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Research Article

A New Proof of Inequality for Continuous Linear Functionals

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Gavrea and Ivan (2010) obtained an inequality for a continuous linear functional which annihilates all polynomials of degree at most $k-1$ for some positive integer k. In this paper, a new functional proof by Riesz representation theorem is provided. Related results and further applications of the inequality are also brought together.

1. Introduction

Let $k \geq 1$ be an integer and $f \in C^k[a,b]$. Denote by $\mathbb{P}_k$ the set of all polynomials of degree not exceeding k. Let $\mathcal{L} : C[a,b] \to \mathbb{R}$ be a continuous linear functional which annihilates all polynomials of degree at most $k-1$; that is,

$$\mathcal{L}(f) = 0, \quad \forall f \in \mathbb{P}_{k-1}. \tag{1.1}$$

It is well known that a continuous linear functional is bounded, and finding the bound or norm of a continuous linear functional is a fundamental task in functional analysis. Recently, in light of the Taylor formula and the Cauchy-Schwarz inequality, Gavrea and Ivan in [1] obtained an inequality for the continuous linear functional $\mathcal{L}$ satisfying (1.1). In order to state their result, we need some more symbols. Recall that the L_2 norm of a square integrable function f on $[a,b]$ is defined by

$$\|f\|_{L_2[a,b]} = \left(\int_a^b |f(x)|^2 \mathrm{d}x\right)^{1/2}, \tag{1.2}$$

and denote by $t_+ := \max\{t, 0\}$ the truncated power function. The notation $\mathcal{L}_t(f(t,s))$ means that the functional $\mathcal{L}$ is applied to f considered as a function of t. The main result of [1] can now be stated as follows.

Theorem 1.1. *The functional $\mathcal{L}$ satisfies the following inequality:*

$$|\mathcal{L}(f)| \leq M_k \left\| f^{(k)} \right\|_{L_2[a,b]}, \tag{1.3}$$

where

$$M_k = \sqrt{\frac{(-1)^k}{(2k-1)!} \mathcal{L}_s \mathcal{L}_t (t-s)_+^{2k-1}} \tag{1.4}$$

are the best possible constants. The equality is attained if and only if f is of the form

$$f(s) = C\left(\mathcal{L}_t(t-s)_+^{k-1}\right)^{(-k)}, \quad s \in [a,b], \tag{1.5}$$

where C is an arbitrary constant and the symbol $(-k)$ denotes a kth antiderivative of f.

Remark 1.2. Usually, the functional $\mathcal{L}$ is allowed an interchange with the integral (this is silently assumed throughout this paper). This is true in most interesting cases when, for example, $\mathcal{L}$ is an integral or a derivative or a linear combination of them. If the interchange is permitted, then it is easily verified

$$\left(\mathcal{L}_t(t-s)_+^{k-1}\right)^{(-k)} = \frac{(-1)^k (k-1)!}{(2k-1)!} \mathcal{L}_t (t-s)_+^{2k-1} + p(s), \quad p \in \mathbb{P}_{k-1}. \tag{1.6}$$

It should be pointed out that the inequality (1.3) can be found in Wang and Han [2, Lemma 1] (see also [3]). In this note, we will give a short account of historical background on inequality (1.3). A new functional proof based on the Riesz representation theorem [4, 5] is also given. Furthermore, some related results are brought together, and further applications are also included.

2. Historical Background

It is well known that a Hilbert space can be given a Gaussian measure. Let H be a Hilbert space equipped with Gaussian measure and $\mathcal{L}$ a continuous linear functional acting on H. Smale in [6] (a pioneering work on continuous complexity theory) defined an average (with respect to the Gaussian measure) error for quadrature rules. A result of Smale [6] says that the average error is proportional to $\|\mathcal{L}\|$. More precisely,

$$\operatorname*{Av}_{f \in H} \mathcal{L}(f) = \sqrt{\frac{2}{\pi}} \|\mathcal{L}\|. \tag{2.1}$$

Using (2.1), Smale was able to compute the average error for right rectangle rule, the trapezoidal rule, and Simpson's rule (see [6, Theorem D]).

