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Research Article

On Strong Law of Large Numbers for Dependent Random Variables

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We discuss strong law of large numbers and complete convergence for sums of uniformly bounded negatively associate (NA) random variables (RVs). We extend and generalize some recent results. As corollaries, we investigate limit behavior of some other dependent random sequence.

1. Introduction

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integer, let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables defined on probability space $(\Omega, \mathcal{F}, P)$, and put $S_n = \sum_{k=1}^{n} X_k$. The symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

In [1], Jajte studied a large class of summability method as follows: a sequence $\{X_n, n \geq 1\}$ is summable to X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k}{h(k)} \longrightarrow X, \quad \text{as } n \longrightarrow \infty. \tag{1.1}$$

The main result of Jajte is as follows.

Theorem 1.1. *Let $g(\cdot)$ be a positive, increasing function and $h(\cdot)$ a positive function such that $\phi(y) \equiv g(y)h(y)$ satisfies the following conditions.*

(1) *For some $d \geq 0$, $\phi(\cdot)$ is strictly increasing on $[d, +\infty)$ with range $[0, +\infty)$.*

(2) *There exist C and a positive integer k_0 such that $\phi(y+1)/\phi(y) \leq C$, $y \geq k_0$.*

(3) *There exist constants a and b such that $\phi^2(s) \int_s^{\infty} (1/\phi^2(x))dx \leq as + b$, $s > d$.*

Then, for i.i.d. random variables $\{X_n,\ n \in \mathbb{N}\}$,

$$\frac{1}{g(n)}\sum_{k=1}^{n}\frac{X_k - EX_k\mathbf{1}_{[|X_k|\le\phi(k)]}}{h(k)} \longrightarrow 0 \quad \text{a.s. iff } E\Big[\phi^{-1}(|X|)\Big] < \infty, \tag{1.2}$$

where ϕ^{-1} *is the inverse of function* ϕ, $\mathbf{1}_A$ *is the indicator of event* A.

Motivated by Jajte [1], the present paper is devoted to the study of the limiting behavior of sums when $\{X, X_n,\ n \in \mathbb{N}\}$ are dependent RVs In particular, we willl consider the case when $\{X, X_n,\ n \in \mathbb{N}\}$ are NA RVs and obtain some general results on the complete convergence of dependent RVs First, we shall give some definitions.

Definition 1.2. A finite family of random variables $\{X_n,\ 1 \le i \le n\}$ is said to be negatively associated (abbreviated NA) if, for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$, we have

$$\operatorname{Cov}\big(f_1(X_i,\ i \in A), f_2\big(X_j,\ j \in B\big)\big) \le 0, \tag{1.3}$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists.

Definition 1.3. A finite family of random variables $\{X_n,\ 1 \le i \le n\}$ is said to be positively associated (abbreviated PA) if

$$\operatorname{Cov}\big(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)\big) \ge 0, \tag{1.4}$$

whenever f and g are coordinatewise increasing and the covariance exists.

An infinite family of random variables is NA (resp., PA) if every finite subfamily is NA (resp., PA).

Let $\mathcal{F}_m^n$ be the σ-algebra generated by RVs X_i, $m \le i \le n$.

Definition 1.4. A sequence of random variables $\{X_n,\ n \in \mathbb{N}\}$ is said to be m-dependence if $\mathcal{F}_1^r$ and $\mathcal{F}_{r'}^{\infty}$ are independent for all r and r' such that $1 \le r < r' < \infty,\ r' - r > m$.

Definition 1.5. A sequence of random variables $\{X_n,\ n \in \mathbb{N}\}$ is said to be φ-mixing (or uniformly strong mixing), if

$$\varphi(\tau) = \sup_{t\in\mathbb{N}} \sup_{A\in\mathcal{F}_1^t,\ B\in\mathcal{F}_{t+\tau}^{\infty},\ P(A)>0} |P(B \mid A) - P(B)| \longrightarrow 0 \quad \text{as } \tau \longrightarrow \infty. \tag{1.5}$$

These concepts of dependence were introduced by Esary et al. [2] and Joag-Dev and Proschan [3]. Their basic properties may be found in [2, 3] and the references therein.

