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*Research Article*

# Refinements of Results about Weighted Mixed Symmetric Means and Related Cauchy Means

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A recent refinement of the classical discrete Jensen inequality is given by Horváth and Pečarić. In this paper, the corresponding weighted mixed symmetric means and Cauchy-type means are defined. We investigate the exponential convexity of some functions, study mean value theorems, and prove the monotonicity of the introduced means.

## 1. Introduction and Preliminary Results

A new refinement of the discrete Jensen inequality is given in [1]. The following notations are also introduced in [1].

Let $X$ be a set, $P(X)$ its power set, and $|X|$ denotes the number of elements in $X$. Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$\begin{aligned} S_{v,w} &: \{1,\ldots,u\}^v \longrightarrow \{1,\ldots,u\}^{v-1}, \quad 1 \leq w \leq v, \\ S_v &: \{1,\ldots,u\}^v \longrightarrow P\Big(\{1,\ldots,u\}^{v-1}\Big), \\ T_v &: P(\{1,\ldots,u\}^v) \longrightarrow P\Big(\{1,\ldots,u\}^{v-1}\Big) \end{aligned} \tag{1.1}$$

by

$$S_{v,w}(i_1,\ldots,i_v) := (i_1,\ldots,i_{w-1},i_{w+1},\ldots,i_v), \quad 1 \le w \le v,$$
$$S_v(i_1,\ldots,i_v) = \bigcup_{w=1}^{v}\{S_{v,w}(i_1,\ldots,i_v)\},$$
$$T_v(I) = \begin{cases} \bigcup_{(i_1,\ldots,i_v)\in I} S_v(i_1,\ldots,i_v), & I \ne \phi, \\ \phi, & I = \phi. \end{cases} \tag{1.2}$$

Further, introduce the function

$$\alpha_{v,i} : \{1,\ldots,u\}^v \longrightarrow \mathbb{N}, \quad 1 \le i \le u, \tag{1.3}$$

via

$$\alpha_{v,i}(i_1,\ldots,i_v) := \text{Number of occurrences of } i \text{ in the sequence } (i_1,\ldots,i_v). \tag{1.4}$$

For each $I \in P(\{1,\ldots,u\}^v)$, let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_v)\in I} \alpha_{v,i}(i_1,\ldots,i_v), \quad 1 \le i \le u. \tag{1.5}$$

It is easy to observe from the construction of the functions $S_v, S_{v,w}, T_v$ and $\alpha_{v,i}$ that they do not depend essentially on $u$, so we can write for short $S_v$ for $S_v^u$, and so on.

($H_1$) The following considerations concern a subset $I_k$ of $\{1,\ldots,n\}^k$ satisfying

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n, \tag{1.6}$$

where $n \ge 1$ and $k \ge 2$ are fixed integers.

Next, we proceed inductively to define the sets $I_l \subset \{1,\ldots,n\}^l$ $(k-1 \ge l \ge 1)$ by

$$I_{l-1} := T_l(I_l), \quad k \ge l \ge 2. \tag{1.7}$$

By (1.6), $I_1 = \{1,\ldots,n\}$ and this implies that $\alpha_{I_1,i} = 1$ for $1 \le i \le n$. From (1.6), again, we have $\alpha_{I_l,i} \ge 1$ $(k-1 \ge l \ge 1, 1 \le i \le n)$.

For every $k \ge l \ge 2$ and for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$, let

$$H_{I_l}(j_1,\ldots,j_{l-1}) := \{((i_1,\ldots,i_l),m) \in I_l \times \{1,\ldots,l\} \mid S_{l,m}(i_1,\ldots,i_l) = (j_1,\ldots,j_{l-1})\}. \tag{1.8}$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geq l \geq 1)$ inductively by

$$t_{I_k,k}(i_1,\ldots,i_k) := 1, \quad (i_1,\ldots,i_k) \in I_k,$$
$$t_{I_k,l-1}(j_1,\ldots,j_{l-1}) := \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l). \tag{1.9}$$

Let $J$ be an interval in $\mathbb{R}$, let $\mathbf{x} := (x_1,\ldots,x_n) \in J^n$, let $\mathbf{p} := (p_1,\ldots,p_n)$ such that $p_i > 0$ $(1 \leq i \leq n)$ and $\sum_{i=1}^n p_i = 1$, and let $f : J \to \mathbb{R}$ be a convex function. For any $k \geq l \geq 1$, set

$$A_{l,l} = A_{l,l}(I_k;\mathbf{x};\mathbf{p}) := \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum_{s=1}^{l}(p_{i_s}/\alpha_{I_l,i_s})x_{i_s}}{\sum_{s=1}^{l} p_{i_s}/\alpha_{I_l,i_s}}\right), \tag{1.10}$$

and associate to each $k-1 \geq l \geq 1$ the number

$$A_{k,l} = A_{k,l}(I_k;\mathbf{x};\mathbf{p}) := \frac{1}{(k-1)} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{l}(p_{i_s}/\alpha_{I_k,i_s})x_{i_s}}{\sum_{s=1}^{l} p_{i_s}/\alpha_{I_k,i_s}}\right). \tag{1.11}$$

We need the following hypotheses.

($H_2$) Let $\mathbf{x} := (x_1,\ldots,x_n)$ and $\mathbf{p} := (p_1,\ldots,p_n)$ be positive $n$-tuples such that $\sum_{i=1}^n p_i = 1$.

($H_3$) Let $J \subset \mathbb{R}$ be an interval, let $\mathbf{x} := (x_1,\ldots,x_n) \in J^n$, let $\mathbf{p} := (p_1,\ldots,p_n)$ be a positive $n$-tuples such that $\sum_{i=1}^n p_i = 1$, and let $h$, $g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

($H_4$) Let $J \subset \mathbb{R}$ be an interval, let $\mathbf{x} := (x_1,\ldots,x_n) \in J^n$, and let $\mathbf{p} := (p_1,\ldots,p_2)$ be positive $n$-tuples such that $\sum_{p_i}^n p_i = 1$. Further, let $f : J \to \mathbb{R}$ be a convex function.

