(v^{-1}q_v\right), \\ q_{v+1} = O(q_v), \\ v = O(P_v), \end{gathered} \tag{3.1}$$

are all satisfied, then the series $\sum a_n\lambda_n$ is summable $|N,p,q|_k$, $k \geq 1$.

Proof. We have

$$\begin{aligned} T_n &= \sum_{v=0}^{n} vp_{n-v}q_v a_v\lambda_v \\ &= \sum_{v=0}^{n-1}\left(\sum_{r=0}^{v} a_r\right)\Delta_v(v\, p_{n-v}q_v\lambda_v) + \left(\sum_{v=0}^{n} a_v\right)np_0q_n\lambda_n \\ &= \sum_{v=0}^{n-1} s_v\big(-p_{n-v}q_v\lambda_v + (v+1)\Delta q_v p_{n-v}\lambda_v + (v+1)q_{v+1}\Delta_v p_{n-v}\lambda_v \\ &\qquad\quad + (v+1)q_{v+1}p_{n-v-1}\Delta\lambda_v\big) + np_0q_ns_n\lambda_n \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4} + T_{n5}. \end{aligned} \tag{3.2}$$

In order to prove the result, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \left| T_{nj} \right|^k < \infty, \qquad j = 1, 2, 3, 4, 5. \tag{3.3}$$