Definition 1.6. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of random variables which is said to be: uniformly bounded by a random variable X (we write $\{X_n, n \in \mathbb{N}\} \prec X$) if there exists a constant $C > 0$, for almost every $\omega \in \Omega$, such that

$$\sup_{n\geq 1} P\{|X_n| > t\} \leq CP\{|X| > t\} \quad \forall t > 0. \tag{1.6}$$

Remark 1.7. The uniformly bounded random variables in (1.6) can be insured by moment conditions. For example, if

$$\sup_{n} E|X_n|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \tag{1.7}$$

then there exists a uniformly bounded random variable X such that $EX^2 < \infty$.

The structure of this paper is as follows. Some needed technical results will be presented in Section 2. The strong law of large numbers for NA RVs will be established in Section 3. The Spitzer and Hus-Robbins-type law of large numbers will be presented in Sections 4 and 5, respectively.

2. Preliminaries

We now present some terminologies and lemmas. The following six properties are listed for reference in obtaining the main results in the next sections. Detailed proofs can be founded in the cited references.

Lemma 2.1 (cf. [4]) (three-series theorem for NA). *Let $\{X_n, n \in \mathbb{N}\}$ be NA. Let $C > 0$ and let $X_n^C = X_n \mathbf{1}_{[|X_n|\leq C]}$. In order that $\sum_{n=1}^{\infty} X_n$ converges a.s., it is sufficient that*

(1) $\sum_{n=1}^{\infty} P\{|X_n| > C\} < \infty$,

(2) $\sum_{n=1}^{\infty} EX_n^C$ *converges,*

(3) $\sum_{n=1}^{\infty} var(X_n^C) < \infty$.

Lemma 2.2 (cf. [4]). *Let $\{X_n, n \in \mathbb{N}\}$ be NA with $EX_n = 0$, $EX_n^2 < \infty$, then for $p > 2$, for all $\varepsilon > 0$*

$$\begin{gathered} P\left\{\max_{1\leq k\leq n}|S_k| \geq \varepsilon\right\} \leq 2\varepsilon^{-2}\sum_{i=1}^{n} EX_i^2, \\ E\left(\max_{1\leq k\leq n}|S_k|^2\right) \leq C\sum_{i=1}^{n} EX_i^2, \\ E\left(\max_{1\leq k\leq n}|S_k|^p\right) \leq C\left\{\sum_{i=1}^{n} EX_i^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \end{gathered} \tag{2.1}$$

Lemma 2.3 (cf. [5]). *Let $\{X_n, n \in \mathbb{N}\}$ be m-dependence with $EX_n = 0$, $EX_n^2 < \infty$, then*

$$E(S_n)^2 \leq (m+1)\sum_{i=1}^{n} EX_i^2. \tag{2.2}$$

Lemma 2.4 (cf. [5]). *Let $\{X_n, n \in \mathbb{N}\}$ be φ-mixing with $EX_n = 0$, $EX_n^2 < \infty$, then*

$$E(S_n)^2 \leq \left(1 + 4\sum_{r=1}^{n-1} \varphi^{1/2}(r)\right)\sum_{i=1}^{n} EX_i^2. \tag{2.3}$$

Lemma 2.5 (cf. [6]). *Let $\{X_n,\ n \in \mathbb{N}\}$ be PA with $EX_n = 0$, $EX_n^2 < \infty$, then*

$$E\max_{1\leq k\leq n} S_k^2 \leq 2ES_n^2. \tag{2.4}$$

Furthermore, if

$$\sum_{k=1}^{n} \mu^{1/2}(2^n) < \infty, \tag{2.5}$$

then

$$E\max_{1\leq k\leq n} S_k^2 \leq Cn\max_{1\leq k\leq n} EX_k^2, \tag{2.6}$$

where $\mu(n) = \sup_{i\geq 1}\sum_{j:j-i\geq n} \operatorname{cov}(X_i, X_j)$.