Assume ($H_1$) and ($H_2$). The power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1,\ldots,i_n) \in I_1$ $(l = 1,\ldots,k)$ are given as

$$M_r(I_k,\mathbf{i}^l) := \begin{cases} \left(\dfrac{\sum_{s=1}^{l}(p_{i_s}/\alpha_{I_k,i_s})x_{i_s}^r}{\sum_{s=1}^{l} p_{i_s}/\alpha_{I_k,i_s}}\right)^{1/r}, & r \neq 0, \\ \left(\displaystyle\prod_{s=1}^{l} x_{i_s}^{p_{i_s}/\alpha_{I_k,i_s}}\right)^{1/\sum_{s=1}^{l}(p_{i_s}/\alpha_{I_k,i_s})}, & r = 0. \end{cases} \tag{1.12}$$

We also use the means

$$M_r := \begin{cases} \left(\displaystyle\sum_{i=1}^{n} p_i x_i^r\right)^{1/r}, & r \neq 0, \\ \displaystyle\prod_{i=1}^{n} x_i^{p_i}, & r = 0. \end{cases} \tag{1.13}$$

For $\gamma$, $\eta \in \mathbb{R}$, we introduce the mixed symmetric means with positive weights as follows:

$$M^1_{\eta,\gamma}(I_k,k) := \begin{cases} \left[\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)(M_\gamma(I_k,\mathbf{i}^k))^\eta\right]^{1/\eta}, & \eta\neq 0,\\ \prod_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}(M_\gamma(I_k,\mathbf{i}^k))^{(\sum_{s=1}^{k}p_{i_s}/\alpha_{I_k,i_s})}, & \eta=0, \end{cases} \tag{1.14}$$

and, for $k-1 \geq l \geq 1$,

$$M^1_{\eta,\gamma}(I_k,l) := \begin{cases} \left[\frac{1}{(k-1)\ldots l}\sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)(M_\gamma(I_k,\mathbf{i}^l))^\eta\right]^{1/\eta}, & \eta\neq 0,\\ \left[\prod_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l}(M_\gamma(I_k,\mathbf{i}^l))^{t_{I_k,l}(\mathbf{i}^l)(\sum_{s=1}^{l}p_{i_s}/\alpha_{I_k,i_s})}\right]^{1/(k-1)\ldots l}, & \eta=0. \end{cases} \tag{1.15}$$

We deduce the monotonicity of these means from the following refinement of the discrete Jensen inequality in [1].

**Theorem 1.1.** *Assume ($H_1$) and ($H_4$). Then,*

$$f\left(\sum_{i=1}^{n}p_i x_i\right) \leq A_{k,k} \leq A_{k,k-1} \leq \cdots \leq A_{k,2} \leq A_{k,1} = \sum_{i=1}^{n}p_i f(x_i), \tag{1.16}$$

*where the numbers $A_{k,l}$ $(k \geq l \geq 1)$ are defined in* (1.10) *and* (1.11). *If $f$ is a concave function, then the inequalities in* (1.16) *are reversed.*

Under the conditions of the previous theorem,

$$\begin{aligned} \Upsilon^1(\mathbf{x},\mathbf{p},f) &:= A_{k,m} - A_{k,l} \geq 0, \quad k \geq l > m \geq 1,\\ \Upsilon^2(\mathbf{x},\mathbf{p},f) &:= A_{k,l} - f\left(\sum_{i=1}^{n}p_i x_i\right) \geq 0, \quad k \geq l \geq 1. \end{aligned} \tag{1.17}$$

**Corollary 1.2.** *Assume ($H_1$) and ($H_2$). Let $\eta$, $\gamma \in \mathbb{R}$ such that $\eta \leq \gamma$, then*

$$M_\gamma = M^1_{\gamma,\eta}(I_k,1) \geq \cdots \geq M^1_{\gamma,\eta}(I_k,k) \geq M_\eta, \tag{1.18}$$

$$M_\eta = M^1_{\eta,\gamma}(I_k,1) \leq \cdots \leq M^1_{\eta,\gamma}(I_k,k) \leq M_\gamma. \tag{1.19}$$

*Proof.* Assume $\eta, \gamma \neq 0$. To obtain (1.18), we can apply Theorem 1.1 to the function $f(x) = x^{\gamma/\eta}$ $(x > 0)$ and the $n$-tuples $(x_1^{\eta}, \ldots, x_n^{\eta})$ to get the analogue of (1.16) and to raise the power $1/\gamma$. Equation (1.19) can be proved in a similar way by using $f(x) = x^{\eta/\gamma}$ $(x > 0)$ and $(x_1^{\gamma}, \ldots, x_n^{\gamma})$ and raising the power $1/\eta$.

When $\eta = 0$ or $\gamma = 0$, we get the required results by taking limit. □

Assume $(H_1)$ and $(H_3)$. Then, we define the quasiarithmetic means with respect to (1.10) and (1.11) as follows:

$$M_{h,g}^{1}(I_k, k) := h^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{k}(p_{i_s}/\alpha_{I_k,i_s})g(x_{i_s})}{\sum_{s=1}^{k}p_{i_s}/\alpha_{I_k,i_s}}\right)\right), \tag{1.20}$$

and, for $k-1 \geq l \geq 1$,

$$M_{h,g}^{1}(I_k, l) = h^{-1}\left(\frac{1}{(k-1)\ldots l}\sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}\left(\mathbf{i}^l\right)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{l}(p_{i_s}/\alpha_{I_k,i_s})g(x_{i_s})}{\sum_{s=1}^{l}p_{i_s}/\alpha_{I_k,i_s}}\right)\right). \tag{1.21}$$

The monotonicity of these generalized means is obtained in the next corollary.

**Corollary 1.3.** *Assume $(H_1)$ and $(H_3)$. For a continuous and strictly monotone function $q : J \to \mathbb{R}$, one defines*

$$M_q := q^{-1}\left(\sum_{i=1}^{n} p_i q(x_i)\right). \tag{1.22}$$

*Then,*

$$M_h = M_{h,g}^{1}(I_k, 1) \geq \cdots \geq M_{h,g}^{1}(I_k, k) \geq M_g, \tag{1.23}$$

*if either $h\circ g^{-1}$ is convex and $h$ is increasing or $h\circ g^{-1}$ is concave and $h$ is decreasing,*

$$M_g = M_{g,h}^{1}(I_k, 1) \leq \cdots \leq M_{g,h}^{1}(I_k, k) \leq M_h, \tag{1.24}$$

*if either $g\circ h^{-1}$ is convex and $g$ is decreasing or $g\circ h^{-1}$ is concave and $g$ is increasing.*

*Proof.* First, we can apply Theorem 1.1 to the function $h\circ g^{-1}$ and the $n$-tuples $(g(x_1), \ldots, g(x_n))$, then we can apply $h^{-1}$ to the inequality coming from (1.16). This gives (1.23). A similar argument gives (1.24): $g\circ h^{-1}$, $(h(x_1), \ldots, h(x_n))$ and $g^{-1}$ can be used. □

Throughout Examples 1.4-1.5, 1.9–1.12, which are based on examples in [1], the conditions $(H_2)$, in the mixed symmetric means, and $(H_3)$, in the quasiarithmetic means, will be assumed.