Later on, Wang and Han in [2] extended and unified results in [6, Theorem D], and they also simplified the corresponding analysis given in [6]. The main observations in [2] are

(i) any quadrature rule has its algebraic precision, or equivalently, the corresponding quadrature error functional annihilates some polynomials,

(ii) and hence the Peano kernel theorem applies.

In fact, more can be stated. The quadrature rule in the above observations can be replaced by any continuous linear one. The main result and its elegant proof deserve to be better known. For reader's convenience, they are recorded here. To do this, we need more notations. For brevity, let $[0,1] = [a,b]$. Denote by L_2 the square integrable functions on $[0,1]$, and

$$\mathscr{H}_k = \left\{ f \in C[0,1] \mid f^{(k-1)} \text{ is absolutely continuous and } f^{(k)} \in L_2 \right\}. \tag{2.2}$$

The inner product on $\mathscr{H}_k$ is defined by

$$\langle f, g \rangle_{\mathscr{H}_k} = \sum_{j=0}^{k-1} f^{(j)}(0) g^{(j)}(0) + \left\langle f^{(k)}, g^{(k)} \right\rangle_{L_2}. \tag{2.3}$$

Then, $\mathscr{H}_k$ is a Hilbert space of functions. The result in [2] can be now stated as follows.

Theorem 2.1. *Let $\mathcal{L} : \mathscr{H}_k \to \mathbb{R}$ be a continuous linear functional satisfying $\mathcal{L}(f) = 0$, for all $f \in \mathbb{P}_{k-1}$. Then,*

$$\|\mathcal{L}\|^2 = \frac{(-1)^k}{(2k-1)!} \mathcal{L}_s \mathcal{L}_t (t-s)_+^{2k-1}. \tag{2.4}$$

It is easily seen that Theorem 1.1 is a rediscovery of Theorem 2.1. For completeness, we record the original short but beautiful proof of (2.4) in [2].

Proof. We have by the Peano kernel theorem

$$\mathcal{L}(f) = \int_0^1 G_k(s) f^{(k)}(s) \mathrm{d}s, \tag{2.5}$$

where

$$G_k(s) = \frac{1}{(k-1)!} \mathcal{L}_t (t-s)_+^{k-1}. \tag{2.6}$$

And hence,

$$\|\mathcal{L}\| = \|G_k\|_{L_2}. \tag{2.7}$$

Obviously,

$$\|G_k\|_{L_2}^2 = \mathcal{L}(f_1), \tag{2.8}$$

where $f_1 \in \mathcal{H}_k$ satisfying

$$f_1^{(k)} = G_k, \quad \text{almost everywhere.} \tag{2.9}$$

Solving the above equality, we have

$$f_1(s) = \frac{(-1)^k}{(2k-1)!}\mathcal{L}_t(t-s)_+^{2k-1} + p(s), \quad p \in \mathbb{P}_{k-1}. \tag{2.10}$$

Applying the functional $\mathcal{L}$ to both sides of (2.10) and noting (2.7) and (2.8), we obtain (2.4) as required. □

3. A Functional Proof

It seems that the original proof of Wang and Han recorded in the previous section does not fully utilize the space $\mathcal{H}_k$. We now provide an alternative functional proof.