Lemma 2.6. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables and X a random variable. If $\{X_n\} \prec X$, then for all $t > 0$, $p \geq 2$*

$$\mathbb{E}X_n^p \mathbf{1}_{[|X_n|\leq t]} \leq C\left[t^p \mathbb{P}\{|X| > t\} + \mathbb{E}X^p \mathbf{1}_{[|X|\leq t]}\right]. \tag{2.7}$$

Proof. By the integral equality

$$p\int_0^t s^{p-1}\mathbb{P}(|X| > s)ds = t^p\mathbb{P}(|X| > t) + \mathbb{E}|X|^p\mathbf{1}_{[|X|\leq t]}, \tag{2.8}$$

it follows that

$$\begin{aligned}\mathbb{E}|X_n|^p\mathbf{1}_{[|X_n|\leq t]} &\leq p\int_0^t s^{p-1}\mathbb{P}(X_n > s)ds \\ &\leq Cp\int_0^t s^{p-1}\mathbb{P}(|X| > s)ds = C\left[t^p\mathbb{P}(|X| > t) + \mathbb{E}|X|^p\mathbf{1}_{[|X|\leq t]}\right].\end{aligned} \tag{2.9}$$

□

Lemma 2.7 (cf. [7]). *Let $\{A_n, n \geq 1\}$ be a sequence of events defined on $(\Omega, \mathcal{F}, P)$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$, if $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $P(A_k \cap A_m) \leq P(A_k)P(A_m)$ for $k \neq m$, then $P(\limsup_n A_n) = 1$.*

3. Strong Law of Large Numbers

Theorem 3.1. *Let $g(\cdot), h(\cdot)$, and $\phi(\cdot)$ be as in Theorem 1.1, and let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of negatively associated random variables with $EX_n = 0$. Assume that $\{X_n, n \in \mathbb{N}\} \prec X$. If $E[\phi^{-1}(|X|)] < \infty$, then*

$$\lim_n \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k}{h(k)} = 0, \quad a.s. \tag{3.1}$$

Conversely, let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of identically distributed NA random variables, if (3.1) *is true, then $E[\phi^{-1}(|X|)] < \infty$.*

Proof. Assume that $E[\phi^{-1}(X)] < \infty$. To prove (3.1) by applying the Kronecker lemma, it suffices to show that

$$\text{the series } \sum_{k=1}^{\infty} \frac{X_k}{\phi(k)} \quad \text{converges a.s.} \tag{3.2}$$

Here, we shall use the three-series theorem for NA RVs.

Let $Y_k = X_k/\phi(k)\mathbf{1}_{[|X_k/\phi(k)| \leq 1]}$. Then, by $E[\phi^{-1}(|X|)] < \infty$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} P\left\{\left|\frac{X_k}{\phi(k)}\right| > 1\right\} &\leq C\sum_{k=1}^{\infty} P\left\{\left|\frac{X}{\phi(k)}\right| > 1\right\} \\ &= C\sum_{k=1}^{\infty} P\left\{\phi^{-1}(|X|) > k\right\} \leq CE\left[\phi^{-1}(|X|)\right] < \infty, \end{aligned} \tag{3.3}$$

which shows that $P\{\{|X_k/\phi(k)| > 1\}, i.o.\} = 0$, and

$$\sum_{k=1}^{\infty} \left|\frac{X_k}{\phi(k)}\right| \mathbf{1}_{[|X_k/\phi(k)| > 1]} < \infty \quad \text{a.s.} \tag{3.4}$$

Therefore, from $EX_k = 0$, it follows that

$$\sum_{k=1}^{\infty} |E(Y_k)| \leq \sum_{k=1}^{\infty} E\left|\frac{X_k}{\phi(k)}\right| \mathbf{1}_{[|X_k/\phi(k)| > 1]} < \infty. \tag{3.5}$$

To this end we estimate the series

$$\begin{aligned}
\sum_{k=1}^{\infty}\operatorname{Var}(Y_k) &\le \sum_{k=1}^{\infty}E\left(Y_k^2\right) = E\sum_{k=1}^{\infty}\frac{X_k^2}{\phi^2(k)}\mathbf{1}_{[|X_k/\phi(k)|\le 1]} \\
&\le C\sum_{k=1}^{\infty}\left[E\mathbf{1}_{[|X|>\phi(k)]} + E\frac{X^2}{\phi^2(k)}\mathbf{1}_{[|X/\phi(k)|\le 1]}\right] \\
&\le C\sum_{k=1}^{\infty}P\{|X|>\phi(k)\} + C\left[k_0 + C\sum_{k=k_0+1}^{\infty}\frac{EX^2}{\phi^2(k+1)}\mathbf{1}_{[|X|\le\phi(k)]}\right] \\
&\le CE\left[\phi^{-1}(|X|)\right] + Ck_0 + CE\left[X^2\int_{\phi^{-1}(|X|)}^{\infty}\frac{1}{\phi^2(x)}dx\right] \\
&\le CE\left[\phi^{-1}(|X|)\right] + Ck_0 + CaE\left[\phi^{-1}(|X|)\right] + Cb < \infty.
\end{aligned} \tag{3.6}$$