*Example 1.4.* Suppose

$$I_2 := \left\{(i_1, i_2) \in \{1, \dots, n\}^2 \mid i_1 | i_2\right\}, \tag{1.25}$$

where $i_1|i_2$ means that $i_1$ divides $i_2$. Since $i|i$ $(i = 1, \dots, n)$, therefore (1.6) holds. We note that

$$\alpha_{I_2,i} = \left[\frac{n}{i}\right] + d(i), \quad i = 1, \dots, n, \tag{1.26}$$

where $[n/i]$ is the largest positive integer not greater than $n/i$, and $d(i)$ means the number of positive divisors of $i$. Then, (1.14) gives for $\eta, \gamma \in \mathbb{R}$

$$M^1_{\eta,\gamma}(I_2, 2) = \begin{cases} \left[\displaystyle\sum_{\mathbf{i}^2=(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{[n/i_s] + d(i_s)}\right) \left(M_\gamma(I_2, \mathbf{i}^k)\right)^n\right]^{1/n}, & \eta \neq 0, \\ \displaystyle\prod_{\mathbf{i}^2=(i_1,i_2)\in I_2} \left(M_\gamma(I_2, \mathbf{i}^2)\right)^{\sum_{s=1}^2 p_{i_s}/([n/i_s]+d(i_s))} & \eta \neq 0, \end{cases} \tag{1.27}$$

while (1.20) gives

$$M^1_{h,g}(I_2, 2) = h^{-1}\left(\sum_{(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{[n/i_s] + d(i_s)}\right) h \circ g^{-1}\left(\frac{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))g(x_{i_s})}{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))}\right)\right). \tag{1.28}$$

Assume (H$_4$) holds, and consider the set $I_2$ in Example 1.4. Then, Theorem 1.1 implies that

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq \sum_{(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{[n/i_s] + d(i_s)}\right) f\left(\frac{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))x_{i_s}}{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))}\right) \leq \sum_{r=1}^{n} p_r f(x_r), \tag{1.29}$$

and thus

$$\Upsilon^3(\mathbf{x}, \mathbf{p}, f) := \sum_{(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{[n/i_s] + d(i_s)}\right) f\left(\frac{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))x_{i_s}}{\sum_{s=1}^2 (p_{i_s}/([n/i_s] + d(i_s)))}\right) - f\left(\sum_{r=1}^{n} p_r x_r\right) \geq 0,$$

$$\Upsilon^4(\mathbf{x}, \mathbf{p}, f) := \sum_{r=1}^{n} p_r f(x_r) - \sum_{(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{[n/i_s] + d(i_s)}\right) f\left(\frac{\sum_{s=1}^2 (p_{i_s}/[n/i_s] + d(i_s))x_{i_s}}{\sum_{s=1}^2 (p_{i_s}/[n/i_s] + d(i_s))}\right) \geq 0. \tag{1.30}$$

*Example 1.5.* Let $c_i \geq 1$ be an integer ($i = 1, \ldots, n$), let $k := \sum_{i=1}^{n} c_i$, and also let $I_k = P_k^{c_1,\ldots,c_n}$ consist of all sequences $(i_1, \ldots, i_k)$ in which the number of occurrences of $i \in \{1, \ldots, n\}$ is $c_i$ ($i = 1, \ldots, n$). Obviously, (1.6) holds, and, by simple calculations, we have

$$I_{k-1} = \bigcup_{i=1}^{n} P_{k-1}^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1! \cdots c_n!} c_i, \quad i = 1, \ldots, n. \tag{1.31}$$

Moreover, $t_{I_k,k-1}(i_1, \ldots, i_{k-1}) = k$ for

$$(i_1, \ldots, i_{k-1}) \in P_{k-1}^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad i = 1, \ldots, n. \tag{1.32}$$

Under the above settings, (1.15) can be written as

$$M_{\eta,\gamma}^{1}(I_k, k-1) = \begin{cases} \left[ \dfrac{1}{k-1} \displaystyle\sum_{i=1}^{n} (c_i - p_i) \left( \dfrac{\sum_{r=1}^{n} p_r x_r^{\gamma} - (p_i/c_i) x_i^{\gamma}}{1 - (p_i/c_i)} \right)^{\eta/\gamma} \right]^{1/\eta}, & \eta \neq 0,\ \gamma \neq 0, \\ \left( \displaystyle\prod_{i=1}^{n} \left( \dfrac{\sum_{r=1}^{n} p_r x_r^{\gamma} - (p_i/c_i) x_i^{\gamma}}{1 - (p_i/c_i)} \right)^{(c_i - p_i)/\gamma} \right)^{1/(k-1)}, & \gamma \neq 0,\ \eta = 0, \\ \left( \displaystyle\prod_{i=1}^{n} \left( x_i^{-p_i} \left( \prod_{i=1}^{n} x_r^{p_r} \right) \right)^{c_i} \right)^{1/(k-1)}, & \gamma = 0,\ \eta = 0, \end{cases} \tag{1.33}$$

while (1.21) becomes

$$M_{h,g}^{1}(I_k, k-1) = h^{-1} \left( \frac{1}{k-1} \sum_{i=1}^{n} (c_i - p_i) h \circ g^{-1} \left( \frac{\sum_{r=1}^{n} p_r g(x_r) - (p_i/c_i) g(x_i)}{1 - (p_i/c_i)} \right) \right). \tag{1.34}$$

Assume ($H_4$) holds, and consider the set $I_k$ in Example 1.5. Then, Theorem 1.1 yields that

$$\begin{gathered} A_{k,k-1} = \frac{1}{k-1} \sum_{i=1}^{n} (c_i - p_i) f\left( \frac{\sum_{r=1}^{n} p_r x_r - (p_i/c_i) x_i}{1 - (p_i/c_i)} \right), \\ f\left( \sum_{r=1}^{n} p_r x_r \right) \leq A_{k,k-1} \leq \sum_{r=1}^{n} p_r f(x_r). \end{gathered} \tag{1.35}$$

This shows that

$$\begin{gathered} \Upsilon^{5}(\mathbf{x}, \mathbf{p}, f) := A_{k,k-1} - f\left( \sum_{r=1}^{n} p_r x_r \right) \geq 0, \\ \Upsilon^{6}(\mathbf{x}, \mathbf{p}, f) := \sum_{r=1}^{n} p_r f(x_r) - A_{k,k-1} \geq 0. \end{gathered} \tag{1.36}$$

The following result is also given in [1].