First, we define an equivalence relation ~ on $\mathcal{H}_k$ with respect to its subspace $\mathbb{P}_{k-1}$ since $\mathcal{L}$ vanishes on $\mathbb{P}_{k-1}$. We say that $f \sim g$ if $f - g \in \mathbb{P}_{k-1}$. It is easy to check that the quotient space $\mathcal{H}_k/\mathbb{P}_{k-1}$ is still a Hilbert space. For any $f \in \mathcal{H}_k$, there must exist a function $F \sim f$ such that $F^{(j)}(0) = 0$, $j = 0, 1, \ldots, k-1$. For example,

$$F(x) = f(x) - f(0) - \sum_{j=1}^{k-1} \frac{f^{(j)}(0)}{j!} x^j \tag{3.1}$$

may serve this purpose. So, we may assume that $f^{(j)}(0) = 0$, $j = 0, 1, \ldots, k-1$, for any $f \in \mathcal{H}_k/\mathbb{P}_{k-1}$ and, the inner product on $\mathcal{H}_k/\mathbb{P}_{k-1}$ is

$$\langle f, g\rangle_{\mathcal{H}_k/\mathbb{P}_{k-1}} = \left\langle f^{(k)}, g^{(k)} \right\rangle_{L_2}. \tag{3.2}$$

The functional $\mathcal{L}$ can be viewed as acting on $\mathcal{H}_k/\mathbb{P}_{k-1}$, since it vanishes on $\mathbb{P}_{k-1}$. The Peano kernel theorem can be rewritten as

$$\mathcal{L}(f) = \left\langle f^{(k)}, G_k \right\rangle_{L_2}, \tag{3.3}$$

where G_k is defined by (2.6). By the Riesz representation theorem (see, e.g., [4] or [5]), there exists a unique $f_0 \in \mathcal{H}_k/\mathbb{P}_{k-1}$ such that

$$\mathcal{L}(f) = \langle f, f_0\rangle_{\mathcal{H}_k/\mathbb{P}_{k-1}}, \tag{3.4}$$

$$\|\mathcal{L}\| = \|f_0\|_{\mathcal{H}_k/\mathbb{P}_{k-1}} = \sqrt{\mathcal{L}(f_0)}. \tag{3.5}$$

From (3.2) and (3.4)

$$\mathcal{L}(f) = \langle f, f_0 \rangle_{\mathcal{H}_k/\mathbb{P}_{k-1}} = \left\langle f^{(k)}, f_0^{(k)} \right\rangle_{L_2}. \tag{3.6}$$

From (3.3) and (3.6), we have

$$f_0^{(k)} = G_k, \quad \text{almost everywhere,} \tag{3.7}$$

which gives

$$f_0(s) = \frac{(-1)^k}{(2k-1)!} \mathcal{L}_t (t-s)_+^{2k-1}. \tag{3.8}$$

Applying the linear functional $\mathcal{L}$ to both sides of the above equality gives

$$\mathcal{L}(f_0) = \frac{(-1)^k}{(2k-1)!} \mathcal{L}_s \mathcal{L}_t (t-s)_+^{2k-1}, \tag{3.9}$$

which together with (3.5) yields

$$\|\mathcal{L}\|^2 = \frac{(-1)^k}{(2k-1)!} \mathcal{L}_s \mathcal{L}_t (t-s)_+^{2k-1}, \tag{3.10}$$

as desired.

Remark 3.1. From the above proof, we see that f_0 given by (3.8) is the representer of the Hilbert space $\mathcal{H}_k/\mathbb{P}_{k-1}$.

4. Related Results and Further Applications

Numerical integration and quadrature rules are classical topics in numerical analysis while quadrature error functionals are typical continuous linear functionals on function spaces. It was quadrature error functionals that stimulated study of Smale [6] and Wang and Han [2]. So it is natural to consider the applications of (2.4) to quadrature error estimates.