Conversely, since $\{X, X_n, n\in\mathbb{N}\}$ are identically NA RVs. If (3.1) holds, that is,

$$\frac{1}{\phi(n)}\sum_{k=1}^{n}X_k = o(1), \quad \text{a.s.} \tag{3.7}$$

It follows that

$$\frac{X_n}{\phi(n)} = \frac{\sum_{k=1}^{n}X_k}{\phi(n)} - \frac{\phi(n-1)}{\phi(n)}\frac{\sum_{k=1}^{n-1}X_k}{\phi(n-1)} \longrightarrow o(1) \quad \text{a.s.} \tag{3.8}$$

which shows that $\phi^{-1}(n)X_n \to 0$ a.s. Hence,

$$\phi^{-1}(n)X_n^{\pm} \longrightarrow 0 \quad \text{a.s.,} \tag{3.9}$$

where $x^+ = \max(0,x)$ and $x^- = \max(0,-x)$.

Since $\{X_n^{\pm}, n\in\mathbb{N}\}$ is still an NA sequence. Defining the following events,

$$A_n = \left\{X_n^+ > \frac{\varepsilon\phi(n)}{3}\right\}, \qquad B_n = \left\{X_n^- > \frac{\varepsilon\phi(n)}{3}\right\}, \tag{3.10}$$

we have $P(A_k\cap A_l)\le P(A_k)P(A_l)$, and $P(B_k\cap B_l)\le P(B_k)P(B_l)$, for $k\ne l$. By Lemma 2.7, if $\phi^{-1}(n)X_n^{\pm} \to 0$ a.s., then $\sum_{n=1}^{\infty}P\{X_n^+ > (1/3)\varepsilon\phi(n)\} < \infty$ and $\sum_{n=1}^{\infty}P\{X_n^- > (1/3)\varepsilon\phi(n)\} < \infty$. Therefore,

$$\sum_{n=1}^{\infty}P\{|X_n| > \varepsilon\phi(n)\} \le \sum_{n=1}^{\infty}P\left\{X_n^+ > \frac{1}{3}\varepsilon\phi(n)\right\} + \sum_{n=1}^{\infty}P\left\{X_n^- > \frac{1}{3}\varepsilon\phi(n)\right\} < \infty, \quad \forall\varepsilon>0, \tag{3.11}$$

which is equivalent to $E[\phi^{-1}(|X|)] < \infty$.

These complete the proof of Theorem 3.1. □

Theorem 3.1 also includes a particular case of logarithmic means, we can establish the following.

Corollary 3.2. *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of NA RVs with $EX_n = 0$ and $\{X_n, n \in \mathbb{N}\} \prec X$. If $E|X| < \infty$, then, one has*

$$\lim_{n} \frac{1}{\log n} \sum_{k=1}^{n} \frac{X_k}{k} = 0, \quad a.s. \tag{3.12}$$

Proof. Let $h(y) = y$, $g(y) = \log y$, that is, $\phi(y) = y \log y$. In this case, $\phi^{-1}(y) \sim y/\log y$ as $y \to \infty$, therefore $E(|X|^{\alpha}) \leq E[\phi^{-1}(|X|)] \leq E(|X|)$, for $0 < \alpha < 1$. □

Corollary 3.3. *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of NA RVs with $EX_n = 0$ and $\{X_n, n \in \mathbb{N}\} \prec X$. If $E|X| < \infty$, then, for every $a, b \geq 0$, $a + b > 1/2$, one has*

$$\lim_{n} \frac{1}{n^{a}} \sum_{k=1}^{n} \frac{X_k}{k^{b}} = 0, \quad a.s. \tag{3.13}$$

Remark 3.4. As pointed out by Jajte [1], Theorem 3.1 includes several regular summability methods such as (1) the Kolmogorov SLLN $[h(y) = 1,\ g(y) = y]$; (2) the classical MZ SLLN $[h(y) = 1,\ g(y) = y^{1/\alpha},\ 1 \leq \alpha \leq 2]$.