**Theorem 1.6.** *Assume* ($H_1$) *and* ($H_4$)*, and suppose* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$*. Then,*

$$A_{k,l} = A_{l,l} = \frac{n}{l|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum_{s=1}^{l} p_{i_s} x_{i_s}}{\sum_{s=1}^{l} p_{i_s}}\right), \quad k \geq l \geq 1, \tag{1.37}$$

*and thus*

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{k,k} \leq A_{k-1,k-1} \leq \cdots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{1.38}$$

*If* $f$ *is a concave function then the inequalities* (1.38) *are reversed.*

Under the conditions of the previous theorem, we have, from (1.38), that

$$\begin{aligned} \Upsilon^7(\mathbf{x},\mathbf{p},f) &:= A_{m,m} - A_{l,l} \geq 0, \quad k \geq l > m \geq 1, \\ \Upsilon^8(\mathbf{x},\mathbf{p},f) &:= A_{l,l} - f\left(\sum_{r=1}^{n} p_r x_r\right) \geq 0, \quad k \geq l \geq 1. \end{aligned} \tag{1.39}$$

Assume ($H_1$) and ($H_2$), and suppose $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. In this case, the power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1,\ldots,i_l) \in I_l$ $(l = 1,\ldots,k)$ has the form

$$M_r(I_l,\mathbf{i}^l) = M_r(I_k,\mathbf{i}^l) = \begin{cases} \left(\dfrac{\sum_{s=1}^{l} p_{i_s} x_{i_s}^r}{\sum_{s=1}^{l} p_{i_s}}\right)^{1/r}, & r \neq 0, \\ \left(\displaystyle\prod_{s=1}^{l} x_{i_s}{}^{p_{i_s}}\right)^{1/\sum_{s=1}^{l} p_{i_s}}, & r = 0. \end{cases} \tag{1.40}$$

Now, for $\gamma$, $\eta \in \mathbb{R}$ and $k \geq l \geq 1$, we introduce the mixed symmetric means with positive weights related to (1.37) as follows:

$$M^2_{\eta,\gamma}(I_l) := \begin{cases} \left[\dfrac{n}{l|I_l|} \displaystyle\sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) \left(M_\gamma(I_l,\mathbf{i}^l)\right)^\eta\right]^{1/\eta}, & \eta \neq 0, \\ \left[\displaystyle\prod_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} \left(M_\gamma(I_l,\mathbf{i}^l)\right)^{\left(\sum_{s=1}^{l} p_{i_s}\right)}\right]^{n/l|I_l|}, & \eta = 0. \end{cases} \tag{1.41}$$

**Corollary 1.7.** *Assume* ($H_1$) *and* ($H_2$), *and suppose* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}(k \geq l \geq 2)$. *Let* $\eta$, $\gamma \in \mathbb{R}$ *such that* $\eta \leq \gamma$. *Then,*

$$\begin{aligned} M_\gamma &= M^2_{\gamma,\eta}(I_1) \geq \cdots \geq M^2_{\gamma,\eta}(I_k) \geq M_\eta, \\ M_\eta &= M^2_{\eta,\gamma}(I_1) \leq \cdots \leq M^2_{\eta,\gamma}(I_k) \leq M_\gamma. \end{aligned} \tag{1.42}$$

*Proof.* The proof comes from Corollary 1.2. □

Assume ($H_1$) and ($H_3$), and suppose $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. We define for $k \geq l \geq 1$ the quasiarithmetic means with respect to (1.37) as follows:

$$M^2_{h,g}(I_l) := h^{-1}\left(\frac{n}{l|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{l} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{l} p_{i_s}}\right)\right). \tag{1.43}$$

**Corollary 1.8.** *Assume* ($H_1$) *and* ($H_3$), *and suppose* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then,*

$$M_h = M^2_{h,g}(I_1) \geq \cdots \geq M^2_{h,g}(I_k) \geq M_g, \tag{1.44}$$

*where either* $h\circ g^{-1}$ *is convex and* $h$ *is increasing or* $h\circ g^{-1}$ *is concave and* $h$ *is decreasing,*

$$M_g = M^2_{g,h}(I_1) \leq \cdots \leq M^2_{g,h}(I_k) \leq M_h, \tag{1.45}$$

*where either* $g\circ h^{-1}$ *is convex and* $g$ *is decreasing or* $g\circ h^{-1}$ *is concave and* $g$ *is increasing.*

*Proof.* The proof is a consequence of Corollary 1.3. □

*Example 1.9.* If we set

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \cdots < i_k\right\}, \quad 1 \leq k \leq n, \tag{1.46}$$

then $\alpha_{I_n,i} = 1$ $(i = 1,\ldots,n)$, that is, (1.6) is satisfied for $k = n$. It comes easily that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots,n)$, $|I_k| = \binom{n}{k}$ $(k = 1,\ldots,n)$, and for each $k = 2,\ldots,n$

$$\left|H_{I_k}(j_1,\ldots,j_{k-1})\right| = n-(k-1), \quad (j_1,\ldots,j_{k-1}) \in I_{k-1}. \tag{1.47}$$

In this case, (1.41) becomes for $n \geq k \geq 1$

$$M^2_{\eta,\gamma}(I_k) = \begin{cases} \left[ \dfrac{1}{\binom{n-1}{k-1}} \displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \sum_{s=1}^{k} p_{i_s} \right) (M_\gamma(I_k, \mathbf{i}^k))^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[ \displaystyle\prod_{1 \leq i_1 < \cdots < i_k \leq n} (M_\gamma(I_k, \mathbf{i}^k))^{(\sum_{s=1}^{k} p_{i_s})} \right]^{1/\binom{n-1}{k-1}}, & \eta = 0, \end{cases} \tag{1.48}$$

and (1.43) has the form

$$M^2_{h,g}(I_k) = h^{-1} \left( \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \sum_{s=1}^{k} p_{i_s} \right) h \circ g^{-1} \left( \frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} \right) \right). \tag{1.49}$$

Equation (1.48) is a weighted mixed symmetric mean and (1.49) is a generalized mean, as given in [2]. Therefore, Corollaries 1.7 and 1.8 are more general than the Corollaries 1.2 and 1.3 given in [2].