Applying HÖlder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{nP_n^k} |T_{n1}|^k &= \sum_{n=1}^{m} \frac{1}{nP_n^k} \left| \sum_{v=0}^{n-1} p_{n-v} q_v s_v \lambda_v \right|^k \\
&\leq \sum_{n=1}^{m} \frac{1}{nP_n^k} \sum_{v=0}^{n-1} p_{n-v} q_v^k |s_v|^k |\lambda_v|^k \left(\sum_{v=0}^{n-1} p_{n-v} \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m} \frac{P_n^{k-1}}{nP_n^k} \sum_{v=0}^{n-1} p_{n-v} q_v^k |s_v|^k |\lambda_v|^k \\
&= O(1) \sum_{v=0}^{m} q_v^k |s_v|^k |\lambda_v|^k \sum_{n=v+1}^{\infty} \frac{p_{n-v}}{nP_n} \\
&= O(1) \sum_{v=0}^{m} v^{-1} q_v^k |s_v|^k |\lambda_v|^k \\
&= O(1) \sum_{v=0}^{m} \frac{q_v^k |s_v|^k}{vX_v^{k-1}} |\lambda_v| |\lambda_v|^{k-1} X_v^{k-1} \\
&= O(1) \sum_{v=0}^{m} \frac{q_v^k |s_v|^k}{vX_v^{k-1}} |\lambda_v| \\
&= O(1) \sum_{v=0}^{m-1} \Delta |\lambda_v| \sum_{r=0}^{v} \frac{q_r^k |s_r|^k}{rX_r^{k-1}} + |\lambda_n| \sum_{v=0}^{m} \frac{q_v^k |s_v|^k}{vX_v^{k-1}} \\
&= O(1) \sum_{v=0}^{m-1} |\Delta \lambda_v| X_v + |\lambda_m| X_m = O(1), \\
\sum_{n=1}^{m} \frac{1}{nP_n^k} |T_{n2}|^k &= \sum_{n=1}^{m} \frac{1}{nP_n^k} \left| \sum_{v=0}^{n-1} (v+1) p_{n-v} \Delta q_v s_v \lambda_v \right|^k \\
&\leq \sum_{n=1}^{m} \frac{1}{nP_n^k} \sum_{v=0}^{n-1} v^k p_{n-v} \left| \Delta q_v \right|^k |s_v|^k |\lambda_v|^k \left(\sum_{v=0}^{n-1} p_{n-v} \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m} \frac{P_n^{k-1}}{nP_n^k} \sum_{v=0}^{n-1} v^k p_{n-v} \left| \Delta q_v \right|^k |s_v|^k |\lambda_v|^k \\
&= O(1) \sum_{v=0}^{m} v^k \left| \Delta q_v \right|^k |s_v|^k |\lambda_v|^k \sum_{n=v+1}^{\infty} \frac{p_{n-v}}{nP_n}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=0}^{m} v^{k-1}\left|\Delta q_v\right|^{k}|s_v|^{k}|\lambda_v|^{k}\\
&= O(1)\sum_{v=0}^{m} \frac{q_v^{k}|s_v|^{k}}{vX_v^{k-1}}|\lambda_v|\\
&= O(1), \quad \text{as in the case of } T_{n1},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m}\frac{1}{nP_n^{k}}|T_{n3}|^{k} &= \sum_{n=1}^{m}\frac{1}{nP_n^{k}}\left|\sum_{v=0}^{n-1}(v+1)\Delta_v p_{n-v}q_{v+1}s_v\lambda_v\right|^{k}\\
&\le \sum_{n=1}^{m}\frac{1}{nP_n^{k}}\sum_{v=0}^{n-1} v^{k}\Delta_v p_{n-v}q_{v+1}^{k}|s_v|^{k}|\lambda_v|^{k}\left(\sum_{v=0}^{n-1}\left|\Delta_v p_{n-v}\right|\right)^{k-1}\\
&= O(1)\sum_{n=1}^{m}\frac{1}{nP_n^{k}}\sum_{v=0}^{n-1} v^{k}\Delta_v p_{n-v}q_{v+1}^{k}|s_v|^{k}|\lambda_v|^{k}\\
&= O(1)\sum_{v=0}^{m} v^{k}q_{v+1}^{k}|s_v|^{k}|\lambda_v|^{k}\sum_{n=v+1}^{\infty}\frac{\left|\Delta_v p_{n-v}\right|}{nP_n^{k}}\\
&= O(1)\sum_{v=0}^{m} v^{k-1}P_v^{-k}q_{v+1}^{k}|s_v|^{k}|\lambda_v|^{k}\\
&= O(1)\sum_{v=0}^{m} v^{-1}q_v^{k}|s_v|^{k}|\lambda_v|^{k}\\
&= O(1), \quad \text{as in the case of } T_{n1},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m}\frac{1}{nP_n^{k}}|T_{n4}|^{k} &= \sum_{n=1}^{m}\frac{1}{nP_n^{k}}\left|\sum_{v=0}^{n-1}(v+1)p_{n-v-1}q_{v+1}s_v\Delta\lambda_v\right|^{k}\\
&\le \sum_{n=1}^{m}\frac{1}{nP_n^{k}}\sum_{v=0}^{n-1} v^{k}p_{n-v-1}^{k}q_{v+1}^{k}|s_v|^{k}|\Delta\lambda_v|X_v^{1-k}\left(\sum_{v=0}^{n-1}X_v|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=1}^{m}\frac{1}{nP_n^{k}}\sum_{v=0}^{n-1} v^{k}p_{n-v-1}^{k}q_{v+1}^{k}|s_v|^{k}|\Delta\lambda_v|\\
&= O(1)\sum_{v=0}^{m}\frac{v^{k}q_{v+1}^{k}|s_v|^{k}}{X_v^{k-1}}|\Delta\lambda_v|\sum_{n=v+1}^{\infty}\frac{p_{n-v-1}^{k}}{nP_n^{k}}\\
&= O(1)\sum_{v=0}^{m}\frac{q_{v+1}^{k}|s_v|^{k}}{vX_v^{k-1}}v|\Delta\lambda_v|\\
&= O(1)\sum_{v=0}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=0}^{v}\frac{q_r^{k}|s_r|^{k}}{rX_r^{k-1}} + m|\Delta\lambda_m|\sum_{v=0}^{m}\frac{q_v^{k}|s_v|^{k}}{vX_v^{k-1}}\\
&= O(1)\sum_{v=0}^{m-1}|\Delta\lambda_v|\,X_v + O(1)\sum_{v=0}^{m-1}v\left|\Delta^{2}\lambda_v\right|X_v + O(1)|\Delta\lambda_m|\,X_m = O(1),
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{nP_n^k} |T_{n5}|^k &= \sum_{n=1}^{m} \frac{1}{nP_n^k} \left| np_0 q_n s_n \lambda_n \right|^k \\
&= O(1) \sum_{n=1}^{m} n^{k-1} P_n^{-k} q_n^k |s_n|^k |\lambda_n|^k \\
&= O(1) \sum_{n=1}^{m} n^{-1} q_n^k |s_n|^k |\lambda_n|^k \\
&= O(1), \quad \text{as in the case of } T_{n1}.
\end{aligned} \tag{3.4}$$

This completes the proof of the theorem. □

References

[1] G. Das, "On some methods of summability," *The Quarterly Journal of Mathematics Oxford Series*, vol. 17, no. 2, pp. 244–256, 1966.

[2] W. T. Sulaiman, "Notes on two summability methods," *Pure and Applied Mathematika Sciences*, vol. 31, no. 1-2, pp. 59–69, 1990.

[3] W. T. Sulaiman, "Extension on absolute summability factors of infinite series," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 1224–1230, 2006.

[4] N. Singh and N. Sharma, "On (N, p, q) summability factors of infinite series," *Proceedings of Mathematical Sciences*, vol. 110, no. 1, pp. 61–68, 2000.