4. Spitzer Type Law of Large Numbers

Since the definition of complete convergence was introduced by Hsu and Robins, there have been many authors who devote themselves to the study of the complete convergence for sums of independent and dependent RVs and obtain a series of elegant results, see [4, 8] and reference therein.

We say that the Hsu-Robbins [9] law of large numbers (LLN) is valid if, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{|S_n| \geq n\varepsilon\} < \infty, \tag{4.1}$$

and the Spitzer [10] LLN is valid if, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| \geq n\varepsilon\} < \infty. \tag{4.2}$$

Theorem 4.1. *Let $\phi(\cdot)$ be as in Theorem 1.1, and let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of NA random variables with $EX_n = 0$. Assume that $\{X_n, n \in \mathbb{N}\} \prec X$. If $E[\phi^{-1}(|X|)] < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| \geq \varepsilon\phi(n)\} < \infty. \tag{4.3}$$

Conversely, let $\{X, X_n, n \in \mathbb{N}\}$ *be a sequence of identically distributed NA random variables, if* (4.3) *is true, then* $E[\phi^{-1}(|X|)] < \infty$ *and* $EX = 0$.

Proof. For $1 \le k \le n$, let $U_k^{(n)} = X_k \mathbf{1}_{[|X_k| \le \phi(n)]}$, $V_k^{(n)} = X_k - U_k^{(n)}$, then for every n

$$|S_n| = \left| \sum_{k=1}^{n} \left(U_k^{(n)} + V_k^{(n)} \right) \right| \le \left| \sum_{k=1}^{n} \left(U_k^{(n)} - EU_k^{(n)} \right) \right| + \left| \sum_{k=1}^{n} EU_k^{(n)} \right| + \left| \sum_{k=1}^{n} V_k^{(n)} \right|. \tag{4.4}$$

Note that

$$\begin{aligned} \{|S_n| \ge \varepsilon\phi(n)\} \subset & \left\{ \left| \sum_{k=1}^{n} \left(U_k^{(n)} - EU_k^{(n)} \right) \right| > \frac{\varepsilon\phi(n)}{2} \right\} \cup \left\{ \left| \sum_{k=1}^{n} V_k^{(n)} \right| > \frac{\varepsilon\phi(n)}{2} \right\} \\ & \cup \left\{ \frac{1}{\phi(n)} \max_{1 \le i \le n} \left| \sum_{k=1}^{n} EU_k^{(n)} \right| \longrightarrow 0 \right\}. \end{aligned} \tag{4.5}$$

For the first term on the RHS of (4.5), by Markov inequality and Lemma 2.2 and (3.6), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \left| \sum_{k=1}^{n} \left(U_k^{(n)} - EU_k^{(n)} \right) \right| > \frac{\varepsilon\phi(n)}{2} \right\} \\ & \le \sum_{n=1}^{\infty} \frac{C}{n\phi^2(n)} \sum_{k=1}^{n} E\left(U_k^{(n)} \right)^2 \quad \left(\text{Since } U_k^{(n)},\ 1 \le k \le n \text{ is also NA} \right) \\ & = C \sum_{n=1}^{\infty} \frac{1}{n\phi^2(n)} \sum_{k=1}^{n} EX_k^2 \mathbf{1}_{[|X_k| \le \phi(n)]} \\ & \le C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \left[E\mathbf{1}_{[|X| \ge \phi(n)]} + \frac{EX^2 \mathbf{1}_{[|X| \le \phi(n)]}}{\phi^2(n)} \right] \\ & = C \sum_{n=1}^{\infty} \left[E\mathbf{1}_{[|X| \ge \phi(n)]} + \frac{EX^2 \mathbf{1}_{[|X| \le \phi(n)]}}{\phi^2(n)} \right] < \infty. \end{aligned} \tag{4.6}$$

For the second term on the RHS of (4.5), since

$$\left\{ \left| \sum_{k=1}^{n} V_k^{(n)} \right| > \frac{\varepsilon\phi(n)}{2} \right\} \subset \bigcup_{k=1}^{n} \{|X_k| > \phi(n)\}, \tag{4.7}$$

hence,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}P\left\{\left|\sum_{k=1}^{n}V_k^{(n)}\right|>\frac{\varepsilon\phi(n)}{2}\right\}
&\le\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}P\{|X_k|>\phi(n)\}\\
&\le\sum_{n=1}^{\infty}\frac{C}{n}\sum_{k=1}^{n}P\{|X|>\phi(n)\}\\
&=C\sum_{n=1}^{\infty}P\{|X|>\phi(n)\}<\infty.
\end{aligned}\tag{4.8}$$