Assume $(H_4)$ holds, and consider the set $I_k$ in Example 1.9. Then, Theorem 1.6 shows that

$$\begin{gathered} A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \sum_{s=1}^{k} p_{i_s} \right) f\left( \frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right), \quad k = 1, \ldots, n, \\ f\left( \sum_{r=1}^{n} p_r x_r \right) = A_{n,n} \leq A_{n-1,n-1} \leq \cdots \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r). \end{gathered} \tag{1.50}$$

Thus, we have

$$\Upsilon^9(\mathbf{x}, \mathbf{p}, f) := A_{m,m} - A_{l,l} \geq 0, \quad n \geq l > m \geq 1. \tag{1.51}$$

*Example 1.10.* If we set

$$I_k := \left\{ (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \cdots \leq i_k \right\}, \quad k \geq 1, \tag{1.52}$$

then $\alpha_{I_k,i} \geq 1$ $(i = 1, \ldots, n)$ and thus (1.6) is satisfied. It is easy to see that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots)$, $|I_k| = \binom{n+k-1}{k}$ $(k = 1, \ldots)$, and for each $l = 2, \ldots, k$

$$\left| H_{I_l}(j_1, \ldots, j_{l-1}) \right| = n, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1}. \tag{1.53}$$

Under these settings (1.41) becomes

$$M_{\eta,\gamma}^{2}(I_k) = \begin{cases} \left[ \dfrac{1}{\binom{n+k-1}{k-1}} \displaystyle\sum_{1\le i_1\le\cdots\le i_k\le n} \left( \sum_{s=1}^{k} p_{i_s} \right) \left( M_\gamma(I_k,\mathbf{i}^k) \right)^{\eta} \right]^{1/\eta}, & \eta \neq 0, \\ \left[ \displaystyle\prod_{1\le i_1\le\cdots\le i_k\le n} \left( M_\gamma(I_k,\mathbf{i}^k) \right)^{(\sum_{s=1}^{k} p_{i_s})} \right]^{1/\binom{n+k-1}{k-1}}, & \eta = 0, \end{cases} \tag{1.54}$$

and (1.43) has the form

$$M_{h,g}^{2}(I_k) = h^{-1}\left( \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\cdots\le i_k\le n} \left( \sum_{s=1}^{k} p_{i_s} \right) h \circ g^{-1}\left( \frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} \right) \right). \tag{1.55}$$

Equation (1.54) represents weighted mixed symmetric means, and (1.55) defines generalized means, as given in [2]. Therefore, Corollaries 1.7 and 1.8 are more general than the Corollaries 1.9 and 1.10 given in [2].

Assume ($H_4$) holds, and consider the set $I_k$ in Example 1.10. Then, it follows from Theorem 1.6 that

$$f\left( \sum_{r=1}^{n} p_r x_r \right) \le \cdots \le A_{k,k} \le \cdots \le A_{1,1} = \sum_{r=1}^{n} p_r f(x_r), \tag{1.56}$$

where

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\cdots\le i_k\le n} \left( \sum_{s=1}^{k} p_{i_s} \right) f\left( \frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right), \quad k \ge 1. \tag{1.57}$$

This yields that

$$\begin{gathered} \Upsilon^{10}(\mathbf{x},\mathbf{p},f) = A_{k,k} - A_{l,l} \ge 0, \quad l > k \ge 1, \\ \Upsilon^{11}(\mathbf{x},\mathbf{p},f) := A_{k,k} - f\left( \sum_{r=1}^{n} p_r x_r \right) \ge 0, \quad k \ge 1. \end{gathered} \tag{1.58}$$

*Example 1.11.* We set

$$I_k := \{1,\ldots,n\}^k, \quad k \ge 1. \tag{1.59}$$

Then, $\alpha_{I_k,i} \ge 1$ ($i = 1,\ldots,n$), hence (1.6) holds. It is not hard to see that $T_k(I_k) = I_{k-1}$ ($k = 2,\ldots$), $|I_k| = n^k$ ($k = 1,\ldots$), and for each $l = 2,\ldots,k$,

$$\left| H_{I_l}(j_1,\ldots,j_{l-1}) \right| = n^l, \quad (j_1,\ldots,j_{l-1}) \in I_{l-1}. \tag{1.60}$$

Therefore, under these settings, for $k \geq 1$, (1.41) leads to

$$M^2_{\eta,\Upsilon}(I_k) = \begin{cases} \left[ \dfrac{1}{kn^{k-1}} \displaystyle\sum_{\mathbf{i}^k=(i_l,\ldots,i_k)\in I_k} \left( \sum_{s=1}^{k} p_{i_s} \right) \left( M_\Upsilon(I_k, \mathbf{i}^k) \right)^{\eta} \right]^{1/\eta}, & \eta \neq 0, \\ \left[ \displaystyle\prod_{\mathbf{i}^k=(i_l,\ldots,i_k)\in I_k} \left( M_\Upsilon(I_k, \mathbf{i}^k) \right)^{(\sum_{s=1}^{k} p_{i_s})} \right]^{1/kn^{k-1}}, & \eta = 0, \end{cases} \tag{1.61}$$

and (1.43) gives

$$M^2_{h,g}(I_k) = h^{-1}\left( \frac{1}{kn^{k-1}} \sum_{\mathbf{ik}=(i_1,\ldots,i_k)\in I_k} \left( \sum_{s=1}^{k} p_{i_s} \right) h \circ g^{-1}\left( \frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} \right) \right), \tag{1.62}$$

respectively.

Assume ($H_4$) holds, and consider the set $I_k$ in Example 1.11. Then, Theorem 1.6 implies that

$$f\left( \sum_{r=1}^{n} p_r x_r \right) \leq \cdots \leq A_{k,k} \leq \cdots \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r), \tag{1.63}$$

where

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k)\in I_k} \left( \sum_{s=1}^{k} p_{i_s} \right) f\left( \frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right), \quad k \geq 1. \tag{1.64}$$

Therefore, we have

$$\begin{gathered} \Upsilon^{12}(\mathbf{x}, \mathbf{p}, f) = A_{k,k} - A_{l,l} \geq 0, \quad l > k \geq 1, \\ \Upsilon^{13}(\mathbf{x}, \mathbf{p}, f) := A_{k,k} - f\left( \sum_{r=1}^{n} p_r x_r \right) \geq 0, \quad k \geq 1. \end{gathered} \tag{1.65}$$