Example 4.1. Let n be a positive integer, $f \in \mathcal{H}_k$ and $x \in [0,1]$. Let the Euler-Maclaurin remainder functional $\mathcal{L}^{\mathrm{EM}}$ be defined by

$$\mathcal{L}^{\mathrm{EM}}(f) = \int_0^1 f(t)dt - \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i+x}{n}\right) + \sum_{\nu=1}^{k-1} \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{n^\nu \nu!} B_\nu(x), \tag{4.1}$$

where $B_\nu(t)$ is the νth Bernoulli polynomial. It is not hard to verify that $\mathcal{L}^{\mathrm{EM}}$ vanishes on $\mathbb{P}_{k-1}$. So the norm of $\mathcal{L}^{\mathrm{EM}}$ can be calculated according to (2.4). It can be found in [2] (cf. [7]) which gave a bound in terms of Bernoulli number $B_{2k} = B_{2k}(0)$; that is,

$$\left\|\mathcal{L}^{\mathrm{EM}}\right\| = \left[\frac{(-1)^{k-1}}{(2k)!}B_{2k} + \left(\frac{B_k(x)}{k!}\right)^2\right]^{1/2}\frac{1}{n^k}. \tag{4.2}$$

Example 4.2. Let m, n be positive integers and $f \in \mathcal{H}_k$. Suppose that the following quadrature rule

$$\int_0^1 f(t)\mathrm{d}t = \sum_{j=0}^{m-1} p_j f(x_j) \tag{4.3}$$

is exact for any polynomial of degree $\le k-1$ for some positive integer k. Then,

$$\mathcal{L}^{\mathrm{CQ}}(f) = \int_0^1 f(t)\mathrm{d}t - \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} p_j f\left(\frac{i+x_j}{n}\right) \tag{4.4}$$

defines a composite quadrature error functional which annihilates any $f \in \mathbb{P}_{k-1}$. So Theorem 3 applies. The expression for the norm of $\mathcal{L}^{\mathrm{CQ}}$ can be found in [2]. A different but easy-to-use expression can also be found in [7]

$$\left\|\mathcal{L}^{\mathrm{CQ}}\right\| = \frac{1}{k!n^k}\left\{\sum_{i,j=0}^{m-1} p_i p_j \left(\frac{(-1)^{k-1}k!^2}{(2k)!}\widetilde{B}_{2k}(x_i - x_j) + B_k(x_i)B_k(x_j)\right)\right\}^{1/2}, \tag{4.5}$$

where $\widetilde{B}_k$ is the Bernoulli function, defined by $\widetilde{B}_k(t) = B_k(\{t\})$. Here $\{t\}$ stands for the fractional part of t.

Example 4.3. The error functionals $\mathcal{L}^M$, $\mathcal{L}^T$, and $\mathcal{L}^S$ for the midpoint rule, trapezoidal quadrature and Simpson's rule are, respectively,

$$\begin{aligned}
\mathcal{L}^M(f) &= \int_0^1 f(t)\mathrm{d}t - f\left(\frac{1}{2}\right),\\
\mathcal{L}^T(f) &= \int_0^1 f(t)\mathrm{d}t - \frac{1}{2}(f(0) + f(1)),\\
\mathcal{L}^S(f) &= \int_0^1 f(t)\mathrm{d}t - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].
\end{aligned} \tag{4.6}$$

They vanishes on $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$, respectively. So, (4.5) applies (see [7] for details). It is a routine computation to find their norms and they can be found in [2] (some of them can also be found in [6–8]). In the following, $\mathscr{H}_k^*$ stands for the dual space of $\mathscr{H}_k$

$$\begin{gathered}\left\|\mathcal{L}^M\right\|_{\mathscr{H}_1^*}=\left\|\mathcal{L}^T\right\|_{\mathscr{H}_1^*}=\frac{1}{2\sqrt{3}},\\ \left\|\mathcal{L}^M\right\|_{\mathscr{H}_2^*}=\frac{1}{8\sqrt{5}},\qquad \left\|\mathcal{L}^T\right\|_{\mathscr{H}_2^*}=\frac{1}{2\sqrt{30}},\\ \left\|\mathcal{L}^S\right\|_{\mathscr{H}_1^*}=\frac{1}{6},\qquad \left\|\mathcal{L}^S\right\|_{\mathscr{H}_2^*}=\frac{1}{12\sqrt{30}},\\ \left\|\mathcal{L}^S\right\|_{\mathscr{H}_3^*}=\frac{1}{48\sqrt{105}},\qquad \left\|\mathcal{L}^S\right\|_{\mathscr{H}_4^*}=\frac{1}{576\sqrt{14}}.\end{gathered}\tag{4.7}$$