For the third term on the RHS of (4.5), we have, by (3.5),

$$\begin{aligned}
\frac{1}{\phi(n)}\sum_{k=1}^{n}\left|EU_k^{(n)}\right|&=\frac{1}{\phi(n)}\sum_{k=1}^{n}\left|EV_k^{(n)}\right|\quad(\text{By } EX_k=0)\\
&=\frac{\sum_{k=1}^{n}\left|EX_k\mathbf{1}_{[|X_k|>\phi(n)]}\right|}{\phi(n)}\\
&\le\frac{\sum_{k=1}^{n}E|X_k|\mathbf{1}_{[|X_k|>\phi(n)]}}{\phi(n)}\\
&\le CE\left[\phi^{-1}(|X|)\right]<\infty.
\end{aligned}\tag{4.9}$$

Therefore, (4.3) follows.

Conversely, since $\sum_{n=1}^{\infty}(1/n)P\{|S_n|\ge\varepsilon\phi(n)\}<\infty$ imply that $S_n/\phi(n)=o(1)$ a.s., hence from Theorem 3.1, we have $E[\phi^{-1}(|X|)]<\infty$. These complete the proof of Theorem 4.1. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, one has*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left\{\max_{1\le k\le n}|S_k|\ge\varepsilon\phi(n)\right\}<\infty.\tag{4.10}$$

Proof. Denote $\overline{S}_k^{(n)}=\sum_{i=1}^{k}U_k^{(n)}$, and noticing that

$$\left\{\max_{1\le k\le n}|S_k|\ge\varepsilon\phi(n)\right\}\subset\left\{\max_{1\le k\le n}\left|\overline{S}_k^{(n)}\right|\ge\varepsilon\phi(n)\right\}\cup\left\{\bigcup_{k=1}^{n}(|X_k|>\phi(n))\right\},\tag{4.11}$$

hence, similarly to the proof of Theorem 4.1, we obtain (4.10). □

Analogously, we can prove the following corollaries, and omit the details.

Corollary 4.3. *Let* $\{X, X_n,\ n \in \mathbb{N}\}$ *be a sequence of* φ*-mixing random variables with* $EX_n = 0$*. Assume that* $\{X_n,\ n \in \mathbb{N}\} \prec X$*. If* $E[\phi^{-1}(|X|)] < \infty$*, and*

$$\sum_{r=1}^{\infty} \varphi^{1/2}(r) < \infty, \tag{4.12}$$

then (4.3) *holds.*

Corollary 4.4. *Let* $\{X, X_n,\ n \in \mathbb{N}\}$ *be a sequence of* m*-dependent random variables with* $EX_n = 0$*. Assume that* $\{X_n,\ n \in \mathbb{N}\} \prec X$*. If* $E[\phi^{-1}(|X|)] < \infty$*, then* (4.3) *holds.*

Corollary 4.5. *Let* $\{X, X_n,\ n \in \mathbb{N}\}$ *be a sequence of PA random variables with* $EX_n = 0$*. Assume that* $\{X_n,\ n \in \mathbb{N}\} \prec X$*. If* $E[\phi^{-1}(|X|)] < \infty$*, and*

$$\sum_{n=1}^{\infty} \mu^{\frac{1}{2}}(2^n) < \infty, \tag{4.13}$$

then (4.3) *holds.*

5. Hsu-Robbins Type Law of Large Numbers

Theorem 5.1. *Let* $\phi(\cdot)$*, be define as in Theorem 1.1, but the following condition (3) is replaced by*

(3′) *There exist constants* a *such that* $\phi^p(s)\int_s^{\infty}(x^{p/2}/\phi^p(x))dx \le as^{1+p/2}$*,* $s > d$*,* $p > 2$*, and let* $\{X, X_n,\ n \in \mathbb{N}\}$ *be a sequence of NA random variables with* $EX_n = 0$*. Assume that* $\{X_n,\ n \in \mathbb{N}\} \prec X$*. If* $E[\phi^{-1}(|X|)]^{1+p/2} < \infty$*, then for all* $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} P\{|S_n| \ge \varepsilon\phi(n)\} < \infty. \tag{5.1}$$

Proof. From the previous section, we know that to prove Theorem 5.1, we need only to prove the convergence of the following three series.