*Example 1.12.* Let $1 \leq k \leq n$ and let $I_k$ consist of all sequences $(i_1, \ldots, i_k)$ of $k$ distinct numbers from $\{1, \ldots, n\}$. Then, $\alpha_{I_n,i} \geq 1$ $(i = 1, \ldots, n)$, hence (1.6) holds. It is immediate that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots)$, $|I_k| = n(n-1)\cdots(n-k+1)$ $(k = 1, \ldots, n)$, and for every $k = 2, \ldots, n$,

$$\left| H_{I_k}(j_1, \ldots, j_{k-1}) \right| = (n-k+1)k, \quad (j_1, \ldots, j_{k-1}) \in I_{k-1}. \tag{1.66}$$

Therefore under these settings, for $k = 1, \ldots, n$, (1.41) gives

$$M^2_{\eta,\Upsilon}(I_k) = \begin{cases} \left[ \dfrac{n}{kn(n-1)\cdots(n-k+1)} \displaystyle\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) \left(M_\Upsilon(I_k, \mathbf{i}^k)\right)^\eta \right]^{1/\eta}, & \eta \neq 0, \\ \left[ \displaystyle\prod_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k} \left(M_\Upsilon(I_k, \mathbf{i}^k)\right)^{(\sum_{s=1}^{k} p_{i_s})} \right]^{n/kn(n-1)\cdots(n-k+1)}, & \eta = 0, \end{cases} \tag{1.67}$$

and (1.43) has the form

$$M^2_{h,g}(I_k) = h^{-1}\left( \frac{n}{kn(n-1)\cdots(n-k+1)} \sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) h \circ g^{-1}\left( \frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} \right) \right), \tag{1.68}$$

respectively.

Assume $(H_4)$ holds, and consider the set $I_k$ in Example 1.12. Then, Theorem 1.6 yields that

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{n,n} \leq \cdots \leq A_{k,k} \leq \cdots \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r), \tag{1.69}$$

where for $k = 1, \ldots, n$,

$$A_{k,k} = \frac{n}{kn(n-1)\cdots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left( \frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}} \right). \tag{1.70}$$

Therefore, we have

$$\begin{aligned} \Upsilon^{14}(\mathbf{x}, \mathbf{p}, f) &= A_{m,m} - A_{l,l} \geq 0, \quad n \geq l > m \geq 1, \\ \Upsilon^{15}(\mathbf{x}, \mathbf{p}, f) &= A_{l,l} - f\left(\sum_{r=1}^{n} p_r x_r\right) \geq 0, \quad n \geq l \geq 1. \end{aligned} \tag{1.71}$$

## 2. Main Results

We have seen that

$$\Upsilon^i(\mathbf{x}, \mathbf{p}, f) \geq 0, \quad i = 1, \ldots, 15. \tag{2.1}$$

From now on, $(H_1)$ and $(H_4)$ are assumed if we consider $\Upsilon^i(\mathbf{x}, \mathbf{p}, f)$ $(i = 1, 2)$, further, the hypothesis $|H_{I_l}(j_1, \ldots, j_{l-1})| = \beta_{l-1}$ for any $(j_1, \ldots, j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$ is also assumed if we consider $\Upsilon^i(\mathbf{x},\mathbf{p},f)$ $(i=7,8)$. The numbers $\Upsilon^i(\mathbf{x},\mathbf{p},f)$ $(i=3,\ldots,6,9,\ldots,15)$ are generated by concrete examples, and $(H_4)$ is assumed.

We need the following subclass of convex functions (see [3]).

*Definition 2.1.* A function $f:(a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n}\xi_i\xi_j f\left(x_i+x_j\right)\geq 0, \tag{2.2}$$

for all $n\in\mathbb{N}$ and all choices $\xi_i\in\mathbb{R}$ and $x_i+x_j\in(a,b)$ $(1\leq i,j\leq n)$.

We quote here useful propositions from [3].

**Proposition 2.2.** *Let $f:(a,b)\to\mathbb{R}$ be a function. Then, the following statements are equivalent*

(i) *$f$ is exponentially convex.*

(ii) *$f$ is continuous and*

$$\sum_{i,j=1}^{n}\xi_i\xi_j f\left(\frac{x_i+x_j}{2}\right)\geq 0, \tag{2.3}$$

*for every $\xi_i\in\mathbb{R}$ and every $x_i\in(a,b)$ $(1\leq i\leq n)$.*

**Proposition 2.3.** *If $f:(a,b)\to(0,\infty)$ is an exponentially convex function, then $f$ is log-convex which means that for every $x,y\in(a,b)$ and all $\lambda\in[0,1]$*

$$f(\lambda x+(1-\lambda)y)\leq f(x)^{\lambda}f(y)^{1-\lambda}. \tag{2.4}$$

First, we introduce a special class of functions.

$(H_5)$ Let $\{f_s:(c,d)(\subset\mathbb{R})\to\mathbb{R}\mid s\in(a,b)\subset\mathbb{R}\}$ be a set of twice differentiable convex functions such that the function $s\to f_s''(x)s\in(a,b)$ is exponentially convex for every fixed $x\in(c,d)$.

As examples, consider two classes of functions $\varphi_s:(0,\infty)\to\mathbb{R}$ defined by

$$\varphi_s(x)=\begin{cases}\dfrac{x^s}{s(s-1)}, & s\neq 0,\\ -\log x, & s=0,\\ x\log x & s=1,\end{cases} \tag{2.5}$$

and $\phi_s:\mathbb{R}\to[0,\infty)$ defined by

$$\phi_s(x)=\begin{cases}\dfrac{1}{s^2}e^{sx}, & s\neq 0,\\ \dfrac{1}{2}x^2, & s=0.\end{cases} \tag{2.6}$$

It is easy to see that the sets of functions $\{\varphi_s\mid s\in\mathbb{R}\}$ and $\{\phi_s\mid s\in\mathbb{R}\}$ satisfy $(H_5)$.

Assume ($H_5$). If $f$ is replaced by $f_s$ in (2.1), we obtain

$$\widehat{\Upsilon}^i_s := \Upsilon^i(\mathbf{x},\mathbf{p},f_s) \geq 0, \quad s \in (a,b),\ i = 1,\ldots,15. \tag{2.7}$$

Especially,

$$\Upsilon^i_s := \Upsilon^i(\mathbf{x},\mathbf{p},\varphi_s) \geq 0, \quad s \in \mathbb{R},\ i = 1,\ldots,15. \tag{2.8}$$

In this paper we prove the exponential convexity of the functions $s \to \widehat{\Upsilon}^i_s$ $(s \in (a,b))$, and we give mean value theorems for $\Upsilon^i(\mathbf{x},\mathbf{p},f)$ $(i = 1,\ldots,15)$. We also define the respective means of Cauchy type and study their monotonicity. The results for $\Upsilon^i$ $(i = 3,\ldots,6)$ are special cases of the results for $\Upsilon^i$ $(i = 1,2)$, and the results for $\Upsilon^i$ $(i = 9,\ldots,15)$ are special cases of results for $\Upsilon^i$ $(i = 7,8)$. Especially, the results for $\Upsilon^i$ $(i = 9,10,11)$ are also given in [2].