From these and (1.4), or equivalently (2.4), we immediately obtain

$$\begin{gathered}\left|\mathcal{L}^M(f)\right|=\left|\int_0^1 f(t)\mathrm{d}t-f\left(\frac{1}{2}\right)\right|\le\begin{cases}\dfrac{1}{2\sqrt{3}}\|f'\|_{L_2}, & \text{if } f\in\mathscr{H}_1,\\[2mm] \dfrac{1}{8\sqrt{5}}\|f''\|_{L_2}, & \text{if } f\in\mathscr{H}_2.\end{cases}\\ \left|\mathcal{L}^T(f)\right|=\left|\int_0^1 f(t)\mathrm{d}t-\frac{1}{2}(f(0)+f(1))\right|\le\begin{cases}\dfrac{1}{2\sqrt{3}}\|f'\|_{L_2}, & \text{if } f\in\mathscr{H}_1,\\[2mm] \dfrac{1}{2\sqrt{30}}\|f''\|_{L_2}, & \text{if } f\in\mathscr{H}_2.\end{cases}\\ \left|\mathcal{L}^S(f)\right|=\left|\int_0^1 f(t)\mathrm{d}t-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]\right|\le\begin{cases}\dfrac{1}{6}\|f'\|_{L_2}, & \text{if } f\in\mathscr{H}_1,\\[2mm] \dfrac{1}{12\sqrt{30}}\|f''\|_{L_2}, & \text{if } f\in\mathscr{H}_2,\\[2mm] \dfrac{1}{48\sqrt{105}}\|f'''\|_{L_2}, & \text{if } f\in\mathscr{H}_3,\\[2mm] \dfrac{1}{576\sqrt{14}}\|f^{(4)}\|_{L_2}, & \text{if } f\in\mathscr{H}_4.\end{cases}\end{gathered}\tag{4.8}$$

Note that there is a mistake in Example 9 in [1]. The constant $1/576\sqrt{14}$ in the last inequality is mistaken to be $1/1152\sqrt{14}$ there.

Recently, there is a flurry of interest in the so-called Ostrowski-Grüss-type inequalities. Some authors, for example, see Ujević [9], consider to bound a quadrature error functional in terms of the Chebyshev functional, that is, $\|f^{(k)}\|_{L_2}^2-(\int_0^1 f^{(k)}(t)\mathrm{d}t)^2$, for some appropriate integer k (see, e.g., [9]). It is worth mentioning that these Ostrowski-Grüss-type inequalities are related to inequality (1.3). Actually, we have the following general result.

Proposition 4.4. *Suppose that a continuous linear functional $\mathcal{L} : \mathcal{H}_k \to \mathbb{R}$ vanishes on $\mathbb{P}_{k-1}$. Then for any nonnegative $j < k$, we have*

$$|\mathcal{L}(f)| \leq \sqrt{\frac{(-1)^j}{(2j-1)!}\mathcal{L}_s\mathcal{L}_t(t-s)_+^{2j-1}}\left(\left\|f^{(j)}\right\|_{L_2}^2 - \left(\int_0^1 f^{(j)}(t)\mathrm{d}t\right)^2\right)^{1/2}. \tag{4.9}$$

Proof. Let p be a polynomial in $\mathbb{P}_{k-1}$ such that $p^{(j)}(t) = 1$. Let

$$F(t) = f(t) - p(t)\int_0^1 f^{(j)}(t)\mathrm{d}t, \quad f \in \mathcal{H}_k. \tag{4.10}$$