First, note that $E[\phi^{-1}(|X|)]^{1+p/2} < \infty \Rightarrow E[\phi^{-1}(|X|)]^2 < \infty \Rightarrow nP(|X| > \phi(n)) \to 0$, we have

$$\begin{aligned}
\frac{1}{\phi(n)}\sum_{k=1}^{n}\left|EU_k^{(n)}\right| &= \frac{1}{\phi(n)}\sum_{k=1}^{n}\left|EX_k\mathbf{1}_{[|X_k|\le\phi(n)]}\right| \\
&= \frac{1}{\phi(n)}\sum_{k=1}^{n}\left|EX_k\mathbf{1}_{[|X_k|>\phi(n)]}\right| \quad (\text{Since } EX_k = 0) \\
&\le \frac{1}{\phi(n)}\sum_{k=1}^{n}E|X_k|\mathbf{1}_{[|X_k|>\phi(n)]}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\phi(n)} \sum_{k=1}^{n} \int_0^\infty P\left\{|X_k| \mathbf{1}_{[|X_k| > \phi(n)]} > t\right\} dt \\
&\leq \frac{Cn}{\phi(n)} \cdot \phi(n) P\{|X| > \phi(n)\} + \frac{Cn}{\phi(n)} \int_n^\infty P\left(\phi^{-1}(|X|) > t\right) dt \\
&\leq CnP\{|X| > \phi(n)\} + \frac{Cn}{\phi(n)} \int_n^\infty \frac{E\left[\phi^{-1}(|X|)\right]^2}{t^2} dt \\
&= CnP\{|X| > \phi(n)\} + \frac{CnE\left[\phi^{-1}(|X|)\right]^2}{\phi(n)} \cdot \frac{1}{n} \\
&= CnP\{|X| > \phi(n)\} + \frac{CE\left[\phi^{-1}(|X|)\right]^2}{\phi(n)} \longrightarrow 0.
\end{aligned}
\tag{5.2}
$$

Hence, $(1/\phi(n))\max_j \sum_{k=1}^{j} E|U_k^{(n)}| \to 0$ as $n \to \infty$. Next, by (4.4), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n} V_k^{(n)}\right| > \frac{\varepsilon \phi(n)}{2}\right\} &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\{|X_k| > \phi(n)\} \leq \sum_{n=1}^{\infty} C \sum_{k=1}^{n} P\{|X| > \phi(n)\} \\
&\leq C \sum_{n=1}^{\infty} nP\{|X| > \phi(n)\} < CE\left[\phi^{-1}(|X|)\right]^2 < \infty.
\end{aligned}
\tag{5.3}
$$

Last, from the definition of $U_k^{(n)}$ and the NA's property, we know that $\{U_k^{(n)},\ 1 \leq k \leq n,\ n \geq 1\}$ remains a sequence of NA RVs. By applying Lemma 2.2 and C_r inequality, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n}\left(U_k^{(n)} - EU_k^{(n)}\right)\right| > \frac{\varepsilon \phi(n)}{2}\right\} &\leq \sum_{n=1}^{\infty} \frac{C}{\phi^p(n)} E\left|\sum_{k=1}^{n}\left(U_k^{(n)} - EU_k^{(n)}\right)\right|^p \\
&= \sum_{n=1}^{\infty} \frac{C}{\phi^p(n)} \sum_{k=1}^{n} E\left(U_k^{(n)} - EU_k^{(n)}\right)^p \\
&\quad + \sum_{n=1}^{\infty} \frac{C}{\phi^p(n)} \left[\sum_{k=1}^{n} E\left(U_k^{(n)} - EU_k^{(n)}\right)^2\right]^{p/2} \\
&:= I_1 + I_2.
\end{aligned}
\tag{5.4}
$$