**Theorem 2.4.** *Assume ($H_5$), and suppose that the functions* $s \mapsto \widehat{\Upsilon}^i_s (s \in (a,b))$ *are continuous. The following statements hold for* $\widehat{\Upsilon}^i_s$ $(i = 1,\ldots,15)$.

(a) *For every* $q \in \mathbb{N}$ *and* $s_1,\ldots,s_q \in (a,b)$, *the matrix* $[\widehat{\Upsilon}^i_{(s_l+s_m)/2}]^q_{l,m=1}$ *is positive semidefinite. Particularly,*

$$\det \left[\widehat{\Upsilon}^i_{(s_l+s_m)/2}\right]^k_{l,m=1} \geq 0, \quad \textit{for } k = 1,2,\ldots,q. \tag{2.9}$$

(b) *The function* $s \mapsto \widehat{\Upsilon}^i_s$ $(s \in (a,b))$ *is exponentially convex.*

*Proof.* Fix $1 \leq i \leq 15$.

(a) Let $u_l \in \mathbb{R}$ $(l = 1,\ldots,q)$, and define the functions $\mu_k : (0,\infty) \to \mathbb{R}$ by $\mu_k(x) := \sum_{l,m=1}^k u_l u_m f_{s_{lm}}(x)$ for $k = 1,\ldots,q$, where $s_{lm} = (s_l + s_m)/2$ $(1 \leq l,m \leq q)$. Then $\mu_k$ $(k = 1,\ldots,q)$ is a convex function since

$$\mu_k''(x) = \sum_{l,m=1}^k u_l u_m f''_{s_{lm}}(x) \geq 0, \quad x \in (c,d). \tag{2.10}$$

By taking $f = \mu_k$ in (2.1), we have

$$\sum_{l,m=1}^k u_l u_m \widehat{\Upsilon}^i_{s_{lm}} \geq 0, \quad k = 1,\ldots,q. \tag{2.11}$$

This means that the matrix $[\widehat{\Upsilon}^i_{(s_l+s_m)/2}]^q_{l,m=1}$ is positive semidefinite, that is, (2.9) is valid.

(b) It is assumed that the function $s \mapsto \widehat{\Upsilon}^i_s$ $(s \in (a,b))$ is continuous. By using Poposition 2.2 and (a), we get the exponential convexity of the function $s \mapsto \widehat{\Upsilon}^i_s$ $(s \in (a,b))$. □

Since the functions $s \mapsto \Upsilon_s^i$ $(s \in \mathbb{R})$ are continuous $(i = 1, \ldots, 15)$, we have the following.

**Corollary 2.5.** *The function* $s \mapsto \Upsilon_s^i$ $(s \in \mathbb{R},\ i = 1, \ldots, 15)$ *are exponentially convex. This remains valid if we replace* $\varphi_s$ *by* $\phi_s$ *in* (2.8).

## 3. Cauchy Means

In this section, first, we are interested in mean value theorems.

**Theorem 3.1.** *Assume* $f \in C^2[a,b]$ *and* $\Upsilon^i(\mathbf{x}, \mathbf{p}; x^2) \neq 0$ $(i = 1, \ldots, 15)$. *Then, there exists* $\xi_i \in [a,b]$ *such that*

$$\Upsilon^i(\mathbf{x}, \mathbf{p}, f) = \frac{1}{2} f''(\xi_i) \Upsilon^i\left(\mathbf{x}, \mathbf{p}, x^2\right), \quad i = 1, \ldots, 15. \tag{3.1}$$

**Theorem 3.2.** *Assume* $f, g \in C^2[a,b]$. *Then, there exists* $\xi_i \in [a,b]$ *such that*

$$\frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)} = \frac{f''(\xi_i)}{g''(\xi_i)}, \quad i = 1, \ldots, 15, \tag{3.2}$$

*provided that the denominators are nonzero.*

The idea of the proofs of Theorems 3.1 and 3.2 is the same as the proofs of Theorems 2.3 and 2.4 in [2].

**Corollary 3.3.** *Let* $f, g : [a,b](\subset (0,\infty)) \to \mathbb{R}$, $f(x) = x^p$ *and* $g(x) = x^q$. *Then, for distinct real numbers* $p$ *and* $q$, *different from 0 and 1, there exists* $\xi_i \in [a,b]$ *such that*

$$\xi_i^{p-q} = \frac{q(q-1)}{p(p-1)} \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)}, \quad i = 1, \ldots, 15. \tag{3.3}$$

*Proof.* Theorem 3.2 can be applied. □

*Remark 3.4.* Suppose the conditions of Corollary 3.3 are satisfied.

(a) Since the function $\xi \to \xi^{p-q}$ $(\xi \in (0,\infty))$, $p \neq q$ is invertible, then we get, from (3.3), that for $i = 1, \ldots, 15$

$$a \leq \left( \frac{q(q-1)}{p(p-1)} \frac{\Upsilon^i(\mathbf{x}, \mathbf{p}, f)}{\Upsilon^i(\mathbf{x}, \mathbf{p}, g)} \right)^{1/(p-q)} \leq b. \tag{3.4}$$

(b) By choosing $a := \min_{1 \leq i \leq n} x_i$ and $b := \max_{1 \leq i \leq n} x_i$, we can see that the expression between $a$ and $b$ in (3.4) defines a mean.

**Corollary 3.5.** *Assume* ($H_1$) *and* ($H_2$), *and suppose* $x_i \in [a,b] \subset (0,\infty)$ $(1 \leq i \leq n)$. *In* (3.6), *it is also supposed that* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}(k \geq l \geq 2)$. *Then, for distinct real numbers* $p$, $q$, *and* $r$, *all are different from 0 and 1, there exists* $\xi_1, \xi_2 \in [a,b]$, *such that*

$$\xi_1^{p-q} = \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^1(I_k,l-1)\right)^p - \left(M_{p,r}^1(I_k,l)\right)^p}{\left(M_{q,r}^1(I_k,l-1)\right)^q - \left(M_{q,r}^1(I_k,l)\right)^q}, \tag{3.5}$$

$$\xi_2^{p-q} = \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^2(I_{l-1})\right)^p - \left(M_{p,r}^2(I_l)\right)^p}{\left(M_{q,r}^2(I_{l-1})\right)^q - \left(M_{q,r}^2(I_l)\right)^q}. \tag{3.6}$$

*Proof.* We can apply Theorem 3.2 to the functions $f, g : [a,b] \to \mathbb{R}$, $f(x) = x^{p/r}$, and $g(x) = x^{q/r}$, and the $n$-tuples $(x_1^r,\ldots,x_n^r)$. □

*Remark 3.6.* Suppose the conditions of Corollary 3.5 are satisfied.