Then, $F \in \mathcal{H}_k$ and

$$\mathcal{L}(F) = \mathcal{L}\left(f - p\int_0^1 f^{(j)}(t)\mathrm{d}t\right) = \mathcal{L}(f), \tag{4.11}$$

since $p \in \mathbb{P}_{k-1}$ and $\mathcal{L}$ vanishes on $\mathbb{P}_{k-1}$. Obviously,

$$|\mathcal{L}(f)| = |\mathcal{L}(F)| \leq \|\mathcal{L}\|\left\|F^{(j)}\right\|_{L_2}. \tag{4.12}$$

Moreover, by noting $p^{(j)}(t) = 1$, we have

$$\left\|F^{(j)}\right\|_{L_2} = \left\|f^{(j)} - \int_0^1 f^{(j)}(t)\mathrm{d}t\right\|_{L_2}. \tag{4.13}$$

It is trivial to check that

$$\left\|f^{(j)} - \int_0^1 f^{(j)}(t)\mathrm{d}t\right\|_{L_2}^2 = \left\|f^{(j)}\right\|_{L_2}^2 - \left(\int_0^1 f^{(j)}(t)\mathrm{d}t\right)^2. \tag{4.14}$$

From (4.12)–(4.14) and (2.4), follows (4.9). This completes the proof. □

Note that Proposition 4.4 shows that we have a corresponding inequality (4.9) for every $j < k$ whenever we have inequality (2.4). It should be mentioned, however, (4.9) does not hold for k in general especially when the kernel of $\mathcal{L}$ is exactly $\mathbb{P}_{k-1}$.

Proposition 4.4 can be reformulated in a slightly different language as follows.

Corollary 4.5. *Suppose that $\mathcal{L}$ is a continuous linear functional acting on $\mathcal{H}_k$ and* $\ker\mathcal{L} = \{f \mid \mathcal{L}(f) = 0\} = \mathbb{P}_{k-1}$. *Then for any nonnegative $j < k$, both* (2.4) *and* (4.9) *hold while only* (2.4) *is also valid for k.*

Finally, we end this paper with an inequality of the above-mentioned Grüss-type. More examples are left to the interested readers.

Example 4.6 (see also [7]). From Example 4.3 and Proposition 4.4, we have

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]\right| \leq \frac{1}{6}\left(\|f'\|_{L_2}^2 - \left(\int_0^1 f'(t)\mathrm{d}t\right)^2\right)^{1/2}. \tag{4.15}$$

In view of Proposition 4.4 or Corollary 4.5, the above inequality is still valid with f' replaced by f'' and f''', respectively, and with obvious change in the coefficients. We omit the details.

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References

[1] I. Gavrea and M. Ivan, "An inequality for continuous linear functionals," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 381–384, 2010.
[2] X. H. Wang and D. F. Han, "Computational complexity on numerical integrals," *Science in China. Series A*, vol. 34, no. 1, pp. 42–47, 1991.
[3] X. H. Wang, "Some results relevant to Smale's reports," in *From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990)*, M. W. Hirsch, J. E. Marsden, and M. Shub, Eds., pp. 456–465, Springer, New York, NY, USA, 1993.
[4] J. B. Conway, *A course in Functional Analysis*, vol. 96 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1990.
[5] B. Bollobás, *Linear Analysis: An Introductory Course*, Cambridge University Press, Cambridge, UK, 2nd edition, 1999.
[6] S. Smale, "On the efficiency of algorithms of analysis," *Bulletin of the American Mathematical Society*, vol. 13, no. 2, pp. 87–121, 1985.
[7] H. Ren and S. Yang, "A general L_2 inequality of Grüss type," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, article 54, pp. 1–6, 2006.
[8] S. Yang, "A unified approach to some inequalities of Ostrowski-Grüss type," *Computers & Mathematics with Applications*, vol. 51, no. 6-7, pp. 1047–1056, 2006.
[9] N. Ujević, "Sharp inequalities of Simpson type and Ostrowski type," *Computers & Mathematics with Applications*, vol. 48, no. 1-2, pp. 145–151, 2004.