It is easy to see that

$$
\begin{aligned}
I_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{\phi^p(n)} \sum_{k=1}^{n} E\left(U_k^{(n)}\right)^p \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{\phi^p(n)} \left[\sum_{k=1}^{n} \phi^p(n) P(|X| > \phi(n)) + \sum_{k=1}^{n} E|X|^p \mathbf{1}_{[|X| \leq \phi(n)]}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} nP\{|X| > \phi(n)\} + C\sum_{n=1}^{\infty}\frac{n}{\phi^p(n)}E|X|^p\mathbf{1}_{[|X|\le\phi(n)]} \\
&\le CE\Big[\phi^{-1}(|X|)\Big]^2 + \frac{1}{2}k_0(k_0+1)C + C\sum_{k=k_0+1}^{\infty}\frac{(k+1)^{p/2}}{\phi^p(k+1)}E|X|^p\mathbf{1}_{[|X|\le\phi(k)]} \\
&\le CE\Big[\phi^{-1}(|X|)\Big]^{1+p/2} + Ck_0^2 + CE\left[|X|^p\int_{\phi^{-1}(|X|)}^{\infty}\frac{x^{p/2}}{\phi^p(x)}dx\right] \\
&\le CE\Big[\phi^{-1}(|X|)\Big]^{1+p/2} + Ck_0^2 < \infty,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty}\frac{1}{\phi^p(n)}\left[\sum_{k=1}^{n}E\Big(U_k^{(n)}\Big)^2\right]^{p/2} \\
&\le C\sum_{n=1}^{\infty}\frac{1}{\phi^p(n)}\Big[n\phi^2(n)P\{|X| > \phi(n)\} + nEX^2\mathbf{1}_{[|X|\le\phi(n)]}\Big]^{p/2} \\
&\le C\sum_{n=1}^{\infty}\frac{1}{\phi^p(n)}\Big[\phi^p(n)n^{p/2}P\big(|X| > \phi(n)\big)\Big] + C\sum_{n=1}^{\infty}\frac{n^{p/2}}{\phi^p(n)}\Big[EX^2\mathbf{1}_{[|X|\le\phi(n)]}\Big]^{p/2} \\
&\le C\sum_{n=1}^{\infty}n^{p/2}P\{|X| > \phi(n)\} + C\sum_{n=1}^{\infty}\frac{n^{p/2}}{\phi^p(n)}E|X|^p\mathbf{1}_{[|X|\le\phi(n)]} \\
&\le CE\Big[\phi^{-1}(|X|)\Big]^{1+p/2} < \infty.
\end{aligned}
\tag{5.6}
$$

These complete the proof of Theorem 5.1. □

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References

[1] R. Jajte, “On the strong law of large numbers,” *The Annals of Probability*, vol. 31, no. 1, pp. 409–412, 2003.

[2] J. D. Esary, F. Proschan, and D. W. Walkup, “Association of random variables, with applications,” *Annals of Mathematical Statistics*, vol. 38, pp. 1466–1474, 1967.

[3] K. Joag-Dev and F. Proschan, “Negative association of random variables, with applications,” *The Annals of Statistics*, vol. 11, no. 1, pp. 286–295, 1983.

[4] C. Su and Y. B. Wang, “Strong convergence for identically distributed negatively associated sequences,” *Chinese Journal of Applied Probability and Statistics*, vol. 14, no. 2, pp. 131–140, 1998 (Chinese).

[5] J. Sunklodas, “On the law of large numbers for weakly dependent random variables,” *Lietuvos Matematikos Rinkinys*, vol. 44, no. 3, pp. 359–371, 2004.

[6] S. C. Yang, “Moment ineqaulities for partial sums of random variables,” *Science in China Series A*, vol. 44, no. 3, pp. 218–223, 2003.

[7] P. Matuła, “A note on the almost sure convergence of sums of negatively dependent random variables,” *Statistics & Probability Letters*, vol. 15, no. 3, pp. 209–213, 1992.

[8] J. Bingyi and H. Y. Liang, “Strong limit theorems for weighted sums of negatively associated random sequence,” *Journal of Theoretical Probability*, vol. 21, no. 4, pp. 890–909, 2008.

[9] P. L. Hsu and H. Robbins, “Complete convergence and the law of large numbers,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 33, pp. 25–31, 1947.

[10] F. Spitzer, “A combinatorial lemma and its application to probability theory,” *Transactions of the American Mathematical Society*, vol. 82, pp. 323–339, 1956.