(a) Since the function $\xi \to \xi^{p-q}$ $(\xi \in (0,\infty))$ is invertible, then we get, from (3.5) and (3.6) that

$$\begin{gathered} a \leq \left( \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^1(I_k,l-1)\right)^p - \left(M_{p,r}^1(I_k,l)\right)^p}{\left(M_{q,r}^1(k,l-1)\right)^q - \left(M_{q,r}^1(k,l)\right)^q} \right)^{1/(p-q)} \leq b, \\ a \leq \left( \frac{q(q-r)}{p(p-r)} \frac{\left(M_{p,r}^2(I_{l-1})\right)^p - \left(M_{p,r}^2(I_l)\right)^p}{\left(M_{q,r}^2(I_{l-1})\right)^q - \left(M_{q,r}^2(I_l)\right)^q} \right)^{1/(p-q)} \leq b. \end{gathered} \tag{3.7}$$

(b) As in Remark 3.4 (b), the expressions in (3.7) define means.

By Remark 3.4 (b), we can define Cauchy means for $p,\ q \in \mathbb{R}$ as follows:

$$M_{p,q}^i := \left( \frac{\Upsilon^i(\mathbf{x},\mathbf{p},\varphi_p)}{\Upsilon^i(\mathbf{x},\mathbf{p},\varphi_q)} \right)^{1/(p-q)}, \quad p \neq q,\ i = 1,\ldots,15. \tag{3.8}$$

Moreover, we have continuous extensions of these means in other cases. By taking the limit, we have

$$\begin{gathered} M_{q,q}^i := \exp\left( \frac{1-2q}{q(q-1)} - \frac{\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_q\varphi_0)}{\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_q)} \right), \quad q \neq 0,1, \\ M_{1,1}^i := \exp\left( -1 - \frac{\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_0\varphi_1)}{2\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_1)} \right), \\ M_{0,0}^i := \exp\left( 1 - \frac{\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_0^2)}{2\Upsilon^i(\mathbf{x},\mathbf{p};\varphi_0)} \right). \end{gathered} \tag{3.9}$$

Now, we deduce the monotonicity of these means in the form of Dresher's inequality as follows.

**Theorem 3.7.** *Let $p, q, u, v \in \mathbb{R}$ such that $p \leq v,\ q \leq u$. Then*

$$M^i_{p,q} \leq M^i_{v,u}, \quad i = 1, \dots, 15. \tag{3.10}$$

*Proof.* Fix $1 \leq i \leq 15$. Corollary 2.5 shows that the function $p \to \Upsilon^i_p$ ($p \in \mathbb{R}$) is exponentially convex, and hence, by Proposition 2.3, it is log-convex. Therefore, the function $p \to \log(\Upsilon^i_p)$ ($p \in \mathbb{R}$) is convex, which implies (see [4]) that

$$\frac{\log \Upsilon^i_p - \log \Upsilon^i_q}{p - q} \leq \frac{\log \Upsilon^i_v - \log \Upsilon^i_u}{v - u}. \tag{3.11}$$

This gives (3.10) if $p \neq q$ and $v \neq u$. The other cases come from this by taking limit. □

By Remark 3.6 (b), we can define Cauchy means in the following form:

$$M^i_{p,q;r} := \left( \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{p/r})}{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r})} \right)^{1/(p-q)}, \quad p \neq q,\ i = 1, 2, \tag{3.12}$$

where $\mathbf{x}^r = (x_1^r, \dots, x_n^r)$. By taking the limit, we have

$$\begin{aligned}
M^i_{q,q;r} &:= \exp\left( \frac{(r-2q)}{q(q-r)} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r}\varphi_0)}{r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_{q/r})} \right), \quad q(q-r) \neq 0,\ r \neq 0, \\
M^i_{0,0;r} &= \exp\left( \frac{1}{r} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0^2)}{2r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0)} \right), \quad r \neq 0, \\
M^i_{r,r;r} &= \exp\left( \frac{-1}{r} - \frac{\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_0\varphi_1)}{2r\Upsilon^i(\mathbf{x}^r, \mathbf{p}; \varphi_1)} \right), \quad r \neq 0, \\
M^i_{q,q;0} &= \exp\left( \frac{-2}{q} + \frac{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; x\phi_q)}{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; \phi_q)} \right), \quad q \neq 0, \\
M^i_{0,0;0} &= \exp\left( \frac{\Upsilon^i(\log \mathbf{x}, \mathbf{p}; x\phi_0)}{3\Upsilon^i(\log \mathbf{x}, \mathbf{p}; \phi_0)} \right),
\end{aligned} \tag{3.13}$$

where $\log \mathbf{x} = (\log x_1, \dots, \log x_n)$.

Now, we give the monotonicity of these new means.

**Theorem 3.8.** *Let $p, q, u, v \in \mathbb{R}$ such that $p \leq v,\ q \leq u$. Then,*

$$M^i_{p,q;r} \leq M^i_{v,u;r}, \quad i = 1, 2. \tag{3.14}$$

*Proof.* Suppose $i = 1, 2$ is fixed. The function $p \to \Upsilon_p^i$ ($p \in \mathbb{R}$) is exponentially convex, and hence, by Proposition 2.3, it is log-convex. Therefore, the function $p \to \log(\Upsilon_p^i)$ ($p \in \mathbb{R}$) is convex, which implies (as in the proof of Theorem 3.7) that

$$\left(\frac{\Upsilon_p^i}{\Upsilon_q^i}\right)^{1/(p-q)} \leq \left(\frac{\Upsilon_v^i}{\Upsilon_u^i}\right)^{1/(v-u)}. \tag{3.15}$$

If $r \neq 0$, set $x_i := x_i^r,\ p := p/r,\ q := q/r,\ u := u/r,\ v := v/r$ in (3.15) to obtain (3.14).

For $r = 0$, we get the required result by limit. □


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